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GRAPH AUTOMORPHISMS OF A FINITE MODULAR LATTICE

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G. Birkhoff ([2], Problem 6) proposed the following problem:

To find all finite lattices L such that each automorphism of the unoriented graph corresponding to L turns out to be a lattice automorphism.

Let us denote by $\mathcal{C}$ the class of all lattices which satisfy the condition mentioned.

In the present note we give a partial solution to this problem concerning modular lattices. By applying the methods and the results of [3] and [4] we prove

($*$) Let L be a finite modular lattice. Then the following conditions are equivalent:

(i) L belongs to $\mathcal{C}$.

(ii) No direct factor of L having more than one element is self-dual.

Let us remark that the related Problem 5 in [2] (proposed already in [1] as Problem 8 and dealing with unoriented graphs of finite lattices) was solved in [3] for the particular case of modular lattices and remains unsolved for the general case.

1. Preliminaries

In the whole paper L denotes a finite lattice. For $a, b \in L$ we put $a \prec b$ or $b \succ a$ if $a < b$ and the interval $[a, b]$ of L is a two-element set.

Let $G(L)$ be the unoriented graph such that

(i) L is the set of all vertices of $G(L)$;

(ii) a pair $(x, y) \in L \times L$ is an edge in $G(G)$ if and only if either $x \prec y$ or $x \succ y$.

For each lattice A we denote by $A^{\sim}$ the lattice which is dual to A. If there exists an isomorphism of A onto $A^{\sim}$, then A is called self-dual.

Let us have a direct product $A \times B$ of finite lattices A and B. Then for (a_1, b_1), $(a_2, b_2) \in A \times B$ the relation

$$(a_1, b_1) \prec (a_2, b_2)$$

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is valid if and only if either $a_1 \prec a_2$ and $b_1 = b_2$, or $a_1 = a_2$ and $b_1 \prec b_2$. From this we conclude

1.1. Lemma. *Let ψ be an isomorphism of L onto the direct product $A \times B$. Further suppose that χ is an isomorphism of B onto $B^{\sim}$. For each $x \in L$ we put $\varphi(x) = y$, where*

$$\psi(x) = (a, b), \quad y = \psi^{-1}((a, \chi(b))).$$

Then φ is an automorphism of the graph $G(L)$.

1.2. Lemma. *Let the assumptions of* 1.1 *be satisfied. Further suppose that B has more than one element. Then φ fails to be a lattice automorphism on L.*

Proof. Choose $a \in A$. There exist $b_1, b_2 \in B$ with $b_1 \prec b_2$. Put

$$x = \psi^{-1}((a, b_1)), \quad y = \psi^{-1}((a, b_2)).$$

Then $x \prec y$. We have

$$\varphi(x) = \psi^{-1}((a, \chi(b_1))), \quad \varphi(y) = \psi^{-1}((a, \chi(b_2)))$$

and $\chi(b_1) \succ \chi(b_2)$. Therefore $\varphi(x) \succ \varphi(y)$. □

1.3. Corollary. *If L belongs to $\mathcal{C}$, then no direct factor of L having more than one element is self-dual.*

2. Internal direct product decompositions

Let A, B be lattices and let

$$\psi\colon\ L \to A \times B$$

be an isomorphism of L onto the direct product $A \times B$. For $x \in L$ with $\psi(x) = (a, b)$ we put $a = x_A$, $b = x_B$.

Let x^0 be a fixed element of L. We denote

$$A_0 = \{x \in L\colon\ x_B = x^0_B\}, \quad B_0 = \{x \in L\colon\ x_A = x^0_A\}.$$

Then A_0 and B_0 are convex sublattices of L with $A_0 \cap B_0 = \{x^0\}$. Moreover, A_0 is isomorphic to A and B_0 is isomorphic to B.

Consider the mapping

$$\psi_0\colon\ L \to A_0 \times B_0 \tag{1}$$

defined by

$$\psi(x) = (x(A_0), x(B_0)),$$

where $x(A_0)$ is an element of A_0 such that

$$(x(A_0))_A = x_A;$$

similarly, $x(B_0)$ is an element of B_0 such that

$$(x(B_0))_B = x_B.$$

Then the mapping ψ_0 is an isomorphism of L onto the lattice $A_0 \times B_0$. We say that ψ_0 is an internal direct product decomposition of L with the central element x^0. The lattices A_0 and B_0 are called internal direct factors of L. (Cf. [4].)

2.1. Lemma. (Cf. [4], Lemma 2.4.) *Suppose that* (1) *is an internal direct product decomposition of* L *with the central element* x^0 *and that, moreover,*

$$\psi_1\colon\ L \to A_0 \times C_0$$

is also an internal direct product decomposition of L *with the central element* x^0. *Then* $B_0 = C_0$.

Now suppose that L_1 and L_2 are finite modular lattices and that φ is an isomorphism of $G(L_1)$ onto $G(L_2)$. Such situation was investigated in [3].

We denote by $\mathcal{A}_1$ the set of all intervals $[x, y]$ of L_1 such that

$$x \prec y \quad \text{and} \quad \varphi(x) \prec \varphi(y).$$

Further let $\mathcal{B}_1$ be the set of all intervals $[u, v]$ of L_1 such that

$$u \prec v \quad \text{and} \quad \varphi(u) \succ \varphi(v).$$

Analogously we define the sets $\mathcal{A}_2$ and $\mathcal{B}_2$ of intervals of L_2 (with φ^{-1} instead of φ).

Let x_1^0 be a fixed element of L_1. We denote by A_1^0 the set of all elements $x \in L_1$ such that either $x = x_1^0$, or there exist $y_1, y_2, \ldots, y_n \in L_1$ which satisfy the following conditions:

(i) $y_1 = x_1^0$, $y_n = x$,

(ii) if $i \in \{1, 2, \ldots, n-1\}$, then the elements y_i, y_{i+1} are comparable and the corresponding interval of L_1 belongs to $\mathcal{A}_1$.

Similarly we define the set $B_1^0 \subseteq L_1$ (taking $\mathcal{B}_1$ instead of $\mathcal{A}_1$).

Further let x_2^0 be an arbitrary element of L_2. In an analogous way we define the subsets A_2^0 and B_2^0 of L_2 (taking φ^{-1} instead of φ).

Looking at the construction performed in [3] (cf. the lemmas used for proving Theorem 1 in [3]) and applying the notion of the internal direct product decomposition we arrive at the following lemma:

2.2. Lemma. *Under the assumptions as above, there exist internal direct product decompositions*

$$\psi_1\colon\ L_1 \to A_1^0 \times B_1^0 \quad \textit{(with the central element } x_1^0\textit{)},$$
$$\psi_2\colon\ L_2 \to A_2^0 \times B_2^0 \quad \textit{(with the central element } x_2^0\textit{)}$$

such that

(i) *the lattices A_1^0 and A_2^0 are isomorphic,*
(ii) *the lattice B_1^0 is isomorphic to $(B_2^0)^\sim$.*

3. Proof of (*)

Suppose that no direct factor of L having more than one element is self-dual.

Let φ be an automorphism of the graph $G(L)$. We put $L = L_1 = L_2$ and apply Lemma 2.2 above. Choose x^0 in L and put $x^0 = x_1^0 = x_2^0$. Then under the notation as in Section 2 we have

$$\mathcal{A}_1 = \mathcal{A}_2, \quad \mathcal{B}_1 = \mathcal{B}_2.$$

Thus, in the set-theoretical sense, we get $A_1^0 = A_2^0$. Further, since A_1^0 and A_2^0 are sublattices of L, we obtain that A_1^0 and A_2^0 are equal as lattices. Put $A_1^0 = A = A_2^0$. Then in view of 2.2 we obtain internal direct product decompositions

$$\psi_1\colon\ L \to A \times B_1^0,$$
$$\psi_2\colon\ L \to A \times B_2^0$$

with the same central element x^0. Thus according to 2.1,

$$B_1^0 = B_2^0.$$

Moreover, in view of 2.2 (ii), B_1^0 is dually isomorphic to B_2^0, hence B_1^0 is self-dual. Then the assumption yields that B_1^0 is a one element set.

Since the element x^0 of L was arbitrarily chosen, we conclude that the set $\mathcal{B}_1$ must be empty and thus all prime intervals of L belong to $\mathcal{A}_1$.

Let $x, y \in L$. If $x < y$, then there are $y_1, y_2, \ldots, y_n$ in L such that $x = y_1 \prec y_2 \prec \ldots \prec y_n = y$, whence $\varphi(x) = \varphi(y_1) \prec \varphi(y_2) \prec \ldots \prec \varphi(y_n) = \varphi(y)$, thus $\varphi(x) < \varphi(y)$. Conversely, by applying φ^{-1} instead of φ we get that $\varphi(x) < \varphi(y)$ implies $x < y$. Hence φ is a lattice isomorphism.

Therefore we have

3.1. Lemma. *Suppose that L is a modular lattice such that none of its direct factors having more than one element is self-dual. Then each automorphism of $G(L)$ is an automorphism of the lattice L.*

Now, $(*)$ is a consequence of 1.3 and 3.1.

We conclude by remarking that all the above considerations remain valid if the assumption that L is a finite modular lattice is replaced by the assumption that L is a modular lattice such that each bounded chain in L is finite.

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