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ON THE TOPOLOGICAL BOUNDARY OF THE ONE-SIDED SPECTRUM

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Abstract. It is well-known that the topological boundary of the spectrum of an operator is contained in the approximate point spectrum. We show that the one-sided version of this result is not true. This gives also a negative answer to a problem of Schmoeger.

Denote by $\mathscr{L}(X)$ the algebra of all bounded linear operators acting in a Banach space X. For $T \in \mathscr{L}(X)$ denote by $\sigma(T)$, $\sigma_l(T)$ and $\sigma_\pi(T)$ the spectrum, left spectrum and the approximate point spectrum of T, respectively:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\},$$
$$\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left invertible}\},$$
$$\sigma_\pi(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}.$$

It is well-known that $\partial\sigma(T) \subset \sigma_\pi(T) \subset \sigma_l(T) \subset \sigma(T)$. This implies in particular that the outer topological boundaries (= the boundaries of the polynomially convex hull) of $\sigma(T), \sigma_l(T)$ and $\sigma_\pi(T)$ coincide.

The aim of this paper is to show that the inner topological boundaries of σ_l and σ_π can be different.

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We use the following notations. If X is a closed subspace of a Banach space Y then we denote $c(X,Y) = \inf\{\|P\|\colon\ P \in \mathscr{L}(Y) \text{ is a projection with range } X\}$ (if X is not complemented in Y then we set $c(X,Y) = \infty$).

For Banach spaces X and Y denote by $X\hat{\otimes}Y$ and $X\check{\otimes}Y$ the projective and injective tensor products (see [2]). Thus $X\hat{\otimes}Y$ and $X\check{\otimes}Y$ are the completions of the algebraic

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tensor product $X \otimes Y$ endowed with the projective (injective) norms

$$\|u\|_{X\hat{\otimes}Y} = \inf\Big\{\sum_i \|x_i\| \cdot \|y_i\| \colon\ u = \sum_i x_i \otimes y_i\Big\}$$

and

$$\|u\|_{X\check{\otimes}Y} = \sup\big\{|(x^* \otimes y^*)(u)| \colon\ x^* \in X^*, y^* \in Y^*, \|x^*\| \leqslant 1, \|y^*\| \leqslant 1\big\}.$$

Clearly elements of $Y\hat{\otimes}X^*$ can be identified with the trace class operators $X \to Y$ (with the trace norm).

If $\{Y_i\}$ is a family of Banach spaces then we denote by $\bigoplus_i Y_i$ the direct sum of Y_i's with the ℓ_1 norm, $\|\bigoplus y_i\| = \sum_i \|y_i\|$.

Lemma 1. *Let X_i, Y_i $(i \in \mathbb{Z})$ be Banach spaces, $X_i \subset Y_i$. Then*

$$c\Big(\bigoplus_i X_i, \bigoplus_i Y_i\Big) = \sup_i\{c(X_i, Y_i)\}.$$

P r o o f. Denote $X = \bigoplus_i X_i$ and $Y = \bigoplus_i Y_i$.

$\leqslant$: If $P_i \in \mathscr{L}(Y_i)$ are projections with ranges X_i and $\sup_i \|P_i\| < \infty$ then $P = \bigoplus_i P_i$ is a projection onto X with the norm $\|P\| = \sup_i \|P_i\|$.

$\geqslant$: Suppose $P \in \mathscr{L}(Y)$ is a projection with range X. Denote $P_k = Q_k P J_k$ $(k \in \mathbb{Z})$ where $J_k \colon\ Y_k \to Y$ is the natural embedding and $Q_k \colon\ X \to X_k$ the canonical projection. It is easy to check that P_k is a projection with range X_k and $\|P_k\| \leqslant \|P\|$ so that $c(X_k, Y_k) \leqslant c(X, Y)$. □

Lemma 2. *Let E be a finite dimensional subspace of a Banach space X. Then*

$$c(E, X) = \sup\{|\mathrm{tr}(S)| \colon\ S \in \mathscr{L}(E), \|JS\|_{X\hat{\otimes}E^*} \leqslant 1\}$$

where $J \colon\ E \to X$ is the natural embedding.

P r o o f. $\geqslant$: Let P be a projection from X onto E and let $S \in \mathscr{L}(E)$. Then

$$|\mathrm{tr}(S)| = |\mathrm{tr}(PJS)| \leqslant \|PJS\|_{E\hat{\otimes}E^*} \leqslant \|P\| \cdot \|JS\|_{X\hat{\otimes}E^*}.$$

$\leqslant$: Consider $\mathscr{M} = \{JS \colon\ S \in \mathscr{L}(E)\}$ as a subspace of $X\hat{\otimes}E^*$. Define $f \in \mathscr{M}^*$ by $f(JS) = \mathrm{tr}(S)$. The norm of f is equal to $k = \sup\{|\mathrm{tr}(S)| \colon\ S \in \mathscr{L}(E), \|JS\|_{X\hat{\otimes}E^*} \leqslant 1\}$. By the Hahn-Banach theorem there exists an extension $g \in (X\hat{\otimes}E^*)^*$ with the same norm k. Since $(X\hat{\otimes}E^*)^*$ is isometrically isometric to $\mathscr{L}(X,E)$ (see [2], p. 230), there exists $P\in\mathscr{L}(X,E)$ with $\|P\|=k$ and, for all $x\in X$ and $e^*\in E^*$, $\langle Px,e^*\rangle = g(x\otimes e^*)$. In particular, for $e\in E$ and $e^*\in E^*$,

$$\langle Pe,e^*\rangle = g(e\otimes e^*) = f(e\otimes e^*) = \mathrm{tr}(e\otimes e^*) = \langle e,e^*\rangle$$

so that $Pe=e$ and P is a projection with range E. Hence $c(E,X)\leqslant k$. □

Proposition 3. *Let X_1 and X_2 be Banach spaces, let $E_1\subset X_1$ and $E_2\subset X_2$ be finite dimensional subspaces. Then*

$$c(E_1\check{\otimes}E_2, X_1\check{\otimes}X_2) = c(E_1,X_1)\cdot c(E_2,X_2).$$

Proof. It is well-known that $E_1\check{\otimes}E_2$ is a subspace of $X_1\check{\otimes}X_2$ (see [2], p. 225).

$\leqslant$: If $P_i\in\mathscr{L}(X_i)$ is a projection with range E_i $(i=1,2)$ then it is easy to check that $P_1\otimes P_2\in\mathscr{L}(X_1\check{\otimes}X_2)$ is a projection onto $E_1\check{\otimes}E_2$ with $\|P_1\otimes P_2\|\leqslant\|P_1\|\cdot\|P_2\|$.

$\geqslant$: Denote by $J_i\colon E_i\to X_i$ $(i=1,2)$ the natural embedding. Then $J=J_1\otimes J_2$ is the natural embedding of $E_1\check{\otimes}E_2$ into $X_1\check{\otimes}X_2$. Let $\varepsilon>0$. By Lemma 2 there exist $S_i\in\mathscr{L}(E_i)$ $(i=1,2)$ such that $\|J_iS_i\|_{X_i\hat{\otimes}E_i^*}=1$ and $|\mathrm{tr}(S_i)|>c(E_i,X_i)-\varepsilon$ $(i=1,2)$. Consider $S=S_1\otimes S_2\in\mathscr{L}(E_1\check{\otimes}E_2)$. It is easy to check that

$$\mathrm{tr}(S)=\mathrm{tr}(S_1)\cdot\mathrm{tr}(S_2) > (c(E_1,X_1)-\varepsilon)\cdot(c(E_2,X_2)-\varepsilon) \tag{1}$$

and

$$\|JS\|_{(X_1\check{\otimes}X_2)\hat{\otimes}(E_1\check{\otimes}E_2)^*}\leqslant\|J_1S_1\|_{X_1\hat{\otimes}E_1^*}\|J_2S_2\|_{\check{\otimes}X_2\hat{\otimes}E_2^*}=1. \tag{2}$$

To see (2), observe that if $\delta>0$, $J_1S_1=\sum\limits_i x_{1i}\otimes e_{1i}^*$ and $J_2S_2=\sum\limits_j x_{2j}\otimes e_{2j}^*$ for some $x_{1i}\in X_1$, $x_{2j}\in X_2$, $e_{1i}^*\in E_1^*$, $e_{2j}\in E_2^*$, $\sum\limits_i\|x_{1i}\|\cdot\|e_{1i}^*\|<1+\delta$ and $\sum\limits_i\|x_{2j}\|\cdot\|e_{2j}^*\|<1+\delta$ then

$$JS=\sum_{i,j}(x_{1i}\otimes x_{2j})\otimes(e_{1i}^*\otimes e_{2j}^*)$$

where $x_{1i}\otimes x_{2j}\in X_1\check{\otimes}X_2$, $e_{1i}^*\otimes e_{2j}^*\in(E_1\check{\otimes}E_2)^*$ and

$$\sum_{i,j}\|x_{1i}\otimes x_{2j}\|_{X_1\check{\otimes}X_2}\cdot\|e_{1i}^*\otimes e_{2j}^*\|_{(E_1\check{\otimes}E_2)^*}<(1+\delta)^2.$$

Thus we have (2) and together with (1) and Lemma 2 we obtain for $\varepsilon\to0$ the required inequality

$$c(E_1\check{\otimes}E_2, X_1\check{\otimes}X_2)\geqslant c(E_1,X_1)\cdot c(E_2,X_2).$$

□

Theorem 4. *There exists a Banach space Z and an operator $T \in \mathscr{L}(Z)$ such that* $\operatorname{dist}\{0, \sigma_\pi(T)\} > \operatorname{dist}\{0, \sigma_l(T)\} > 0$.

Proof. Fix a Banach space X and a finite dimensional subspace $E \subset X$ such that $c(E, X) = a > 1$ (it is well-known that such a pair exists, see e.g. [11], $ 32). Set

$$\begin{aligned} Y_0 &= X \oplus X \check{\otimes} X \oplus X \check{\otimes} X \check{\otimes} X \oplus \ldots, \\ Y_1 &= E \oplus E \check{\otimes} X \oplus E \check{\otimes} X \check{\otimes} X \oplus \ldots, \\ Y_2 &= E \oplus E \check{\otimes} E \oplus E \check{\otimes} E \check{\otimes} X \oplus \ldots, \\ &\vdots \\ Y_k &= \bigoplus_{i=1}^{\infty} \underbrace{E \check{\otimes} \ldots \check{\otimes} E}_{\min\{k,i\}} \check{\otimes} \underbrace{X \check{\otimes} \ldots \check{\otimes} X}_{\max\{i-k,0\}}. \\ &\vdots \end{aligned}$$

We can consider Y_{k+1} as a subspace of Y_k so that $Y_0 \supset Y_1 \supset Y_2 \supset \ldots$. By Lemma 1 and Proposition 3, $c(Y_j, Y_k) = a^{j-k}$ $(k < j)$. Set $Z = \ldots \oplus Y_0 \oplus \ldots \oplus Y_0 \oplus Y_1 \oplus Y_2 \oplus \ldots$ and let $T \in \mathscr{L}(Z)$ be the shift operator to the left,

$$T(\ldots y_{-2} \oplus y_{-1} \oplus \boxed{y_0} \oplus y_1 \oplus y_2 \ldots) = (\ldots y_{-2} \oplus y_{-1} \oplus y_0 \oplus \boxed{y_1} \oplus y_2 \ldots)$$

(the box denotes the zero position). Clearly T is an isometry so that $\sigma_\pi(T) = \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$ and $\operatorname{dist}\{0, \sigma_\pi(T)\} = 1$.

Further

$$c(T^k Z, Z) = c(\ldots Y_{k-1} \oplus \boxed{Y_k} \oplus Y_{k+1} \oplus \ldots, \ldots Y_0 \oplus \boxed{Y_0} \oplus Y_1 \oplus \ldots) = a^k.$$

In particular TZ is complemented in Z so that T is left invertible.

Denote $t = \operatorname{dist}\{0, \sigma_l(T)\}$ and $U = \{\lambda \in \mathbb{C}\colon |\lambda| < t\}$. By [1] there exists an analytic function $F\colon U \to \mathscr{L}(Z)$ such that $F(\lambda)(T - \lambda) = I$ $(\lambda \in U)$. Let

$$F(\lambda) = \sum_{i=0}^{\infty} F_i \lambda^i \quad (\lambda \in U)$$

be the Taylor expansion of F. Since $F(\lambda)(T - \lambda) = I$ we have $F_0 T = I, F_i T = F_{i-1}$ $(i \geqslant 1)$ so that $F_i T^{i+1} = I$ $(i = 0, 1, \ldots)$. It is easy to check that $T^{i+1} F_i$ is a projection onto $T^{i+1} Z$. Thus

$$a^i = c(T^i Z, Z) \leqslant \|T^i F_{i-1}\| = \|F_{i-1}\|$$

so that the radius of convergence of the function $F(\lambda)=\sum_{i=0}^{\infty}F_i\lambda^i$ is

$$t=\Bigl(\limsup_{i\to\infty}\|F_i\|^{1/i}\Bigr)^{-1}\leqslant a^{-1}<1.$$

Hence $0<\operatorname{dist}\{0,\sigma_l(T)\}<\operatorname{dist}\{0,\sigma_\pi(T)\}$. □

Corollary 5. *In general $\partial\sigma_l(T)\not\subset\sigma_\pi(T)$.*

Remark 6. An operator $T\in\mathscr{L}(X)$ is called semiregular if T has closed range and $\ker(T)\subset\bigcap_{n\geqslant 0}T^nX$. A semiregular operator with a generalized inverse (i.e., with $\ker(T)$ and the range TX complemented) is called regular. Semiregular and regular operators have been studied by many authors, see e.g. [4], [6], [7], [8], [9], [10].

Denote by $\sigma_{sr}(T)=\{\lambda\colon\ T-\lambda \text{ is not semiregular}\}$ and $\sigma_{\rm reg}(T)=\{\lambda\colon\ T-\lambda \text{ is not regular}\}$ the corresponding spectra. The sets $\sigma_{sr}(T)$ and $\sigma_{\rm reg}(T)$ are non-empty compact sets and $\partial\sigma(T)\subset\sigma_{sr}(T)\subset\sigma_{\rm reg}(T)\subset\sigma(T)$.

The previous example shows that in general $\partial\sigma_{\rm reg}(T)\not\subset\sigma_{sr}(T)$. Indeed, let T be the operator constructed in Theorem 4. For $|\lambda|<1$ the operator $T-\lambda$ is bounded below and so semiregular. Further T has a left inverse so that it is regular. On the other hand there exists $\mu\in\mathbb{C}$ with $|\mu|=a^{-1}<1$ such that $T-\mu$ is not left invertible. This means that the range of $T-\mu$ is not complemented and so $T-\mu$ is not regular. Hence $\operatorname{dist}\{0,\sigma_{sr}\}>\operatorname{dist}\{0,\sigma_{\rm reg}\}>0$ and $\partial\sigma_{\rm reg}(T)\not\subset\sigma_{sr}(T)$. This gives a negative answer to Question 1 of [11] (note that by [5], $\operatorname{dist}\{0,\sigma_{sr}(T)\}=\lim\gamma(T^n)^{1/n}$ where γ denotes the Kato reduced minimum modulus).

Remark 7. Let A be a unital Banach algebra and $a\in A$. Denote by

$$\sigma_l(a)=\{\lambda\colon\ A(a-\lambda)\not\ni 1\}$$

and

$$\tau_l(a)=\bigl\{\lambda\colon\ \inf\{\|(a-\lambda)x\|\colon\ x\in A,\|x\|=1\}=0\bigr\}$$

the left spectrum and the left approximate point spectrum of a, respectively. The right spectrum σ_r and the right approximate point spectrum τ_r can be defined analogously. For the algebra $\mathscr{L}(X)$ of operators in a Banach space X, τ_l coincides with σ_π and τ_r coincides with σ_δ. Thus in general $\partial\sigma_l(a)\not\subset\tau_l(a)$ and $\partial\sigma_r(a)\not\subset\tau_r(a)$. In fact, it is much simpler to construct the corresponding example in the context of Banach algebras:

Let A be the Banach space of all formal power series $u = \sum\limits_{i,j=0}^{\infty} \alpha_{ij} a^i b^j$ in two variables a, b with complex coefficients α_{ij} such that

$$\|u\| = \sum_{i,j=0}^{\infty} |\alpha_{ij}| 2^i < \infty.$$

The algebra multiplication in A is determined uniquely by setting $ba = 1_A$ so that

$$(a^i b^j) \cdot (a^k b^l) = \begin{cases} a^{i+k-j} b^l & (k \geqslant j), \\ a^i b^{l+j-k} & (k < j). \end{cases}$$

With this multiplication A becomes a unital Banach algebra.

Clearly $\|a\| = 2$, $\|b\| = 1$ and a is left invertible since $ba = 1$. Further $\|ax\| = 2\|x\|$ for every $x \in A$ so that $\operatorname{dist}\{0, \tau_l(a)\} = 2$.

We show that $\operatorname{dist}\{0, \sigma_l(a)\} = 1$. Since $ba = 1$ and $\|b\| = 1$ it is easy to check that $\operatorname{dist}\{0, \sigma_l(a)\} \geqslant 1$. On the other hand we show that $a - 1$ is not left invertible. Suppose on the contrary that

$$\left(\sum_{i,j=0}^{\infty} \alpha_{ij} a^i b^j\right)(a - 1) = 1 \tag{3}$$

for some α_{ij} with $\sum |\alpha_{ij}| 2^i < \infty$. This means

$$1 = \sum_{i,j=0}^{\infty} a^i b^j (\alpha_{i,j+1} - \alpha_{ij})$$

so that $\alpha_{i,j+1} = \alpha_{ij}$ if either i or j is nonzero. Since $\sum\limits_{i,j} |\alpha_{ij}| 2^i < \infty$ we conclude that $\alpha_{ij} = 0$ for $(i, j) \neq (0, 0)$. This leads to a contradiction with (3).

On the other hand, the following "mixed" result can be proved in a standard way:

Theorem 8. *Let a be an element of a unital Banach algebra A. Then $\partial\sigma_l(a) \subset \tau_r(a)$ and $\partial\sigma_r(a) \subset \tau_l(a)$.*

Proof. Let $\lambda \in \partial\sigma_l(a)$, let $\lambda_n \notin \sigma_l(a)$ and $\lambda_n \to \lambda$. Then $b_n(a - \lambda_n) = 1$ for some $b_n \in A$. We distinguish two cases:

(a) Suppose $\sup \|b_n\| = \infty$. Then $c_n = \frac{b_n}{\|b_n\|}$ satisfies $\|c_n\| = 1$ and

$$\|c_n(a - \lambda)\| = \frac{\|b_n(a - \lambda)\|}{\|b_n\|} \leqslant \frac{\|b_n(a - \lambda_n)\|}{\|b_n\|} + \frac{\|b_n(\lambda_n - \lambda)\|}{\|b_n\|} \leqslant \frac{1}{\|b_n\|} + |\lambda_n - \lambda| \to 0$$

so that $\lambda \in \tau_r(a)$.

(b) Suppose $\sup \|b_n\| < \infty$. Then

$$b_n(a-\lambda) = b_n(a-\lambda_n) + b_n(\lambda_n - \lambda) = 1 + b_n(\lambda_n - \lambda)$$

and $b_n(\lambda_n - \lambda) \to 0$ so that $b_n(a-\lambda)$ is invertible for n big enough. Thus $a-\lambda$ has a left inverse, a contradiction with the assumption $\lambda \in \partial\sigma_l(a) \subset \sigma_l(a)$. □

Corollary 9. *Let a be a left invertible element of a unital Banach algebra A. Then*

$$\operatorname{dist}\{0, \sigma_r(a)\} \leqslant \operatorname{dist}\{0, \tau_r(a)\} \leqslant \operatorname{dist}\{0, \sigma_l(a)\} \leqslant \operatorname{dist}\{0, \tau_l(a)\}.$$

If a has a right inverse then

$$\operatorname{dist}\{0, \sigma_l(a)\} \leqslant \operatorname{dist}\{0, \tau_l(a)\} \leqslant \operatorname{dist}\{0, \sigma_r(a)\} \leqslant \operatorname{dist}\{0, \tau_r(a)\}.$$

(if a is invertible then all these four numbers are equal).

Added in proofs. As another example of an operator T with $\partial\sigma_l(T) \not\subset \sigma_\pi(T)$ may serve the operator constructed by A. Pietsch, Zur Theorie der σ-Transformationen in lokalconvexen Vektorräumen, Math. Nachr. 21 (1960), 347–369, see p. 367–368. This operator is bounded below but not left invertible. Further (see L. Burlando, Continuity of spectrum and spectral radius in algebras of operators, Ann. Fac. Sci. Toulouse 9 (1988), 5–54, Example 1.11), $T-\lambda$ is left invertible for all λ in a punctured neighbourhood of 0.

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