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ON UNIFORMLY GÂTEAUX SMOOTH $C^{(n)}$-SMOOTH NORMS ON SEPARABLE BANACH SPACES

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Abstract. Every separable Banach space with $C^{(n)}$-smooth norm (Lipschitz bump function) admits an equivalent norm (a Lipschitz bump function) which is both uniformly Gâteaux smooth and $C^{(n)}$-smooth. If a Banach space admits a uniformly Gâteaux smooth bump function, then it admits an equivalent uniformly Gâteaux smooth norm.

Let $(X, \|\cdot\|)$ be a separable Banach space. Then it is easy to construct an equivalent uniformly Gâteaux smooth norm on it. Indeed, let $\{x_j \colon j \in \mathbb{N}\}$ be a countable set contained and dense in the unit ball of X. Then

$$|\!|\!|x^*|\!|\!|^2 = \|x^*\|^2 + \sum_{j=1}^{\infty} x^*(x_j)^2/2^j, \quad x^* \in X^*,$$

is easily seen to be an equivalent, dual, and weak* uniformly rotund norm on X^*. Hence the corresponding norm $|\!|\!|\cdot|\!|\!|$ on X is uniformly Gâteaux smooth. For more details see [DGZ, Section II.6]. Now assume that X admits an equivalent $C^{(n)}$-smooth norm. A natural question then is whether X admits an equivalent norm such that this norm would be both uniformly Gâteaux smooth and $C^{(n)}$-smooth. If $n = 1$, then X^* is separable [Ph, Corollary 4.15, Theorem 2.19] and we can assume that the dual norm $\|\cdot\|$ on X^* is locally uniformly rotund. Then the norm $|\!|\!|\cdot|\!|\!|$ on X constructed above is both uniformly Gâteaux smooth and $C^{(1)}$-smooth. However, if $n > 1$, we seriously doubt that $|\!|\!|\cdot|\!|\!|$ would be $C^{(n)}$-smooth provided that $\|\cdot\|$ is.

The aim of this note is to construct such a norm:

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Theorem 1. *Let $(X, \|\cdot\|)$ be a separable Banach space admitting an equivalent $C^{(n)}$-smooth norm, where $n \in \{1, 2, \ldots\} \cup \{\infty\}$. Then X admits an equivalent norm which is both uniformly Gâteaux smooth and $C^{(n)}$-smooth.*

We start with some preliminaries. The sets of positive integers, and real numbers are denoted by $\mathbb{N}$, and $\mathbb{R}$, respectively. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces and $n \in \mathbb{N}$. The symbol $\mathscr{L}^n(X, Y)$ denotes the (Banach) space of n-linear bounded mappings from X to Y endowed with the norm

$$\|L\| = \sup\{\|L(h_1, \ldots, h_n)\| \colon\ h_1, \ldots, h_n \in B_X\}, \quad L \in \mathscr{L}^n(X, Y).$$

If $n = 1$ we write $\mathscr{L}(X, Y)$. If $Y = \mathbb{R}$ we simply write $\mathscr{L}^n(X) = \mathscr{L}(X, \mathbb{R})$. We use the symbol X^* instead of $\mathscr{L}^1(X)$. The closed and open unit balls in X are denoted by B_X and $\mathring{B}_X$ respectively.

Let f be a mapping from X to Y and $x \in X$. We say that f is *Gâteaux differentiable* at x if there exists $L \in \mathscr{L}(X, Y)$ such that

$$\left\|\frac{1}{\tau}[f(x + \tau h) - f(x)] - L(h)\right\| \to 0 \quad \text{as} \quad \tau \to 0$$

for every $h \in X$. Then we denote $f'(x) = L$. Let Ω be an open subset in X. We say that f is *uniformly Gâteaux smooth* on Ω if f is Gâteaux differentiable at every point in Ω and for every $h \in X$

$$\left\|\frac{1}{\tau}[f(x + \tau h) - f(x)] - f'(x)(h)\right\| \to 0 \quad \text{as} \quad \tau \to 0$$

uniformly for $x \in \Omega$. It is easy to check that if f is Lipschitz on Ω, then f is uniformly Gâteaux smooth on Ω if and only if f is Gâteaux differentiable at every point of Ω and for every $\varepsilon > 0$ and every $h \in X$ there exists $\delta > 0$ such that

$$\left\|f'(x)(h) - f'(z)(h)\right\| < \varepsilon$$

whenever $x, z \in \Omega$ and $\|x - z\| < \delta$. We say that the norm $\|\cdot\|$ is *uniformly Gâteaux smooth* if it is uniformly Gâteaux smooth on the set $\{x \in X \colon\ \|x\| > r\}$ where r is some (actually any) positive number.

We say that f is 1-*times Fréchet differentiable* at x if it is Gâteaux differentiable at x and

$$\left\|\frac{1}{\tau}[f(x + \tau h) - f(x)] - f'(x)(h)\right\| \to 0 \quad \text{as} \quad \tau \to 0$$

uniformly for $h \in B_X$. Now let $n \in \{2, 3, \ldots\}$ and assume that we have already defined the $(n-1)$-times Fréchet differentiability and the symbol $f^{(n-1)}(x) \in$

$\mathscr{L}^{(n-1)}(X,Y)$. Assume that the mapping f is defined and $(n-1)$-times Fréchet differentiable at the points of a neighbourhood of x and let there exist $L \in \mathscr{L}^{(n)}(X,Y)$ such that

$$\Big\|\frac{1}{\tau}\big[f^{(n-1)}(x+\tau h_1)(h_2,\dots,h_n) - f^{(n-1)}(x)(h_2,\dots,h_n)\big] - L(h_1,\dots,h_n)\Big\| \to 0 \quad \text{as } \tau \to 0$$

uniformly for $h_1, h_2, \dots, h_n \in B_X$. Then we say that f is *n-times Fréchet differentiable* at x and we denote $f^{(n)}(x) = L$. (It is known that $f^{(n)}(x)$ is then symmetric with respect to the variables $h_1, \dots, h_n \in X$ [C, Chapitre 1, Théorème 5.3.1]. But we shall not need this fact.)

Let $\Omega \subset X$ be an open set and $n \in \mathbb{N}$. We say that the mapping f is $C^{(n)}$-smooth on Ω if it is n-times Fréchet differentiable at every $x \in \Omega$ and the mapping $x \mapsto f^{(n)}(x)$ from Ω to $\mathscr{L}^n(X,Y)$ is continuous on Ω. We can easily check that f is $C^{(n)}$-smooth on Ω if and only if it is n-times Fréchet differentiable at every point of Ω and for every $x \in \Omega$ and every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\Big\|\frac{1}{\tau}\big[f^{(n-1)}(z+\tau h_1)(h_2,\dots,h_n) - f^{(n-1)}(z)(h_2,\dots,h_n)\big] - L(h_1,\dots,h_n)\Big\| < \varepsilon$$

whenever $0 \neq \tau \in (-\delta, \delta)$, $z \in \Omega$, $\|z - x\| < \delta$, and $h_1, \dots, h_n \in B_X$. We say that f is C^∞-smooth on Ω if it is $C^{(n)}$-smooth on Ω for every $n \in \mathbb{N}$. The 0-smoothness means just the continuity of f and we put $f^{(0)} = f$.

P r o o f of Theorem 1. A rough scheeme of the proof: Applying integral convolutions on the norm $\|\cdot\|$ on X countably many times (see [FWZ, Theorem 3.1]), we construct a convex uniformly Gâteaux smooth function f. If $\|\cdot\|$ is $C^{(n)}$-smooth, then f will also be $C^{(n)}$-smooth. Now an implicit function theorem produces from f a $C^{(n)}$-smooth norm. However we are affraid that the implicit function theorem does not work for the uniform Gâteaux smoothness in general. Hence we need more work: From f we construct, via an integration, a new, better function g. Applying then the implicit function theorem to g, we get a norm which satisfies the conclusion of our theorem.

Step 1. Basic construction. Let $\{x_j\colon\ j \in \mathbb{N}\}$ be a countable set which is contained and dense in the unit ball of X. Denote

$$T = \big[-\tfrac14, \tfrac14\big] \times \big[-\tfrac18, \tfrac18\big] \times \big[-\tfrac1{16}, \tfrac1{16}\big] \times \dots,$$

$Q = [-1,1]^{\mathbb{N}}$, and

$$K = \Big\{\sum_{j=1}^{\infty} t_j x_j\colon\ t = (t_1, t_2, \dots) \in T\Big\}.$$

Note that T, Q and K are compact spaces. Let $\varphi\colon \mathbb{R} \to [0, +\infty)$ be a C^1-smooth function, with support in $[-\frac{1}{2}, \frac{1}{2}]$ and such that $\int_{\mathbb{R}} \varphi = 1$. For $m \in \mathbb{N}$, and $t = (t_1, t_2, \dots) \in Q$ we define

$$\psi_m(t) = \sum_{j=1}^{m} t_j x_j \quad \text{and} \quad \varphi_m(t) = 2\varphi(2t_1)4\varphi(4t_2)\dots 2^m\varphi(2^m t_m).$$

Let μ be the product of countable many Lebesgue measures on $[-1, 1]$. For $m \in \mathbb{N}$ we define

$$f_m(x) = \int_Q \|x - \psi_m(t)\| \varphi_m(t) \,\mathrm{d}\mu(t), \quad x \in X.$$

Note that the integrand here is continuous on (the compact) Q. Hence $f_m(x)$ is well defined for every $x \in X$. Observe also that f_m is a convex function and that

$$|f_m(x) - f_m(y)| \leqslant \|x - y\| \int_Q \varphi_m(t) \,\mathrm{d}\mu(t) = \|x - y\| \quad \text{for all} \quad x, y \in X.$$

Observe further that

$$\begin{aligned} |f_{m_1}(x) - f_{m_2}(x)| &\leqslant \int_Q \Big\| \sum_{j=m_1+1}^{m_2} t_j x_j \Big\| \varphi_{m_2}(t) \,\mathrm{d}\mu(t) \\ &\leqslant \sum_{j=m_1+2}^{m_2+1} 2^{-j} \to 0 \text{ as } m_1 \leqslant m_2 \to \infty \end{aligned}$$

uniformly for $x \in X$. Hence we can put

$$f(x) = \lim_{m\to\infty} f_m(x), \quad x \in X.$$

This is a convex 1-Lipschitz function on X and $\|\cdot\| - \frac{1}{2} \leqslant f \leqslant \|\cdot\| + \frac{1}{2}$. $\square$

Step 2. For a function $g\colon X \to \mathbb{R}$, $x \in X$, and $h \in X$ we put

$$Dg(x)(h) = \lim_{\tau\to 0} \frac{1}{\tau}[g(x + \tau h) - g(x)]$$

if this limit exists and is finite. Fix $x \in X$ and $i, m \in \mathbb{N}$, $i \leqslant m$. Then

$$\begin{aligned}
Df_m(x)(x_i) &= \lim_{\tau\to 0} \frac{1}{\tau}[f_m(x+\tau x_i) - f_m(x)] \\
&= \lim_{\tau\to 0} \frac{1}{\tau}\Bigg[\int_Q \Bigg\| x - \sum_{j\neq i}^{m} t_j x_j - (t_i - \tau)x_i \Bigg\| 2\varphi(2t_1)4\varphi(4t_2)\dots 2^m\varphi(2^m t_m)\,\mathrm{d}\mu(t) \\
&\quad - \int_Q \Bigg\| x - \sum_{j\neq i}^{m} t_j x_j - t_i x_i \Bigg\| 2\varphi(2t_1)4\varphi(4t_2)\dots 2^m\varphi(2^m t_m)\,\mathrm{d}\mu(t)\Bigg] \\
&= \lim_{\tau\to 0} \int_Q \Bigg\| x - \sum_{j=1}^{m} t_j x_j \Bigg\| 2\varphi(2t_1)\dots 2^{i-1}\varphi(2^{i-1}t_{i-1})2^i \frac{1}{\tau}\big[\varphi(2^i(t_i+\tau)) - \varphi(2^i t_i)\big] \\
&\quad \times 2^{i+1}\varphi(2^{i+1}t_{i+1})\dots 2^m\varphi(2^m t_m)\,\mathrm{d}\mu(t) \\
&= \int_Q \|x - \psi_m(t)\| \varphi_m^i(t)\,\mathrm{d}\mu(t),
\end{aligned}$$

where

$$\varphi_m^i(t) = 2\varphi(2t_1)\dots 2^{i-1}\varphi(2^{i-1}t_{i-1})2^{2i}\varphi'(2^i t_i)2^{i+1}\varphi(2^{i+1}t_{i+1})\dots 2^m\varphi(2^m t_m),$$

$t = (t_1, t_2, \dots)$. Here we used the substitution $t_i - \tau \mapsto t_i$, the fact that φ is a $C^{(1)}$-smooth function and the Lebesgue dominated convergence theorem. □

Step 3. Fix $x \in X$ and $i \in \mathbb{N}$ and denote $I = (-1, 1)$. Put

$$\varphi(s) = f(x + s x_i), \quad s \in I,$$

$$\varphi_m(s) = f_m(x + s x_i), \quad s \in I, \quad m \in \mathbb{N}.$$

Then

$$\varphi_m(s) - \varphi(s) = f_m(x + s x_i) - f(x + s x_i) \to 0 \quad \text{as} \quad m \to \infty$$

(uniformly) for $s \in I$. Also, since ${\varphi_m}'(s) = Df_m(x + s x_i)(x_i)$, we have, by Step 2, that

$$\big|{\varphi_{m_1}}'(s) - {\varphi_{m_2}}'(s)\big| \leqslant \sum_{j=m_1+2}^{m_2+1} 2^{-j}\cdot 2^{2i}\int_{\mathbb{R}} |\varphi'(2^i t_i)|\mathrm{d}t_i \to 0 \quad \text{as} \quad i \leqslant m_1 \leqslant m_2 \to \infty$$

uniformly for $s \in I$. By a well known theorem from real analysis we then get that φ is a $C^{(1)}$-smooth function on I and that

$$\varphi'(s) = \lim_{m\to\infty} {\varphi_m}'(s) \quad \text{for} \quad s \in I.$$

Thus, in particular,

$$\begin{aligned} Df(x)(x_i) &= \varphi'(0) = \lim_{m\to\infty} {\varphi_m}'(0) \\ &= \lim_{m\to\infty} \int_Q \|x-\psi_m(t)\| \varphi_m^i(t)\,\mathrm{d}\mu(t). \end{aligned}$$

From this and the definition of φ_m^i it follows that the function $x \mapsto Df(x)(x_i)$ is C_i-Lipschitz, where $C_i = 2^i \int_{\mathbb{R}} |\varphi'(s)|\,\mathrm{d}s$. $\square$

Step 4. Fix $x \in X$ and take any $h \in B_X$ and any $\varepsilon > 0$. We find $i \in \mathbb{N}$ so that $\|h - x_i\| < \varepsilon/2$. Since f is 1-Lipschitz, we have

$$\begin{aligned} &\limsup_{\tau,\tau'\to 0} \Big| \frac{1}{\tau}[f(x+\tau h) - f(x)] - \frac{1}{\tau'}[f(x+\tau' h) - f(x)] \Big| \\ &\leqslant \limsup_{\tau,\tau'\to 0} \Big| \frac{1}{\tau}[f(x+\tau x_i) - f(x)] - \frac{1}{\tau'}[f(x+\tau' x_i) - f(x)] \Big| + 2\|h-x_i\| < \varepsilon. \end{aligned}$$

Hence $Df(x)(h)$ exists. Now, since f is convex and Lipschitz, we conclude that f *is Gâteaux differentiable at* x. $\square$

Step 5. Fix $h \in B_X$. Then for $x, y \in X$ we have by Step 3

$$\begin{aligned} |f'(x)(h) - f'(y)(h)| &< |f'(x)(x_i) - f'(y)(x_i)| + \big(\|f'(x)\| + \|f'(y)\|\big)\|h - x_i\| \\ &< C_i\|x-y\| + 2\|h-x_i\|. \end{aligned}$$

So, if $i \in \mathbb{N}$ is such that $\|h - x_i\| < \varepsilon/4$ and if $\|x - y\| < \varepsilon/(2C_i)$, we get $|f'(x)(h) - f'(y)(h)| < \varepsilon$. From this it follows that f is *uniformly Gâteaux smooth on all of* X. $\square$

Step 6. Using an implicit function theorem, we could produce an equivalent norm from f. However, we are not quite sure if such a norm would keep the uniform Gâteaux smoothness. (For a better understanding of our worries, see Step 9.) In what follows we "improve" f. Let $\eta\colon \mathbb{R} \to [0, +\infty)$ be a C^1-smooth function with support in $[1, 2]$ and such that $\int_{\mathbb{R}} \eta = 1$. Put

$$g(x) = \int_{\mathbb{R}} f(sx)\eta(s)\,\mathrm{d}s, \quad x \in X.$$

We can easily check that the function g is well defined on all of X, and that g is convex and 2-Lipschitz. Moreover, as $\|\cdot\| - \frac{1}{2} \leqslant f \leqslant \|\cdot\| + \frac{1}{2}$, we have

$$\|x\| - \frac{1}{2} \leqslant g(x) \leqslant 2\|x\| + \frac{1}{2}, \quad x \in X.$$

$\square$

Step 7. Fix $x \in X$ and put

$$L(h) = \int_{\mathbb{R}} f'(sx)(h) s\eta(s)\,\mathrm{d}s, \quad h \in X.$$

It is easy to verify that $L(h)$ is well defined and that L is a linear bounded functional on X, with $|L(h)| \leqslant \|h\| \int_{\mathbb{R}} s\eta(s)\,\mathrm{d}s \leqslant 2\|h\|$, $h \in X$. Fix any $\varepsilon > 0$ and any $h \in B_X$. Since f is uniformly Gâteaux smooth (Gâteaux differentiability is actually enough), there is $\delta > 0$ so that

$$|f'(sx+z)(h) - f'(sx)(h)| < \varepsilon/2 \quad \text{whenever} \quad s \in [1,2] \quad \text{and} \quad z \in X,\ \|z\| < 2\delta.$$

Thus for $0 \neq \tau \in (-\delta, \delta)$ we have

$$\begin{aligned}
&\Big|\frac{1}{\tau}[g(x+\tau h) - g(x)] - L(h)\Big| \\
&= \left|\int_{\mathbb{R}} \Big(\frac{1}{\tau}[f(s(x+\tau h)) - f(sx)] - f'(sx)(h)s\Big)\eta(s)\,\mathrm{d}s\right| \\
&= \left|\int_{\mathbb{R}} \int_0^1 [f'(sx+\theta s\tau h)(h) - f'(sx)(h)]\,\mathrm{d}\theta s\eta(s)\,\mathrm{d}s\right| \\
&< \int_{\mathbb{R}} (\varepsilon/2) s\eta(s)\,\mathrm{d}s < 2\varepsilon/2 = \varepsilon.
\end{aligned}$$

This means that g *is Gâteaux differentiable at* x and

$$g'(x)(h) = \int_{\mathbb{R}} f'(sx)(h)s\eta(s)\,\mathrm{d}s, \quad x \in X, \quad h \in X.$$

Now, since f is uniformly Gâteaux smooth on X, the above formula yields that g *is also uniformly Gâteaux smooth on* X. Moreover, integrating by parts, we obtain

$$(*) \qquad g'(x)(x) = -\int_{\mathbb{R}} f(sx)(s\eta'(s) + \eta(s))\,\mathrm{d}s, \quad x \in X.$$

□

Step 8. Consider the set

$$U = \{x \in X :\ g(x) \leqslant 1\}.$$

From the properties of g we can easily deduce that U is a convex, closed, and bounded set. Since $g(0) = f(0) \leqslant 1/2$, the interior of U contains 0. Let p denote Minkowski's functional of U. Then p will be positively homogeneous, convex, continuous, and $p(x) = 0$ if and only if $x = 0$.

In what follows we shall show that p is Gâteaux differentiable and we shall derive a formula for p'. We shall proceed as in [BF]. Fix $x \in X$, with $p(x) = 1$ and let ξ be an element of the subdifferential $\partial p(x)$. Note that then $g(x) = 1$. Let $h \in X$ be such that $\xi(h) = 0$. Then $p(x + \tau h) \geqslant p(x) = 1$ for $\tau > 0$. Take any $r > 1$. Then $p(r(x + \tau h)) = rp(x + \tau h) > 1$ and hence $r(x + \tau h) \notin U$, i.e., $g(r(x + \tau h)) > 1$ for all $r > 1$ and all $\tau > 0$. Thus $g(x + \tau h) \geqslant 1$ $(= g(x))$ for all $\tau > 0$. It follows $g'(x)(h) \geqslant 0$. This holds for every $h \in X$ satisfying $\xi(h) = 0$. Therefore $\xi = \lambda g'(x)$ for a suitable $\lambda \in \mathbb{R}$. But

$$1 = p(x) = \xi(x) = \lambda g'(x)(x),$$

so $\xi = [g'(x)(x)]^{-1} g'(x)$. Hence ξ is uniquely determined, which means that p is Gâteaux differentiable at x and $p'(x) = \xi$. Thus, for every $0 \neq x \in X$ we have

$$(**) \qquad p'(x) = \Big[g'\Big(\frac{x}{p(x)}\Big)\Big(\frac{x}{p(x)}\Big)\Big]^{-1} g'\Big(\frac{x}{p(x)}\Big).$$

□

Step 9. It remains to prove that p is uniformly Gâteaux smooth outside of a neighbourhood of the origin, say on $\Omega = \{x \in X \colon p(x) > r\}$, where r is a fixed positive number. So fix $h \in X$ and consider any $x, y \in \Omega$. Write $\tilde{x} = \frac{x}{p(x)}$, $\tilde{y} = \frac{y}{p(y)}$. Then $p(\tilde{x}) = p(\tilde{y}) = 1 = g(\tilde{x}) = g(\tilde{y})$,

$$g'(\tilde{x})(\tilde{x}) \geqslant g(\tilde{x}) - g(0) \geqslant \tfrac{1}{2}, \quad g'(\tilde{y})(\tilde{y}) \geqslant \tfrac{1}{2},$$

and

$$\begin{aligned} p'(x)(h) - p'(y)(h) &= \big[g'(\tilde{x})(\tilde{x})\big]^{-1} g'(\tilde{x})(h) - \big[g'(\tilde{y})(\tilde{y})\big]^{-1} g'(\tilde{y})(h) \\ &= \big[g'(\tilde{x})(\tilde{x})\big]^{-1} \big[g'(\tilde{x})(h) - g'(\tilde{y})(h)\big] \\ &\quad + \big[g'(\tilde{x})(\tilde{x}) g'(\tilde{y})(\tilde{y})\big]^{-1} \big[g'(\tilde{y})(\tilde{y}) - g'(\tilde{x})(\tilde{x})\big] g'(\tilde{y})(h) \\ &\leqslant 2\big|g'(\tilde{x})(h) - g'(\tilde{y})(h)\big| + 4\big|(g'(\tilde{y})(\tilde{y}) - g'(\tilde{x})(\tilde{x})\big|\big|g'(\tilde{x})(h)\big|. \end{aligned}$$

Recall that $|g'(\tilde{x})(h)| \leqslant 2\|h\|$. Also, from $(*)$ we get $\big|g'(\tilde{y})(\tilde{y}) - g'(\tilde{x})(\tilde{x})\big| \leqslant k\|\tilde{y} - \tilde{x}\|$, where $k = 2\int_Q |s\eta'(s) + \eta(s)|\,\mathrm{d}s$. Thus

$$p'(x)(h) - p'(y)(h) \leqslant 2\big|(g'(\tilde{x})(h) - g'(\tilde{y})(h)\big| + 8k\|\tilde{x} - \tilde{y}\|.$$

Now, recalling that g is uniformly Gâteaux smooth on X, it is enough to show that $\|\tilde{x} - \tilde{y}\|$ is majorized by a multiple of $\|x - y\|$. From the definition of p we know that it has all features of an equivalent norm but the symmetry. So there are constants $a, b > 0$ such that $a\|\cdot\| \leqslant p \leqslant b\|\cdot\|$. Thus for $x, y \in \Omega$ we have

$$\begin{aligned}\|\tilde{x} - \tilde{y}\| = \left\|\frac{x}{p(x)} - \frac{y}{p(y)}\right\| &\leqslant \frac{\|x-y\|}{p(x)} + \frac{\|y\|}{p(x)}\frac{|p(y)-p(x)|}{p(y)} \\ &\leqslant \frac{1}{r}\|x-y\| + \frac{b}{ar}\|x-y\|.\end{aligned}$$

Therefore *p is uniformly Gâteaux smooth on Ω.*

Putting $|\!|\!|x|\!|\!| = p(x) + p(-x)$, $x \in X$, we get an equivalent uniformly Gâteaux smooth norm on X. $\square$

Step 10. From now on fix $n \in \mathbb{N}$ and assume that our original norm $\|\cdot\|$ on X is $C^{(n)}$-smooth . We shall show that the norm $|\!|\!|\cdot|\!|\!|$ defined in the previous step is in fact $C^{(n)}$-smooth .

Claim. *The functions f_m, $m \in \mathbb{N}$, from Step 1 are $C^{(n)}$-smooth on $X \setminus \frac{1}{2}B_X$ and*

$$f_m^{(n)}(x)(h_1, \ldots, h_n) = \int_Q \|\cdot\|^{(n)}(x - \psi_m(t))(h_1, \ldots, h_n)\varphi_m(t)\,\mathrm{d}\mu(t)$$

for $x \in X \setminus \frac{1}{2}B_X$ and $h_1, \ldots, h_n \in X$.

P r o o f. Surely, the claim is true for $n := 0$. Assume that the claim was verified for $n-1$. Fix $x \in X$, with $\|x\| > 1/2$, and for $m \in \mathbb{N}$ put

$$L_m(h_1, \ldots, h_n) = \int_Q \|\cdot\|^{(n)}(x - \psi_m(t))(h_1, \ldots, h_n)\varphi_m(t)\,\mathrm{d}\mu(t), \quad h_1, \ldots, h_n \in X.$$

Note that the integrand here is equal to 0 for all $t \in Q \setminus T$. Further $x - \psi_m(t) \neq 0$ for all $t \in T$. And, since $\|\cdot\|^{(n)}$ is continuous on $X \setminus \{0\}$, $L_m(h_1, \ldots, h_n)$ is well defined and $L_m \in \mathscr{L}^n(X)$.

Let $\varepsilon > 0$ be given. From the compactness of K and from the continuity of $\|\cdot\|^{(n)}$ we find $\delta \in (0, \frac{1}{2}\|x\| - \frac{1}{4})$ so that

$$\left\|\|\cdot\|^{(n)}(z - \psi_m(t)) - \|\cdot\|^{(n)}(x - \psi_m(t))\right\| < \varepsilon$$

whenever $z \in X$, $\|z - x\| < 2\delta$, $m \in \mathbb{N}$, and $t \in T$.

Now take any $0 \neq \tau \in (-\delta, \delta)$, $h_1, \ldots, h_n \in B_X$, $m \in \mathbb{N}$, and $y \in X$ with $\|y - x\| < \delta$. Then, by the induction assumption,

$$\Big|\frac{1}{\tau}\big[f_m^{(n-1)}(y + \tau h_1)(h_2, \ldots, h_n) - f_m^{(n-1)}(y)(h_2, \ldots, h_n)\big] - L_m(h_1, \ldots, h_n)\Big|$$

$$= \Big|\int_Q \Big(\frac{1}{\tau}\big[\|\cdot\|^{(n-1)}(y + \tau h_1 - \psi_m(t))(h_2, \ldots, h_n)$$

$$- \|\cdot\|^{(n-1)}(y - \psi_m(t))(h_2, \ldots, h_n)\big] - \|\cdot\|^{(n)}(x - \psi_m(t))(h_1, \ldots, h_n)\Big)\varphi_m(t)\,\mathrm{d}\mu(t)\Big|$$

$$= \Big|\int_Q \int_0^1 \big[\|\cdot\|^{(n)}(y + \theta\tau h_1 - \psi_m(t))(h_1, \ldots, h_n)$$

$$-\|\cdot\|^{(n)}(x - \psi_m(t))(h_1, \ldots, h_n)\big]\,\mathrm{d}\theta\varphi_m(t)\,\mathrm{d}\mu(t)\Big| < \varepsilon.$$

(Here we used the continuity of $\|\cdot\|^{(n)}$ and the integral mean value theorem.) This means that f_m is $C^{(n)}$-smooth at x and that

$$f_m^{(n)}(x)(h_1, \ldots, h_n) = \int_Q \|\cdot\|^{(n)}(x - \psi_m(t))(h_1, \ldots, h_n)\varphi_m(t)\,\mathrm{d}\mu(t).$$

□

Step 11. **Claim.** *The function f defined in Step* 1 *is $C^{(n)}$-smooth on $X \setminus \frac{1}{2}B_X$ and*

$$\|f_m^{(n)}(x) - f^{(n)}(x)\| \to 0 \quad \text{as} \quad m \to \infty$$

for every $x \in X$, with $\|x\| > 1/2$.

Proof. Surely, the claim is true for $n := 0$. Assume that the claim was verified for $n-1$. Fix $x \in X \setminus \frac{1}{2}B_X$. From the compactness of K and the continuity of $\|\cdot\|^{(n)}$ we get that

$$\Big\|\|\cdot\|^{(n)}(x - \psi_{m_1}(t)) - \|\cdot\|^{(n)}(x - \psi_{m_2})(t))\Big\| \to 0 \quad \text{as} \quad m_1, m_2 \to \infty$$

uniformly for $t \in T$. Hence, by Step 10,

$$\Big\|f_{m_1}^{(n)}(x) - f_{m_2}^{(n)}(x)\Big\| \to 0 \quad \text{as} \quad m_1, m_2 \to \infty.$$

Thus, we can define $L = \lim\limits_{m\to\infty} f_m^{(n)}(x)$ and L belongs to $\mathscr{L}^n(X)$.

Fix $\varepsilon > 0$ and let δ be that chosen in Step 10. Take $0 \neq \tau \in (-\delta, \delta)$, $h_1, \ldots, h_n \in B_X$, and $y \in X$, with $\|y - x\| < \delta$. Then, according to Step 10, we have

$$\Big|\frac{1}{\tau}\big[f_m^{(n-1)}(y + \tau h_1)(h_2, \ldots, h_n) - f_m^{(n-1)}(y)(h_2, \ldots, h_n)\big] - f_m^{(n)}(x)(h_1, \ldots, h_n)\Big| < \varepsilon$$

for all $m \in \mathbb{N}$. Now let m go to ∞. By the induction assumption, we thus get that

$$\left|\frac{1}{\tau}\bigl[f^{(n-1)}(y+\tau h_1)(h_2,\ldots,h_n) - f^{(n-1)}(y)(h_2,\ldots,h_n)\bigr] - L(h_1,\ldots,h_n)\right| \leqslant \varepsilon.$$

This means that f is $C^{(n)}$-smooth at x and that $f^{(n)}(x) = L$ $(= \lim\limits_{m\to\infty} f_m^{(n)}(x))$. $\square$

Step 12. **Claim.** *The function g defined in Step 6 is also $C^{(n)}$-smooth on $X\setminus\frac{1}{2}B_X$ and*

$$g^{(n)}(x)(h_1,\ldots,h_n) = \int_{\mathbb{R}} f^{(n)}(sx)(h_1,\ldots,h_n)s^n\eta(s)\,\mathrm{d}s$$

for all $x \in X \setminus \frac{1}{2}B_X$ and all $h_1,\ldots,h_n \in B_X$.

Proof. Surely, the claim is true for $n := 0$. Assume that the claim was verified for $n-1$. Fix $x \in X \setminus \frac{1}{2}B_X$ and put

$$L(h_1,\ldots,h_n) = \int_{\mathbb{R}} f^{(n)}(sx)(h_1,\ldots,h_n)s^n\eta(s)\,\mathrm{d}s, \quad h_1,\ldots,h_n \in X.$$

As earlier, we can check that L is well defined and that $L \in \mathscr{L}^n(X)$. Let $\varepsilon > 0$ be given. We find $\delta \in (0, \frac{1}{2}\|x\| - \frac{1}{4})$, $\delta < 1$, such that

$$\|f^{(n)}(sz) - f^{(n)}(sx)\| < \varepsilon 2^{-n} \quad \text{whenever} \quad z \in X,\ \|z-x\| < 2\delta, \text{ and } s \in [1,2].$$

Then for $0 \neq \tau \in (-\delta,\delta)$, $y \in X$, $\|y-x\| < \delta$ and $h_1,\ldots,h_n \in B_X$ we have

$$\begin{aligned}
&\left|\frac{1}{\tau}\bigl[g^{(n-1)}(y+\tau h_1)(h_2,\ldots,h_n) - g^{(n-1)}(y)(h_2,\ldots,h_n)\bigr] - L(h_1,\ldots,h_n)\right| \\
&= \Biggl|\frac{1}{\tau}\Biggl[\int_{\mathbb{R}} f^{(n-1)}(s(y+\tau h_1))(h_2,\ldots,h_n)s^{n-1}\eta(s)\,\mathrm{d}s \\
&\quad - \int_{\mathbb{R}} f^{(n-1)}(sy)(h_2,\ldots,h_n)s^{n-1}\eta(s)\,\mathrm{d}s\Biggr] - \int_{\mathbb{R}} f^{(n)}(sx)(h_1,\ldots,h_n)s^n\eta(s)\,\mathrm{d}s\Biggr| \\
&= \left|\int_{\mathbb{R}}\int_0^1 \bigl[f^{(n)}(sy+\theta s\tau h_1))(h_1,\ldots,h_n) - f^{(n)}(sx)(h_1,\ldots,h_n)\bigr]\,\mathrm{d}\theta s^n\eta(s)\,\mathrm{d}s\right| \\
&< \int_{\mathbb{R}} \varepsilon 2^{-n}s^n\eta(s)\,\mathrm{d}s < \varepsilon.
\end{aligned}$$

This means that g *is $C^{(n)}$-smooth at x.* $\square$

Step 13. **Claim.** *The function p defined in Step 8 is $C^{(n)}$-smooth on $X \setminus \{0\}$.*

Proof. From the formula $(**)$ we can see that the mapping p' is a composition of g, g' and some elementary mappings. More carefully:

$$(***) \qquad p' = \delta \circ \{\alpha \circ \gamma \circ [g' \circ \beta \circ (\alpha \circ p, \mathrm{id}), \beta \circ (\alpha \circ p, \mathrm{id})], g' \circ \beta \circ (\alpha \circ p, \mathrm{id})\},$$

where

$$\begin{aligned}
\alpha(t) &= \frac{1}{t}, \quad 0 \neq t \in \mathbb{R}, \\
\beta(t,x) &= tx, \quad t \in \mathbb{R}, \quad x \in X, \\
\gamma(\xi,x) &= \xi(x), \quad \xi \in X^*, \quad x \in X, \\
\delta(t,\xi) &= t\xi, \quad t \in \mathbb{R}, \quad \xi \in X^*, \\
\mathrm{id}(x) &= x, \quad x \in X,
\end{aligned}$$

and "$\circ$" means the composition of two mappings. We note that the mappings α, β, γ, δ, id are C^∞-smooth while g' is $C^{(n-1)}$-smooth. The formula $(***)$ guarantees that p is $C^{(1)}$-smooth on $X \setminus \{0\}$. If $n > 1$, then $(***)$ together with [C, Chapitre 1, Théorème 5.4.2] subsequently gives that p is $C^{(2)}$-smooth, $C^{(3)}$-smooth, ..., $C^{(n)}$-smooth on $X \setminus \{0\}$.

Finally, the $C^{(n)}$-smoothness of p trivially implies the $C^{(n)}$-smoothness of the norm $|||\cdot|||$.

If the original norm $\|\cdot\|$ is C^∞-smooth, then our norm $|||\cdot|||$ will be $C^{(n)}$-smooth for every $n \in \mathbb{N}$, and hence C^∞-smooth. $\square$

Remark. We note that in Steps 1–9 we did not need any smoothness of the norm $\|\cdot\|$. If $\|\cdot\|$ is at least $C^{(1)}$-smooth, then we can save some work in Steps 3 and 4. Also, in Step 9, we can use the standard implicit function theorem [C, Chapitre 1, Théorème 4.7.1].

A function f on a Banach space X is called a *bump* if its support $\operatorname{supp} f := \{x \in X\colon\ f(x) \neq 0\}$ is nonempty and bounded.

Checking Steps 1–5, 10, and 11, we get

Theorem 2. *Let a separable Banach space X admit a Lipschitz bump which is $C^{(n)}$-smooth, where $n \in \mathbb{N} \cup \{\infty\}$. Then X admits a Lipschitz bump which is both uniformly Gâteaux smooth and $C^{(n)}$-smooth.*

In its proof we just must pay more attention to showing the linearity of the function $h \mapsto Df(x)(h)$ in Step 4. This can be guaranteed if we require in Step 1 that the set $\{x_j\colon\ j \in \mathbb{N}\}$ is closed under making convex combinations with rational coefficients. The above remark also applies to this proof.

By putting together the proof of [FWZ, Theorem 3.2] and Steps 6–9, we get, in a different way, a recent result due to Tang [T]:

Theorem 3. *If a Banach space X admits a uniformly Gâteaux smooth bump b, then X admits an equivalent uniformly Gâteaux smooth norm.*

Proof. *Step 1.* We shall first show that b is a bounded function.

Find $c > 0$ such that $b(x) = 0$ whenever $x \in X$ and $\|x\| > c$. Find then $h \in X$ such that $\|h\| > 2c$. Since b is uniformly Gâteaux smooth, there is $m \in \mathbb{N}$ so that

$$\left| m\Big[b\Big(x + \frac{1}{m}(\pm h)\Big) - b(x)\Big] - b'(x)(\pm h)\right| < 1 \quad \text{for all} \quad x \in X.$$

Fix any $x_0 \in X$ with $b(x_0) \neq 0$. Then $\|x_0\| \leqslant c$ and so $b(x_0 + h) = 0$. Put

$$t_j = b\Big(x_0 + \frac{j}{m}h\Big), \quad s_j = b'\Big(x_0 + \frac{j}{m}h\Big)(h), \quad j = 0, 1, \ldots, m-1.$$

Then we have

$$\big|m\big(t_{j+1} - t_j\big) - s_j\big| < 1 \quad \text{and} \quad \big|m\big(t_j - t_{j+1}\big) + s_{j+1}\big| < 1, \quad j = 0, 1, \ldots, m-1.$$

Thus, for $j = 0, 1, \ldots, m-1$ we have

$$\begin{aligned} &\big|s_j\big| \leqslant \big|s_{j+1} - s_j\big| + \big|s_{j+1}\big| < 1 + 1 + \big|s_{j+1}\big|, \\ &\big|s_j\big| \leqslant 2 + \big|s_{j+1}\big| < 4 + \big|s_{j+2}\big| < \ldots < 2m + \big|s_m\big| = 2m, \end{aligned}$$

and further,

$$\begin{aligned} m\big|t_j\big| &< 1 + \big|s_j\big| + m\big|t_{j+1}\big| < 1 + 2m + m\big|t_{j+1}\big| \\ &< 2(1+2m) + m\big|t_{j+2}\big| < \ldots < m(1+2m) + m\big|t_m\big| = m(1+2m). \end{aligned}$$

So, in particular, $m\big|t_0\big| < m(1+2m)$, and hence $\big|b(x_0)\big| = \big|t_0\big| < 1 + 2m$. We proved that $\sup\limits_{x \in X} |b(x)| < 1+2m$. (We note that b is then Lipschitz, see [MV, Remark 2.1(a)]. But we shall not use this fact.)

Step 2. Replacing our b by $x \mapsto \alpha b(\beta x + z) + 6$, $x \in X$, with suitable $\alpha, \beta \in \mathbb{R}$ and $z \in X$, we get a new uniformly Gâteaux smooth function, denoted again by b, such that $b\colon X \to [0, +\infty)$, $b(0) < 1/3$, and $b(x) = 6$ whenever $x \in X$ and $\|x\| \geqslant 1/2$. For $x \in 6B_X$ put

$$\begin{aligned} f(x) = \inf\Big\{ &\sum_{j=1}^m \alpha_j b(x_j)\colon\ \alpha_1, \ldots, \alpha_m \geqslant 0,\ x_1, \ldots, x_m \in 6B_X, \\ &\sum_{j=1}^m \alpha_j = 1, \quad \sum_{j=1}^m \alpha_j x_j = x,\ m \in \mathbb{N}\Big\}. \end{aligned}$$

It is immediate to check that $0 \leqslant f(0) < 1/3$ and that $f(x) = 6$ if and only if $x \in X$ and $\|x\| = 6$. Further it is standard to verify that f is a convex function on $6B_X$. Take $0 \neq x \in 6B_X$. Then $x = \big(1 - \frac{\|x\|}{6}\big)0 + \frac{\|x\|}{6}\frac{6x}{\|x\|}$ and the definition of f gives

$$f(x) \leqslant \Big(1 - \frac{\|x\|}{6}\Big)b(0) + \frac{\|x\|}{6}b\Big(\frac{6x}{\|x\|}\Big) < \frac{1}{3} + \|x\|.$$

Also, if $\|y\| < \frac{1}{2}$, then $b(y) \geqslant 0 \geqslant \|y\| - \frac{1}{2}$, and if $\frac{1}{2} \leqslant \|y\| \leqslant 6$, then $b(y) = 6 \geqslant \|y\|$. Therefore $f(x) \geqslant \|x\| - \frac{1}{2}$. Summarizing up, we get that

$$\|x\| - \frac{1}{2} \leqslant f(x) \leqslant \frac{1}{3} + \|x\| \quad \text{for all} \quad x \in 6B_X.$$

We shall show that f *is Lipschitz on* $5B_X$. Take $x, y \in 5B_X$. We find $t \geqslant 1$ such that $z := (1-t)x + ty$ has norm 6. Then $y = \frac{1}{t}z + (1 - \frac{1}{t})x$ and from the convexity of f we get

$$f(y) \leqslant \frac{1}{t}f(z) + \Big(1 - \frac{1}{t}\Big)f(x) = \frac{6}{t} + f(x) - \frac{1}{t}f(x) \leqslant \frac{6}{t} + f(x).$$

hence $f(y) - f(x) \leqslant \frac{6}{t}$. On the other hand

$$1 \leqslant \|z\| - \|x\| \leqslant \|z - x\| = t\|y - x\|.$$

Therefore

$$f(y) - f(x) \leqslant 6\|y - x\| \quad \text{whenever} \quad x, y \in 5B_X.$$

Step 3. We claim that f *is uniformly Gâteaux smooth on* $4\mathring{B}_X$. Fix $h \in B_X$ and $\varepsilon > 0$. We find $\delta \in (0, \frac{1}{6})$ such that

$$(*) \qquad \frac{1}{t}\big[b(x+th) - b(x) - b'(x)(h)\big] < \frac{\varepsilon}{8} \quad \text{whenever} \quad x \in X \quad \text{and} \quad t \in (0, 6\delta).$$

Fix for a while any $x \in 4\mathring{B}_X$ and any $t \in (0, \delta)$. Then $f(x) \leqslant 1/3 + \|x\| < 5$. Find $\varepsilon' \in (0, \varepsilon/4)$ so that $f(x) + \varepsilon' t < 5$. We find $\alpha_1, \ldots, \alpha_m \geqslant 0$ and $x_1, \ldots, x_m \in 6B_X$ satisfying $\sum_{j=1}^{m} \alpha_j = 1$ and $\sum_{j=1}^{m} \alpha_j x_j = x$, and such that $\sum_{j=1}^{m} \alpha_j b(x_j) < f(x) + \varepsilon' t$. We may, and do assume that $b(x_1) \leqslant b(x_2) \leqslant \ldots \leqslant b(x_m)$. Let k be the largest j such that $b(x_j) < 6$. Then, putting $\alpha = \sum_{j=1}^{k} \alpha_j$, we have $\alpha = 1$ if $k = m$, and

$$5 > \sum_{j=k+1}^{m} \alpha_j b(x_j) = (1 - \alpha) \cdot 6$$

otherwise. Hence $\alpha > \frac{1}{6}$. We note that

$$x \pm th = \sum_{j=1}^{k} \alpha_j \Big(x_j \pm \frac{t}{\alpha} h \Big) + \sum_{j=k+1}^{m} \alpha_j x_j$$

and

$$\Big\| x_j \pm \frac{t}{\alpha} h \Big\| \leqslant \|x_j\| + \frac{t}{\alpha} < \frac{1}{2} + 6\delta < 6 \quad \text{for } j = 1, \dots, k.$$

(Recall that $b(x_j) < 6$ implies $\|x_j\| < 1/2$.) Also $\frac{t}{\alpha} < 6\delta$. Thus, using $(*)$, we have

$$\begin{aligned}
&\frac{1}{t}\big[f(x+th) + f(x-th) - 2f(x)\big] \\
&\quad < \sum_{j=1}^{k} \alpha_j \frac{1}{t}\big[b\big(x_j + \tfrac{t}{\alpha}h\big) + b\big(x_j - \tfrac{t}{\alpha}h\big) - 2b(x_j)\big] + 2\varepsilon' \\
&\quad < \sum_{j=1}^{k} \alpha_j \cdot 2\frac{\varepsilon}{8} + 2\frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

This holds for every $x \in 4\mathring{B}_X$ and for every $t \in (0, \delta)$. Hence f is uniformly Gâteaux smooth on $4\mathring{B}_X$.

Step 4. Define

$$g(x) = 3 \int_{\mathbb{R}} f(sx)\eta(s)\,\mathrm{d}s, \quad x \in 2\mathring{B}_X,$$

where η is the function from Step 6. Then g is a convex Lipschitz function on $2\mathring{B}_X$.

Step 5. As in Step 7, we can check that g is uniformly Gâteaux smooth on $2\mathring{B}_X$ and that the function $x \mapsto g'(x)(x)$ is Lipschitz on $2\mathring{B}_X$. Moreover $g(x) \geqslant \frac{3}{2}$ whenever $x \in X$ and $\|x\| = 1$ (as $f \geqslant \|\cdot\| - \frac{1}{2}$) and $g(0) = 3f(0) < 1$.

Step 6. It remains to apply the lemma below; its proof is contained in Steps 8 and 9. $\square$

Lemma. *Assume that there exists a convex, Lipschitz, uniformly Gâteaux smooth function $g\colon 2\mathring{B}_X \to \mathbb{R}$, with $g(0) < 1$ and $g(x) \geqslant \frac{3}{2}$ whenever $x \in X$ and $\|x\| = 1$. Assume moreover that the function $x \mapsto g'(x)(x)$ is uniformly continuous on $2\mathring{B}_X$. Then X admits an equivalent uniformly Gâteaux smooth norm.*

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