Czechoslovak Mathematical Journal

Ida Cattaneo Gasparini

# Geometric properties of a sequence of standard minimal immersions between spheres

*Czechoslovak Mathematical Journal*, Vol. 49 (1999), No. 2, 401–414

Persistent URL: http://dml.cz/dmlcz/127496

Czechoslovak Mathematical Journal, 49 (124) (1999), 401–414

# GEOMETRIC PROPERTIES OF A SEQUENCE OF STANDARD MINIMAL IMMERSIONS BETWEEN SPHERES

Ida Cattaneo Gasparini, Roma

(Received September 30, 1996)

## Introduction

In the theory of minimal isometric immersions of Riemannian manifolds, minimal immersions into spheres have particular interest.

Takahashi in [T ]proved that if $\varphi$ is an isometric immersion of a compact Riemannian manifold $(M,g)$ of dimension $n$ into $\mathbb{R}^{m+1}$, such that all the components of $\varphi$ are eigenfunctions of the Laplace-Beltrami operator $\Delta$ on $(M,g)$ corresponding to the *same eigenvalue* $\lambda$, then $\varphi(M)$ is contained in a sphere $S^m(r) \subset \mathbb{R}^{m+1}$ with radius $r = \sqrt{n/\lambda}$ and is minimal in $S^m(r)$. The manifolds $M$ which admit such immersions are the irreducible Riemannian homogeneous spaces, namely those in which the isotropy group of a point acts irreducibly on the tangent space.

One then has, for these manifolds, an explicit method for building up minimal immersions into spheres: any orthonormal basis of the eigenspace $V_{\lambda_k}$ of dimension $m_k$ associated to the eigenvalue $\lambda_k$ for each integer $k \geqslant 1$, such that the multiplicity of $\lambda_k$ is sufficiently high in order to provide the coordinate functions of the immersion, gives rise to a minimal isometric immersion $\varphi_{n,k}$ of $M$ into a sphere $S_1^{m_k-1} \subset V_{\lambda_k}$ and these immersions are called *standard minimal immersions*: *s.m.i.*.

In this paper we shall consider the particular homogeneous spaces $M = SO(n+1)/SO(n)$.

Our point of view is to study the geometric properties of a single *s.m.i.* $\varphi_{n,k}$ by means of the sequence of the previous maps $\varphi_{n,r}$ where $n$ is fixed and $r < k$, basing our study on the intrinsic properties of these maps, namely that they are full, equivariant, and that [d-W]$_2$ there is a universal isomorphism between the normal bundle of degree $r-1$ of a *s.m.i.* $\varphi_{n,r}$ and the spherical harmonic of degree r on the $n-1$ unitary sphere.

This universal pattern allows us to state that the sequence of *s.m.i.* $\varphi_{n,r}$ defines a sequence of osculating immersions of order $r-1$ which we call *generalized*, and that the normal bundles to the osculating spaces to the image of the immersions are parallel (Theorem 3.3).

From [Sa], [Ts], we know that the standard, minimal immersions are helical geodesic immersions, namely immersions such that for each geodesic $\gamma$ of the domain of $\varphi_{n,k}$ the curve $\varphi_{n,k}\gamma$ has constant curvatures which do not depend on $\gamma$.

In Theorem 3.7 substituting the maps $\varphi_{n,2}, \varphi_{n,3}, \dots, \varphi_{n,k}$, with equivalent maps, namely equal modulo an isometry of the ambient space, we prove that given any geodesic $\gamma$ in the sphere $S^n_{c_k}$, sphere of constant sectional curvature $c_k$, the curve $\varphi_{n,k}\gamma \subset S_1^{m_k-1}$ has exactly $(k-1)$ curvatures which are expressed by the eigenvalues of the Laplacian respectively on the spheres $S^n_{c_2}, S^n_{c_3}, \dots, S^n_{c_k}$. This geometric point of view of considering, for the study of a *s.m.i.*, all the previous immersions like a sort of approximation has not been considered before and can be applied in the study of those spaces which admit *s.m.i.*, for instance in the case of generalized symmetric spaces.

Naturally some modifications are necessary as, in these cases, the spectrum of the Laplacian is generally unknown.

We think moreover that a possible application could be in the field of isoparametric maps. As a matter of fact for $r = 2$ a *s.m.i.* is an isoparametric immersion since the osculating space of order 1 of $S^n_{c_2}$ is the tangent space to $S_1^{m_2-1}$. This is the case of the Veronese surface. For $r > 2$ the map $\varphi_{n,r}$ can be considered at the same time as a particular and also as a generalized isoparametric map since it is the generalized osculating space of order $r-1$ of $S_{c_r}$ which is isomorphic, not equal, to the tangent space to $S_1^{m_r-1}$.

It will be interesting to see which machinery used in the study of isoparametric maps can be used in the case of a *s.m.i.* and which results remain valid.

In n. 1 we will summarize the basic notions on higher fundamental forms, in n. 2 the notion of standard, minimal immersions and spherical harmonics, and in n. 3 we give the theorems on osculating immersions and helical immersions.

In this paper the differentiability of all geometric objects will be $C^\infty$.

Let $(\overline{M}, g)$ be a Riemannian manifold, $n$ and $m$ respectively the dimensions of $M$ and $\overline{M}$, $f\colon M \longrightarrow \overline{M}$ an isometric immersion and $\langle , \rangle$ the inner product.

The pull back bundle $f^{-1}(T(\overline{M}))$ of the tangent bundle $T(\overline{M})$ on $M$ splits into the orthogonal direct sum

$$f^{-1}(T(\overline{M})) = T(M) \oplus N(M) \tag{1.1}$$

of Riemannian vector bundles, where $N(M)$ is the normal bundle to the bundle $T(M)$ tangent to $M$.

We denote by $(\ )^T$ and by $(\ )^N$ the tangential and orthogonal projection associated to the splitting (1.1).

**Remark 1.1.** In a small neighborhood $U$ of any point $p \in M$, $f$ is a topological embedding, thus locally we can identify a vector field $Y$ on $M$ with its image $f \cdot Y$ defined on $f(M) \subset \overline{M}$.

Denote by $\overline{\nabla}$ the Levi Civita connection on $\overline{M}$ and by $\nabla$ the induced connection on $M$ via the projection on $T(M)$.

The second fundamental form $\overset{o}{\mathrm{s}}$ of $f$ at $p$ is defined by

$$\overset{o}{\mathrm{s}}(X_p, Y_p) = (\overline{\nabla}_{X_p} Y)^N \tag{1.2}$$

where $X_p, Y_p \in T_p(M)$ and Y is a generic extension of $Y_p$.

**Definition 1.2.** The *first normal space* $N_p^1$ is the linear space spanned by the second fundamental form at $p$ and *the second osculating space* $(O_2)_p$is given by

$$(O_2)_p = T_pM \oplus N_p^1. \tag{1.3}$$

To simplify the notations, we omit sometimes indicating the points in which the fundamental forms, the normal and osculating spaces are calculated.

The higher fundamental forms are then defined inductively.

**Definition 1.3.** The third fundamental form $\overset{1}{\mathrm{s}}$ at $p$ is

$$\overset{1}{\mathrm{s}}(X, Y, Z) = (\overline{\nabla}_X \overset{o}{\mathrm{s}}(Y, Z))^{{o_2}^{\perp}}. \tag{1.4}$$

The *second normal space* $N_p^2$ is the linear span of $\overset{1}{\mathrm{s}}$ and the *third osculating space* $(O_3)_p$ is

$$(O_3)_p = T_pM \oplus N_p^1 \oplus N_p^2. \tag{1.5}$$

If $k$ is any positive integer, proceeding inductively, one can define the fundamental forms $\overset{r-2}{\mathrm{s}}$ at $p$, the normal spaces of order $r-1$ and the osculating space $O_r$ of order $r$:

$$(O_r)_p = T_p(M) \oplus N_p^1 \ldots \oplus N_p^{r-1}. \tag{1.6}$$

We define $(O_1)_p = T_p(M) = N_p^o$.

We can see that that the $k^{\text{th}}$-osculating space of $f$ at $p$ is the subspace of $f^{-1}(T(\overline{M}))$ spanned by those vectors obtained by taking covariant derivative up to the $(k-1)^{\text{th}}$ order.

(For further informations on higher fundamental forms see [Sp]).

As $\dim(T_p(M) \oplus N_p^1 \ldots \oplus N_p^k) \leqslant \dim T_p(\overline{M})$ this process must end.

**Remark 1.4.** If the manifold $M$ has constant curvature, the fundamental forms are symmetric. The generic fundamental form of order $k$ at $p$ define then a map

$$\overset{k-2}{\mathrm{s}}\colon\ S^k(T_p(M)) \longrightarrow N_p^{k-1} \tag{1.7}$$

where $S^k(T_p(M))$ is the symmetric product of $k$ copies of $T_p(M)$.

**Definition 1.5.** Let $q$ be the first integer $\geqslant 1$ such that $\dim N_p^q \neq 0$, but $\dim N_p^{q+1} = 0$. We call $q$ the *normal degree* of the immersion in $p$.

In general the dimension of the normal and of the associated osculating spaces is not constant.

We call *normally regular domain* an open set $M' \subset M$ such that in any point $p \in M'$ the dimension of all the normal spaces $N^r$ $(1 \leqslant r \leqslant q)$ is maximal.

**Definition 1.6.** If $f\colon\ M \longrightarrow \overline{M}$ is an isometric immersion of a homogeneous Riemannian manifold $M = G/K$ in a manifold $\overline{M}$, we say that $f$ is *equivariant*, if there exists a continuous homomorphism $\varrho$ from $G$ into the group $I(\overline{M})$ of the isometries of $\overline{M}$ such that

$$f(g \cdot p) = \varrho(g) f(p) \quad \forall p \in \overline{M}, \quad g \in G \tag{1.8}$$

If the map is *equivariant* we have

$$\varrho(g) N_p^h = N_{g\cdot p}^h \quad \forall p \in M, \quad h \leqslant q \tag{1.9}$$

In this case the dimensions of the normal spaces are constants and we obtain a decomposition of the tangent bundle $f^{-1}(T(\overline{M}))$ in the Whitney sum

$$f^{-1}(T(\overline{M})) = T(M) \oplus N^1 \ldots \oplus N^q \oplus N \tag{1.10}$$

**Definition 1.7.** The *mean curvature* $H$ of an isometric immersion $f\colon\ M \longrightarrow \overline{M}$ is the trace of the second fundamental form.

If $(e_1, e_2, \dots e_n)$ is a local orthonormal frame field, then

$$H = 1/n \sum_{i=1}^{n} (\overline{\nabla}_{e_i} e_i)^{N^1} = 1/n \sum_{i=1}^{n} \overset{o}{\mathrm{s}}(e_i, e_i). \tag{1.11}$$

**Definition 1.8.** The Weingarten operator $A^k$ of order $k$ is defined by

$$(\overline{\nabla}_X \xi^k)^{N^{k-1}} = -A^k(X, \xi^k)$$

for $\xi^k \in \Gamma N^k$, section of $N^k$.

In the sequel the following generalized Frenet formula [Sp] will be used:

$$\overline{\nabla}_X \xi^k = -A^k(X, \xi^k) + (\overline{\nabla}_X \xi^k)^{N^k} + \overset{k}{\mathrm{s}}(X, \xi^k). \tag{1.12}$$

## 2. Standard immersions and spherical harmonics

Let $M = G/K$ be a compact homogeneous Riemannian manifold with metric $g$, and assume that the linear isotropy group acts irreducibly on the tangent space.

If $\lambda \neq 0$ is a real number, we shall denote by $V_\lambda$ the set of functions solution of the Laplace-Beltrami equations:

$$\Delta f + \lambda f = 0. \tag{2.1}$$

Since $M$ is compact, each $V_\lambda$ is a finite dimensional vector space.

Considering that $G$ is a transitive group of isometries of $M$, to each element $g \in G$ we can associate an operator $L_g$ on $V_\lambda$ which transforms the functions $f(p) \in V_\lambda$ where $p \in M$ under the rule

$$L_g f(p) = (g \cdot f)(p) = f(g^{-1} \cdot p). \tag{2.2}$$

$L_g$ defines a *representation of* $G$ *in* $V_\lambda$.

Moreover $V_\lambda$ can be endowed with the inner product

$$(f, g) = \int_M f \cdot g \,\mathrm{d}v. \tag{2.3}$$

For convenience we shall normalize it in such a way that the integral over $M$ of the canonical measure $\mathrm{d}v$ is the dimension $m_\lambda$ of $V_\lambda$.

An orthonormal basis $(f_1, f_2, \dots, f_{m_\lambda})$ of $V_\lambda$, defines a map: $\varphi\colon M \longrightarrow \mathbb{R}^{m_\lambda}$ by $\varphi(p) = (f_1(p), f_2(p), \dots, f_{m_\lambda}(p))$ with $p \in M$.

As $\Delta\varphi = (\Delta f_1, \Delta f_2, \dots, \Delta f_{m_\lambda})$, we see that $\varphi$ is a solution of (2.1).

Due to the normalization, $\sum_i (f_i)^2 = 1.$ for all $p \in M$. It turns out that $\varphi$ defines a map of M into $\varphi(M) \subset S_1^{m_\lambda - 1} \in \mathbb{R}^{m_\lambda}$. The identification of $V_\lambda$ with $\mathbb{R}^{m_\lambda}$ is possible after the choice of the orthonormal basis.

**Remark 2.1.** It can be proved that the action of $G$ on $M$ defined by (2.2) leaves the eigenspace $V_\lambda$ invariant and is an isometry for the inner product (2.3).

If now we change the metric $g$ on $M$ with the metric $\widetilde{g} = \sum_{i=1}^{m_\lambda} (\,\mathrm{d}f_i)^2$ induced on $M$ by the Euclidean metric on $\mathbb{R}^{m_\lambda}$, the immersion $\varphi\colon (M, \widetilde{g})$ into $S^{m_\lambda - 1}$ becomes an isometry. Since both metrics $g$ and $\widetilde{g}$ are $G$-invariant,and hence in $p$ they are invariant under the irreducible action of the isotropy group $K$, the metric $g$ is a multiple of the metric $\widetilde{g}$,namely

$$\widetilde{g} = cg \tag{2.4}$$

for $c > 0$ as the functions $f_i$ are not constants.

This relation is true in any point of $M$ since $G$ acts transitively and isometrically on $M$

Denoting by $\widetilde{M}$ the manifold M with the metric $\widetilde{g}$, the Laplacian of $\widetilde{M}$ is given by $\widetilde{\Delta} = 1/c\Delta$. Thus $\varphi\colon \widetilde{M} \longrightarrow S_1^{m_\lambda - 1}$ becomes an isometric immersion satisfying $\widetilde{\Delta}\varphi + \widetilde{\lambda}\varphi = 0$ where $\widetilde{\lambda} = \lambda/c$.

From Takahashi's theorem it follows that $\varphi$ is a minimal immersion into a sphere of radius $r = \sqrt{n/\widetilde{\lambda}}$. As here r=1 we obtain $c = \lambda/n$ and this determines $\widetilde{g}$.

This immersion is called *standard, minimal immersion* (s.m.i) of degree $k$ if $k$ is the $k$-th non zero eigenvalue associated to the immersion.

**Definition 2.2.** Two minimal standard immersions are called *equivalent or congruent* if they differ by an isometry of the ambient space. Note that a different choice of the orthonormal basis of $V_\lambda$ gives rise to an equivalent immersion.

Let us now consider a particular homogeneous space, namely $M = SO(n+1)/SO(n)$. $M$ can be realized as a sphere $S^n$ of the Euclidean space $\mathbb{R}^{n+1}$ with a metric $g$ of constant curvature 1. In that case the spectrum and the eigenfunction of $M$ are known.

The eigenspace $V_{\lambda_k}$ of the Laplacian on $(S_1^n, g)$ associated to the eigenvalue $\lambda_k$, for each $k \in \mathbb{Z}_+$, are the restrictions to $S_1^n$ of the homogeneous polynomials $P_n^k$ of degree $k$ defined on $\mathbb{R}^{n+1}$ which satisfy on $S_1^n$ the equation $\Delta P = 0$. Such restrictions are called *spherical harmonics of* $S_1^n$.

It is proved that all the harmonic homogeneous polynomials of degree $k$ restrict to $S_1^n$ are eigenfunctions of $\Delta$ with the same eigenvalue.

The value of the eigenvalues are the following:

$$\lambda_k = k(k+n-1)$$

and the dimension of the associated eigenspaces $V_{\lambda_k}$ are

$$m_k = (n+2k-1)\frac{(n+k-2)!}{k!(n-1)!}. \tag{2.5}$$

From the general considerations that we have seen above, it follows that an orthonormal basis of the vector space $V_{\lambda_k}$ consisting of the spherical harmonics of $S^n$ of order $k$ gives a *standard minimal isometric immersion of order k*. We shall denote it $\varphi_{n,k}$:

$$\varphi_{n,k}\colon\ S^n_{c_k} \longrightarrow S_1^{m_k-1} \in \mathbb{R}^{m_k} \tag{2.6}$$

where $S^n_{c_k}$ is the $n$-sphere with constant sectional curvature $c_k$. The curvature $c_k$ is defined by the fact that the metric $\tilde{g}$ in $S^n_{c_k}$ is $(\lambda_k/n)g$.

We obtain then:

$$c_k = \frac{n}{k(k+n-1)}. \tag{2.7}$$

For odd $k$ the standard minimal isometric immersion is a minimal isometric embedding of $S^n_{c_k}$ into $S_1^{m_k-1}$.

For even $k$ all the components of the immersion are invariant under the antipodal map, and we get a minimal isometric embedding of $RP^n$ into $S^{m_k-1}$

**Remark 2.3.** The standard minimal isometric immersions between spheres

$$\varphi_{n,k}\colon\ S^n_{c_k} \longrightarrow S_1^{m_k-1} \in \mathbb{R}^{m_k}$$

have some nice properties:

1) they are *full*; namely $\varphi_{n,k}(S^n_{c_k})$ is not contained in a proper vector subspace of $\mathbb{R}^{m_k}$ or equivalently, in a totally geodesic submanifold of $S_1^{m_k-1}$

2) they are *equivariant* (see definition 1.6). we have then a decomposition (1.10) for the tangent bundle in normal spaces

Let $\varphi_{n,k}\colon S^n_{c_k} \longrightarrow S^{m_k-1}_1$ be a standard minimal immersion. Fixing an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space to the sphere $S^n_{c_k}$ in a point $p$, we have an isomorphism between $T_p(S^n_{c_k})$ and $\mathbb{R}^n$. This isomorphism extends to an isomorphism from the symmetric product $S^h(T_p(S^n_{c_h}))$ of $h$-copies of the tangent space of $S^n_{c_h}$ to the vector space $P^h_n$ of the homogeneous polynomials of degree $h$ on $\mathbb{R}^n$.

We recall two propositions of do-Carmo-Wallach that will be useful in the following.

**Proposition 3.1.** *Let $\varphi_{n,k}$ be a standard, minimal immersion then* $\ker \overset{k}{\mathrm{s}} \supseteq r^2 \cdot P^k_n$ *where $r^2 = \sum\limits_{i=1}^{n} e^2_i \in P^2_n$ is the distance function from the origin of $\mathbb{R}^n$ and $\overset{k}{\mathrm{s}}$ is the fundamental form of order $k+2$. [d-W]$_2$.*

P r o o f. The proposition is proved by induction on $k$. For $k=0$ the minimality condition gives $\overset{o}{\mathrm{s}}(r^2) = 0$.

Suppose now $\overset{k-1}{\mathrm{s}}(r^2 . S^{k-1}(T_p(S^n_{c_k})) = 0$ . As

$$\overset{k}{\mathrm{s}}(Y, X_1, \ldots, X_{n+1}) = (\nabla_Y \overset{k-1}{\mathrm{s}}(X_1, X_2, \ldots, X_{k+1}))^{N^{k+1}}$$

from the inductive hypothesis we conclude that for every $t \in S^k(T_p(S^n_{c_k}))$

$$\overset{k}{\mathrm{s}}(r^2 . t) = 0, \quad \text{i.e.} \quad \overset{k}{\mathrm{s}}(r^2 . P^k_n) = 0 \tag{3.1}$$

namely $\ker \overset{k}{\mathrm{s}} \supseteq r^2 \cdot P^k_n$. □

**Proposition 3.2.** *Let $\varphi_{n,k}$ be a standard minimal immersion. Then the vector normal space $N^{h-1}$ of order $h-1$ is isomorphic to the vector space $H^h_n$ of the spherical harmonics of order $h$ on $S^{n-1}_1$, for every $h \leqslant k$* [d-W]$_2$.

P r o o f. We recall [B] that the space $P^h_n$ of the homogeneous polynomials of degree $h$ of $\mathbb{R}^n$ admits the following decomposition in the direct sum of the subspace $H^h_n$ of the harmonic polynomials homogeneous of degree $h$ in $n$ variables and of the polynomials of the form $r^2 P^{h-2}_n$:

$$P^h_n = H^h_n \oplus r^2 \cdot P^{h-2}_n. \tag{3.2}$$

Using the isomorphism $S^h(T_p(S^n_{c_h})) \cong P^h_n$, from the epimorphism (cf. 1.7)

$$\overset{h-2}{\mathrm{s}}\colon S^h(T_p(S^n_{c_h})) \longrightarrow N^{h-1}_p \tag{3.3}$$

and from Proposition 3.1 and the decomposition (3.2) of $P_n^h$, we obtain the injective map:

$$\overset{h-2}{\mathrm{s}}\colon\ S^h(T_p(S^n_{c_h}) \longrightarrow H^h_n$$

and thus the bijective map

$$\overset{h-2}{\mathrm{s}}\colon\ H^h_n \longrightarrow N^{h-1} \tag{3.4}$$

of vector spaces. Then the isomorphism between the set of the spherical harmonics of order h on $S_1^{n-1}$ and $N^{h-1}$ for every $h \leqslant k$ follows. □

**Theorem 3.3.** *The sequence $\varphi_{n,2}, \varphi_{n,3}, \ldots, \varphi_{n,k}$ of minimal standard immersions between spheres, with $k > 1$, $n > 1$, defines a sequence of generalized osculating immersions of order $2, 3, \ldots, k$. Moreover the normal bundles $N^1, N^2, \ldots, N^{k-1}$ to the osculating spaces $O_1, O_2, \ldots, O_{k-1}$ of the immersions $\varphi_{n,2}, \varphi_{n,3}, \ldots, \varphi_{n,k}$ are parallel.*

Proof. We exclude the case $k = 1$ as for $k = 1$ the map $\varphi_{n,1}$ gives the standard immersion $S_1^n \longrightarrow S_1^n \subset \mathbb{R}^{n+1}$.

We exclude also the case $n = 1$ as for any $k$ and $n = 1$ we have a map from $S^1_{c_k}$ to $S^1_1$.

For $k = 2$, the map $\varphi_{n,2}\colon\ S^n_{c_2} \longrightarrow S_1^{m_2-1}$, considering that the immersion is *full*, gives

$$T(S_1^{m_2-1}) = N^o \oplus N^1 = T(S^n_{c_2}) \oplus N^1 = O_1 \oplus N^1 \tag{3.5}$$

with $\dim N^1 = m_2 - n - 1$, equal to the dimension of the space of the harmonic polynomials of degree two on $\mathbb{R}^n$ restricted to $S_1^{n-1}$. The decomposition (3.5) is the Whitney sum of the tangent bundle $\varphi^{-1}(T(S_1^{m_2-1}))$, since the standard maps are equivariant (cf. 1.4). We semplify the notations omitting $\varphi^{-1}$.

For $k = 3$ we obtain, considering (3.5),

$$T(S_1^{m_3-1}) \cong T(S_1^{m_2-1}) \oplus N^2 \cong N^o \oplus N^1 \oplus N^2 \cong O_2 \oplus N^2 \tag{3.6}$$

where we have called $N^2$ the supplementary of $S_1^{m_2-1}$ in $S_1^{m_3-1}$ and where the $N^1$, which appears in (3.5), is isomorphic to the space $N^1$ which appears in (3.6) as both for Prop. 3.2 are isomorphic to $H^2_n$, spherical harmonic of order 2 on the sphere $S_1^{n-1}$. With $O_3$ is denoted the generalized osculating bundle of order 3 on $S^n_{c_3}$.

By recurrence, for the pull-back bundle of $T(S_1^{m_k-1})$, we obtain the following decomposition in "generalized normal bundles"

$$T(S_1^{m_k-1}) = N^0 \oplus N^1 \oplus \ldots \oplus N^{k-1}. \tag{3.7}$$

We call $O_s = N^o \oplus N^1 \oplus \dots N^{s-1}$ *generalized osculating bundle of order* $s$. Here the adjective generalized is used to point out that the normal spaces are associated to *different maps* between spheres having the radius of the sphere domain varying homothetically with $r$ ($r \leqslant k-1$) and the dimension of the sphere image varying as well at every step.

The isomorphism between these different normal spaces relative to different maps is possible because the normal spaces $N^1, N^2, \dots, N^{k-1}$ are isomorphic to the spherical harmonics respectively of order $2, 3, \dots, k$ on the sphere of dimension $n-1$.

We can therefore state that the sequence of the maps $\varphi_{n,2}, \varphi_{n,3}, \dots, \varphi_{n,k}$ defines a sequence of *generalized osculating immersions*, namely the map $\varphi_{n,k}$ defines a *generalized osculating immersion of order* $k$ of $S^n_{c_k}$ in a sphere of radius one contained in the eigenspace $V_{\lambda_k}$.

Moreover the immersions $j_r\colon\ S_1^{m_r-1} \longrightarrow S_1^{m_{r+1}-1}$ with $r = 2, 3, \dots, k-1$, are totally geodesic since they are inclusions of the unitary euclidean sphere $S^{m_r-1}$ into the unitary euclidean sphere $S^{m_{r+1}-1}$ induced by the inclusions $i_r\colon\ \mathbb{R}^{m_r} \longrightarrow \mathbb{R}^{m_{r+1}}$.

The normal bundles of the immersions $j_r$ consist of those tangent vectors to $S^{m_{r+1}-1}$ supplementary and orthogonal to the tangent vectors to $S^{m_r-1}$.

We deduce that

$$\overline{\nabla}_X \xi^r = 0 \tag{3.8}$$

for any $X \in \varphi_{n,r}(S^n_{c_r})$, $\xi^r \in \Gamma N^r$, section of the bundle $N^r$ normal to the osculating space $O_r$, and $\overline{\nabla}$ covariant derivative in $S_1^{m_r-1}$.

The normal bundles are then parallel. □

**Corollary 3.4.** *For any $k \in \mathbb{Z}_+$, the dimension of the spherical harmonic of order $k$ on $S_1^{n-1}$ is given by the difference between the dimensions of the eigenspaces $V_{\lambda_k}$ and $V_{\lambda_{k-1}}$ of the Laplacian on the unitary $n$-sphere.*

**Definition 3.5.** We recall [Sa], that an isometric immersion $f\colon\ M \longrightarrow \overline{M}$ of a connected complete Riemannian manifold $M$ into a Riemannian manifold $\overline{M}$ is called *helical geodesic* immersion of order $s$ if, for each geodesic $\gamma$ of $M$, the curve $f \cdot \gamma$ has constant curvatures $k_1, k_2, \dots, k_s$ which do not depend on $\gamma$.

It is known that strongly harmonic manifolds admit a helical geodesic minimal immersion into a sphere [Be], and standard immersions between spheres admit such immersions [Ts].

On the subject of helical geodesic immersions of Riemannian manifolds there are interesting papers of [Sa] and of [Ts]. Our approach of considering a standard map as approximated by the previous standard maps seems to show some new aspects as it gives, in the case of $s.m.i.$ between spheres, the values of the various curvatures expressing them by means of eigenvalues of the Laplacian and the dimension $n$.

**Remark 3.6.** It will be convenient to indicate some notations that will be used in the following theorem.

We will indicate $h_{r,s}$ the homothety

$$h_{r,s}\colon\ S^n_{c_r} \longrightarrow S^n_{c_s}, \tag{3.9}$$

by $j_{r,s}$ the totally geodesic immersion

$$j_{r,s}\colon\ S_1^{m_r-1} \longrightarrow S_1^{m_s-1}, m_r < m_s, \tag{3.10}$$

by $i_r$ the canonical immersion

$$i_r\colon\ S^n_{c_r} \longrightarrow \mathbb{R}^{n+1}, \tag{3.11}$$

and by $\overline{\nabla}$ the covariant derivative in the ambient space.

**Theorem 3.7.** *Let $\varphi_{n,k}$ be a s.m.i. of order k, if $\gamma$ is any geodesic of $S^n_{c_k}$ then, in the equivalence class of the minimal, standard immersions $\varphi_{n,r}$ with $r \leqslant k$, the principal curvatures of the curve $\varphi_{n,k}\gamma$ are $\sqrt{n/\lambda_2}, \sqrt{n/\lambda_3}, \ldots, \sqrt{n/\lambda_k}$ with $\lambda_2, \lambda_3, \ldots, \lambda_k$ eigenvalues of the Laplacian respectively on the spheres $S^n_{c_2}, S^n_{c_3}, \ldots, S^n_{c_k}$.*

P r o o f. We start by considering the geodesic $\gamma'$ obtained from the geodesic $\gamma$ by the homothety $h_{k,2}\colon\ S^n_{c_k} \longrightarrow S^n_{c_2}$. Let $\gamma'\colon\ I \longrightarrow S^n_{c_2} \subset \mathbb{R}^{n+1}$ be parametrized by the arc length t with $\gamma(o) = p$ point of $S^n_{c_2}$ and let X be the unitary tangent vector to $\gamma'$ in $p$.

The value of the second fundamental form of the immersion $i_2$ on the couple of vectors (X,X) is

$$\overset{o}{\mathrm{s}}_{i_2}(X,X) = \overline{\nabla}_X X - \nabla_X X = \sqrt{c_2}\cdot \xi^1 \tag{3.12}$$

as the radius of the osculating circle to $\gamma'$ is the radius of the sphere in which the curve lies and where $\xi^1$ is the first unitary principal normal to $\gamma'$.

As $\varphi_{n,2}$ is *full*, $\varphi_{n,2}(S^n_{c_2})$ cannot be contained in a subspace of $\mathbb{R}^{m_2}$. It turns out then that $\xi^1$ must belong to $N^1$.

The value of the second fundamental form of the immersion $\varphi_{n,2}$ on the same couple of vectors is:

$$\overset{o}{\mathrm{s}}_{\varphi_{n,2}}(X,X) = (\overline{\nabla}_X X)^{N^1} = \sqrt{c_2}\cdot \eta^1 \tag{3.13}$$

with $\eta^1 \in \Gamma N^1$. Moreover as $N^1 \in T_p(S_1^{m_2-1})$, with an isometry in $S^{m_2-1}$ we obtain a map $\widetilde{\varphi}_{n,2}$ equivalent to $\varphi_{n,2}$ satisfying the equality:

$$\overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,2}}(X,X) = \sqrt{c_2}\cdot\xi^1. \tag{3.14}$$

From the factorization of $\varphi_{n,k}$ in the product of an homothety and a totally geodesic immersion,namely $\varphi_{n,k} = j_{2,k}\cdot\varphi_{n,2}\cdot h_{k,2}$ and considering that the fundamental forms of an homothety and of a totally geodesic immersions are zero, from the formula of the fundamental forms of a product of two maps (cf. [E.S], [E.R.] we obtain

$$\overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,X) = \sqrt{c_2}\cdot\xi^1. \tag{3.15}$$

To get information on the second curvature of $\varphi_{n,k}\gamma$, we need to consider $\varphi_{n,3}$, osculating map of third order.

By the homothety $h_{2,3}\colon S^n_{c_2} \longrightarrow S^n_{c_3}$ we obtain the geodesic $h_{2,3}\gamma' \subset S^n_{c_3} \subset \mathbb{R}^{n+1}$.

The value of the third fundamental form for the immersion $i_3$, gives

$$\overset{1}{\mathrm{s}}_{i_2}(X,\xi^1) = \sqrt{c_3}\cdot\xi^2 \tag{3.16}$$

with $\|\xi^2\| = 1$, and, since $\varphi_{n,3}$ is full, $\xi^2$ belongs to $N^2$.

By the generalized Frenet formula (cf. 1.12) for the immersion $\varphi_{n,3}$ we obtain

$$\overline{\nabla}_X\xi^1 = -A^1(X,\xi^1) \oplus (\overline{\nabla}_X\xi^1)^{N^1} \oplus \overset{1}{\mathrm{s}}_{\varphi_{n,3}}(X,\xi^1) \tag{3.17}$$

where from the (3.8) of theorem (3.3) is $(\overline{\nabla}_X\xi^1)^{N^1} = 0$ .

Moreover $\left\langle A^1(X,\xi^1),X\right\rangle = \left\langle \overset{0}{\mathrm{s}}_{\varphi_{n,2}}(X,X),\xi^1\right\rangle = \sqrt{c_2}$.

Considering that $\varphi_{n,3}$ is full, for the same reasons valid in the case $k=2$, with an isometry of $S_1^{m_3-1}$ we can find an equivalent standard map such that

$$\overset{1}{\mathrm{s}}_{\widetilde{\varphi}_{n,3}}(X,\xi^1) = \sqrt{c_3}\cdot\xi^2. \tag{3.18}$$

As $\varphi_{n,k} = j_{3,k}\cdot\varphi_{n,3}\cdot h_{k,3}$, we see that

$$\overset{1}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,\xi^1) = \sqrt{c_3}\cdot\xi^2.$$

Moreover (3.17) gives

$$\overline{\nabla}_X\xi^1 = -\sqrt{c_2}.X + \sqrt{c_3}\cdot\xi^2 \tag{3.19}$$

where $c_i = n/\lambda_i$ $(i = 2,3)$ and $\lambda_i$ is an eigenvalue of $\Delta$ on $S_{c_i}$.

By recurrence, we obtain, for the image of $\gamma$ in $S^n_{c_k}$, the $k-1$ equations:

$$\overline{\nabla}_X \xi^{k-2} = -\sqrt{c_{k-1}} \cdot \xi^{k-3} + \sqrt{c_k} \cdot \xi^{k-1}. \tag{3.21}$$

Taking into account that the immersions $j_r$ are totally geodesic, the covariant derivative which appears in the above $k-1$ equations (3.19), (3.20) can all be evaluated in $S_1^{m_k-1}$.

We conclude that the map $\varphi_{n,k}$ is an helical geodesic minimal immersion and the values of the principal curvatures of the curve image of any geodesic $\gamma$ of $S^n_{c_k}$ by $\varphi_{n,k}$ are $\sqrt{n/\lambda_2}, \sqrt{n/\lambda_3}, \dots, \sqrt{n/\lambda_k}$ with $\lambda_2, \lambda_3, \dots, \lambda_k$ eigenvalues of the Laplacian respectively on the spheres $S^n_{c_2}, S^n_{c_3} \dots S^n_{c_k}$.

The equations (3.21) for $k = 2, 3, \dots, k$ give the Frenet equations for any geodesic of $S^n_{c_k}$ with respect to the Frenet frames $R(t) = (X(t), \xi^1(t), \dots, \xi^k(t))$ □

**Corollary 3.8.** *Given any geodesic of $S^n_{c_k}$ we can always find in the equivalence class of the minimal standard immersions $\varphi_{n,r}$, $(r = 2, 3, \dots, k)$ a s.m.i. $\widetilde{\varphi}_{n,r}$ such that the following equalities are verified*

$$\overset{o}{\mathrm{s}}_{i_2}(X,X) = \overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,2}}(X,X) = \overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,X) = \sqrt{c_2} \cdot \xi^1,$$
$$\overset{s-2}{\mathrm{s}}_{i_s}(X, \xi^{s-2}) = \overset{s-2}{\mathrm{s}}_{\widetilde{\varphi}_{n,s}}(X, \xi^{s-2}) = \overset{s-2}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X, \xi^{s-2}) = \sqrt{c_s} \cdot \xi^{s-1}$$

*with* $s = 2, 3, \dots, k+1$.

We thank G. Romani for some useful remarks in the proof of Theorem 3.7.

*References*

[B] *M. Berger, P. Gauduchon, E. Mazet*: Le spèctre d'une variètè Riemannienne. Lecture Notes in Math. 194. Springer-Verlag, 1971.

[Be] *A.L. Besse*: Manifolds all of whose geodesics are closed. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 93. Springer-Verlag, 1978.

[C] *I. Cattaneo Gasparini*: Some remarks on the second variation formula for the area. Annals of Global Analysis and Geometry *11* (1993), 221–235.

[Ca] *E. Calabi*: Minimal immersions of surfaces in Euclidean spheres. J. Differential Geom. *1* (1967), 111–125.

[Car] *A. Carfagna d'Andrea*: Immersions with parallel higher fundamental forms. Beiträge zur Algebra und Geometrie *37* (1996), 75–86.

[C.S] *S. Console, A. Sanini*: Submanifolds with conformal second fundamental form and isotropic immersions. Rivista di Matematica Univ. Parma (1993).

[C.dC.K] *S.S. Chern, M. do Carmo, S. Kobayashi*: On minimal submanifolds of a sphere with second fundamental form of constant length. Funct. Analysis and Related Fields. Springer, New-York, 1970, pp. 59–75.

[Che] *B.Y. Chen*: Geometry of Submanifolds. Marcel Decker, 1973.

[C.R] *I. Cattaneo Gasparini, G. Romani*: Normal and osculating maps for submanifolds of $\mathbb{R}^n$. Proc. Roy. Soc. Edinburg, Sect. A *114* (1990), 39–55.

[d-W]$_1$ *M. do Carmo, N. Wallach*: Representation of compact groups and minimal immersions into spheres. J. Differential Geom. *4* (1970), 91–104.

[d-W]$_2$ *M. do Carmo, N. Wallach*: Minimal immersions of spheres into spheres. Ann. of Math. *(2) 93* (1971), 43–62.

[De-Z] *D. De Turck, W. Ziller*: Minimal immersions of spherical space forms in spheres. Comment. Math. Helvetici *67* (1992), 428–438.

[E-R] *J. Eells and A. Ratto*: Harmonic maps and minimal immersions with symmetries. Ann. of Math. Studies 130 (1993). Princeton Univ. Press.

[E.S] *J. Eells and A. Sampson*: Harmonic mappings of Riemannian manifolds. American J. Math. *86* (1964), 109–160.

[H] *W.Y. Hsiang*: On compact homogeneous minimal submanifolds. Proc. Nat. Acad. Sci. USA *56* (1966), 5–6.

[K.N] *S. Kobayashi and K. Nomizu*: Foundations of Differential Geometry, Vol. I and II. Interscience, New York, 1963 and 1969.

[K] *O. Kowalski*: Generalized symmetric spaces. Lecture Notes in Math. 805. Springer-Verlag, 1980.

[L] *H.B. Lawson Jr.*: Lectures on minimal submanifolds, Vol I.. Publish or Perisch, Univ. of California, Berkeley, 1980.

[Li] *A. Lichnerowicz*: Gèomètrie des groupes de transformations. Dunod, Paris, 1958.

[Sa] *K. Sakamoto*: Helical immersions into unit sphere. Math. Ann. *261* (1982), 63–80.

[Si] *J. Simons*: Minimal varieties in Riemannian manifolds. Ann. of Math. *88* (1968), 62–105.

[Sp.] *M. Spivak*: A comprehensive introduction to differential geometry. Publish or Perish, (1979).

[T] *T. Takahashi*: Minimal immersion of Riemannian manifolds. J. Math. Soc. Japan *18* (1966), 380–385.

[Th] *G. Thorbergsson*: Isoparametric submanifold and their buildings. Ann. Math. *133* (1991), 493–498.

[To] *G. Toth*: Harmonic and Minimal Maps with Applications in Geometry and Physics. Ellis Horwood Series, 1984.

[Ts] *K. Tsukada*: Isotropic minimal immersions of spheres into spheres. J. Math. Soc. Japan *35* (1983), 355–379.

[V] *N. Vilenkin*: Special Functions and the Theory of Groups Representation. Transl. Math. Monograph, Vol. 22. American Math. Soc. Providence, 1968.

[Vi] *J. Vilms*: Totally geodesic maps. J. Diff. Geom. *4* (1970), 73–79.

[W,H] *T.J. Willmore and N. Hitchin*: Global Riemannian Geometry. Ellis Horwood Series, 1989.



*Author's address*: Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", Via Antonio Scarpa, 10, 00161 Roma, Italy.