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ON THE TAUBERIAN CONSTANT IN THE IKEHARA THEOREM

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Let p denote any prime. Let $\Lambda(n) = \log p$ for $n = p^r$, $r \in \mathbb{N}$, $\Lambda(n) = 0$ otherwise. Then the prime number theorem, which reads

$$\pi(x) = \sum_{p \leq x} 1 = \int_2^x \frac{du}{\log u} + o\left(\frac{x}{\log x}\right), \quad x \rightarrow +\infty,$$

is equivalent to $\psi(x) = \sum_{n \leq x} \Lambda(n) = x + o(x)$, $x \rightarrow +\infty$. The last relation follows from the Ikehara theorem: Let $A(x)$ be a nonnegative nondecreasing function defined for $x \in \langle 0; +\infty \rangle$ and let the integral $f(s) = \int_0^\infty A(x)e^{-xs} dx$, $s = \sigma + it$, converge for $\sigma > 1$. Let $f(s)$ be analytic for $\sigma \geq 1$, except for a simple pole at $s = 1$ with residue 1. Then $\lim_{x \rightarrow +\infty} e^{-x}A(x) = 1$ (cf. [1], p. 124).

Of course, it suffices to put $A(x) = \psi(e^x)$, $x \geq 0$. Since

$$\int_0^\infty A(x)e^{xs} dx = -\frac{\xi'(s)}{s\xi(s)} = \frac{1}{s-1} + h(s) \quad \text{for } \sigma > 1,$$

where $h(s)$ is analytic for $\sigma \geq 1$, we can use the Ikehara theorem to obtain $\lim_{x \rightarrow +\infty} e^{-x}A(x) = \lim_{x \rightarrow +\infty} e^{-x}\psi(e^x) = \lim_{x \rightarrow +\infty} \frac{\psi(x)}{x} = 1$.

To estimate the remainder term $\pi(x) - \int_2^x \frac{du}{\log u}$ we need a more sophisticated way: Let functions $\omega(x)$ and $\frac{1}{6} \log x - \omega(x)$ be both positive and increasing in $\langle 3; +\infty \rangle$. Then the relation $\psi_1(x) = \sum_{n \leq x} \Lambda(n) \log \frac{x}{n} = x + O(x \exp(-2\omega(x)))$, $x \geq 3$, implies $\pi(x) = \int_2^x \frac{du}{\log u} + O(x \exp(-\omega(x)))$, $x \geq 3$ (see [5]). We know that

$$\begin{aligned} h_1(s) &= -\frac{\xi'(s)}{s^2\xi(s)} - \frac{1}{s-1} = \int_0^\infty \psi_1(e^x)e^{-xs} dx - \frac{1}{s-1} \\ &= \int_0^\infty (e^{-x}\psi_1(e^x) - 1)e^{-x(s-1)} dx \end{aligned}$$

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is analytic for $\sigma \geq 1$. However, we now need a better estimate of the remainder term $e^{-x}\psi_1(e^x) - 1$ than $o(1)$, $x \rightarrow +\infty$, which the Ikehara theorem yields. Theorem 2 of [2] is such an Ikehara theorem with the remainder term. If moreover the function $g(t) = f(1 + it) - \frac{1}{it}$, $t \in \mathbb{R}$, satisfies $g^{(n)} \in L^1(\mathbb{R})$, then $e^{-x}A(x) = 1 + O(x^{-n})$, $x \rightarrow +\infty$. This last theorem yields $\psi_1(x) = x + O(x \log^{-n} x)$, $x \rightarrow +\infty$. Theorem 0.1 of the present paper allows us to show how the constant in O in this relation depends on $n \in \mathbb{N}$.

We denote $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ for any function $f: \mathbb{R} \rightarrow \mathbb{C}$ and $\|f\|_1 = \int_{-\infty}^\infty |f(t)| dt$. Further, let $\hat{f}(x) = \int_{-\infty}^\infty f(t)e^{itx} dt$ be the Fourier transform of a function f for $f \in L^1(\mathbb{R})$. Let $M_0 = M_{-1} = 1$ and let $\{M_i\}_{i=1}^\infty$ be a sequence of positive numbers such that

$$(1) \quad C = \sum_{k=0}^\infty \lambda_k < \infty, \quad \text{where} \quad \lambda_k = \frac{M_{k-1}}{M_k}, \quad k = 0, 1, \dots$$

Lemma 0.1. *For any sequence of positive numbers $\{M_i\}_{i=1}^\infty$ satisfying (1), there exists an even function $u: \mathbb{R} \rightarrow \langle 0; +\infty \rangle$ such that $\text{supp } u = \langle -1; 1 \rangle$, $u(x) > 0$ for $x \in \langle -1; 1 \rangle$, $u \in C^\infty(\mathbb{R})$, $\|u^{(n)}\|_\infty \leq \frac{1}{2}C(C+1)^2(2C)^n M_n$ for $n \in \mathbb{N} \cup \{0\}$, and moreover, $\hat{u}(x) \geq 0$, $x \in \mathbb{R}$, $\int_{-\infty}^\infty \hat{u}(x) dx \geq \frac{1}{C}$.*

This lemma can be used for $M_n = (n!)^{1+\varepsilon}$, $\varepsilon > 0$.

P r o o f. We shall first construct an even function $g: \mathbb{R} \rightarrow \langle 0; +\infty \rangle$, nonincreasing on $\langle 0; +\infty \rangle$, $\text{supp } g = \langle -C; C \rangle$, $g \in C^\infty(\mathbb{R})$, $\|g^{(n)}\|_\infty \leq \frac{1}{2}(C+1)M_n$, $n \in \mathbb{N} \cup \{0\}$ (cf. [3], chap. 19, ex. 10).

Let the sequence $\{g_n\}_{n=0}^\infty$, $g_n: \mathbb{R} \rightarrow \langle 0; +\infty \rangle$, be defined inductively by the relation

$$(2) \quad g_n(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x-t) dt, \quad n \in \mathbb{N}.$$

If $g_{n-1} \in C^{(k)}(\mathbb{R})$, $k \in \mathbb{N} \cup \{0\}$, then $g_n \in C^{(k+1)}(\mathbb{R})$ and

$$(3) \quad g_n^{(l)}(x) = \frac{1}{2\lambda_n} (g_{n-1}^{(l-1)}(x + \lambda_n) - g_{n-1}^{(l-1)}(x - \lambda_n)), \quad l = 1, \dots, k+1.$$

The relations (2) and (3) imply

$$(4) \quad g_n^{(l)}(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}^{(l)}(x-t) dt, \quad l = 0, 1, \dots, k.$$

Put $g_0(x) = \max(1 - |x|, 0)$, $x \in \mathbb{R}$. Since $g_0 \in C^{(0)}(\mathbb{R}) = C(\mathbb{R})$, we have $g_n \in C^{(n)}(\mathbb{R})$, $n \in \mathbb{N}$, and the relations (3) and (4) are satisfied with $k = n - 1$.

If g_{n-1} is even, then g_n is even, as $g_n(-x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(-x-t) dt = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x+t) dt = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x-t) dt = g_n(x)$, $x \in \mathbb{R}$. Since g_0 is even, we have by induction g_n is even, $n \in \mathbb{N}$.

Let $\text{supp } g_{n-1} = \langle -s_{n-1}; s_{n-1} \rangle$, $g_{n-1}(x) > 0$ for $x \in (-s_{n-1}, s_{n-1})$, $s_{n-1} > 0$. Then $g_n(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x-t) dt = 0$ for $|x| > s_{n-1} + \lambda_n$, $g_n(x) > 0$ for $|x| < s_{n-1} + \lambda_n$, and so $\text{supp } g_n = \langle -s_{n-1} - \lambda_n; s_{n-1} + \lambda_n \rangle$. Since $\text{supp } g_0 = \langle -1; 1 \rangle$, $g_0(x) > 0$ for $x \in (-1; 1)$, we have $\text{supp } g_n = \langle -\sum_{k=0}^n \lambda_k; \sum_{k=0}^n \lambda_k \rangle$, $g_n(x) > 0$ for $x \in \left(-\sum_{k=0}^n \lambda_k; \sum_{k=0}^n \lambda_k\right)$, $n \in \mathbb{N}$.

Let g_{n-1} be nonincreasing on $\langle 0; +\infty \rangle$. Then g_n is nonincreasing on $\langle 0; +\infty \rangle$, because $g'_n(x) = \frac{1}{2\lambda_n}(g_{n-1}(x+\lambda_n) - g_{n-1}(x-\lambda_n)) \leq 0$ for $x \geq \lambda_n$, $g'_n(x) = \frac{1}{2\lambda_n}(g_{n-1}(x+\lambda_n) - g_{n-1}(\lambda_n-x)) \leq 0$ for $x \in \langle 0; \lambda_n \rangle$ by (3). As g_0 is nonincreasing on $\langle 0; +\infty \rangle$, we obtain that g_n is nonincreasing on $\langle 0; +\infty \rangle$ for $n \in \mathbb{N}$.

Now, we prove by induction that $g_k^{(n)}$ is Lipschitzian with the constant M_n for all $k \geq n$, $n \in \mathbb{N} \cup \{0\}$.

1) $g_n^{(n)}$ is Lipschitzian with the constant M_n : This is clear for $n = 0$. Let

$$|g_{n-1}^{(n-1)}(x) - g_{n-1}^{(n-1)}(x')| \leq M_{n-1}|x - x'|, \quad x, x' \in \mathbb{R}.$$

Then by (3)

$$\begin{aligned} |g_n^{(n)}(x) - g_n^{(n)}(x')| &\leq \frac{1}{2\lambda_n} |g_{n-1}^{(n-1)}(x+\lambda_n) - g_{n-1}^{(n-1)}(x'+\lambda_n)| \\ &\quad + \frac{1}{2\lambda_n} |g_{n-1}^{(n-1)}(x-\lambda_n) - g_{n-1}^{(n-1)}(x'-\lambda_n)| \\ &\leq \frac{2M_{n-1}|x-x'|}{2\lambda_n} = M_n|x-x'|. \end{aligned}$$

2) Let $k-1 \geq n$ and $|g_{k-1}^{(n)}(x) - g_{k-1}^{(n)}(x')| \leq M_n|x-x'|$, $x, x' \in \mathbb{R}$.

Then by (4), $|g_k^{(n)}(x) - g_k^{(n)}(x')| = \frac{1}{2\lambda_k} \int_{-\lambda_k}^{\lambda_k} (g_{k-1}^{(n)}(x-t) - g_{k-1}^{(n)}(x'-t)) dt \leq \frac{1}{2\lambda_k} \int_{-\lambda_k}^{\lambda_k} M_n|x-x'| dt = M_n|x-x'|$.

Let $k, l \in \mathbb{N} \cup \{0\}$, $k > l$. Then by (4),

$$\begin{aligned} (5) \quad |g_k^{(l)}(x) - g_{k-1}^{(l)}(x)| &\leq \frac{1}{2\lambda_k} \int_{-\lambda_k}^{\lambda_k} |g_{k-1}^{(l)}(x-t) - g_{k-1}^{(l)}(x)| dt \\ &\leq \frac{M_l}{2\lambda_k} \int_{-\lambda_k}^{\lambda_k} |t| dt = \frac{M_l}{2} \lambda_k. \end{aligned}$$

Let $n \in \mathbb{N} \cup \{0\}$. Since $\frac{M_l}{2} \sum_{k=n+1}^{\infty} \lambda_k < \infty$ for $l = 0, 1, \dots, n$, the series

$$(6) \quad \tilde{g}_n^{(l)}(x) = \sum_{k=n+1}^{\infty} (g_k^{(l)}(x) - g_{k-1}^{(l)}(x)), \quad l = 0, 1, \dots, n,$$

converges uniformly in \mathbb{R} . It means that for the function \tilde{g}_n , defined by (6) for $l = 0$, we have $\tilde{g}_n \in C^{(n)}(\mathbb{R})$. Put

$$(7) \quad \begin{aligned} g(x) &= \lim_{n \rightarrow \infty} g_n(x) = \sum_{k=n+1}^{\infty} (g_k(x) - g_{k-1}(x)) + g_n(x) \\ &= \tilde{g}_n(x) + g_n(x), \quad n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Since $g_n \in C^{(n)}(\mathbb{R})$, $\tilde{g}_n \in C^{(n)}(\mathbb{R})$ for any $n \in \mathbb{N} \cup \{0\}$, we have $g \in C^\infty(\mathbb{R})$ for $g = g_n + \tilde{g}_n$. By (3) for $l = n$ we obtain

$$(8) \quad \|g_n^{(n)}\|_\infty \leq \frac{1}{2\lambda_n} (\|g_{n-1}^{(n-1)}\|_\infty + \|g_{n-1}^{(n-1)}\|_\infty) = \frac{1}{\lambda_n} \|g_{n-1}^{(n-1)}\|_\infty = \frac{\|g_0\|_\infty}{\lambda_n \dots \lambda_1} = M_n.$$

Further, by (6) and (5) for $l = n$ we get

$$(9) \quad \|\tilde{g}_n^{(n)}\|_\infty \leq \frac{M_n}{2} \sum_{k=n+1}^{\infty} \lambda_k.$$

It follows by (7) from (8) and (9) that

$$(10) \quad \|g^{(n)}\|_\infty \leq \frac{M_n}{2} \left(2 + \sum_{k=n+1}^{\infty} \lambda_k \right) \leq \frac{M_n}{2} (C + 1).$$

The function g is even and nonincreasing on $(0; +\infty)$ as the limit of a sequence of functions g_n with the same properties. Obviously $g(x) = 0$ for $|x| \geq C = \sum_{k=0}^{\infty} \lambda_k$. We

have $g(x_0) > 0$ for $x_0 \in \langle 0; C \rangle$. Indeed, find $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} \lambda_k < \frac{1}{2}(C - x_0)$.

Then $g_n(\frac{C+x_0}{2}) > 0$ and $g_m(x) \geq g_n(\frac{C+x_0}{2})$ for $m \geq n$ and $x \in \langle 0; \frac{C+x_0}{2} - \sum_{k=n+1}^m \lambda_k \rangle$.

Hence $g_m(x_0) \geq g_n(\frac{C+x_0}{2})$ for $m \geq n$ and $g(x_0) = \lim_{m \rightarrow \infty} g_m(x_0) \geq g_n(\frac{C+x_0}{2})$. So we have $\text{supp } g = \langle -C; C \rangle$.

Using Fubini's theorem for the nonnegative function $g_{n-1}(x-t)$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} g_n(x) dx &= \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} \left(\int_{-\infty}^{\infty} g_{n-1}(x-t) dx \right) dt = \int_{-\infty}^{\infty} g_{n-1}(x) dx \\ &= \int_{-\infty}^{\infty} g_0(x) dx = 1 \end{aligned}$$

and by the uniform convergence of the series in (6) and by (7) we get

$$(11) \quad \int_{-\infty}^{\infty} g(x) dx = \int_{-C}^C g(x) dx = 1.$$

The function $g * g(x) = \int_{-\infty}^{\infty} g(t)g(x-t) dt$ is even, nonincreasing on $\langle 0; +\infty \rangle$, positive on $(-2C; 2C)$. Further, $\text{supp } g * g = \langle -2C; 2C \rangle$, $g * g \in C^\infty(\mathbb{R})$ and by (10)

$$\begin{aligned} \|(g * g)^{(n)}\|_\infty &= \sup_{x \in \mathbb{R}} |(g * g)^{(n)}(x)| = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{\infty} g(t)g^{(n)}(x-t) dt \right| \\ &\leq 2C \|g\|_\infty \|g^{(n)}\|_\infty \leq \frac{C}{2} (C+1)^2 M_n. \end{aligned}$$

Put $u(x) = g * g(2Cx)$. Then u is even, nonincreasing on $\langle 0; +\infty \rangle$, $u \in C^\infty(\mathbb{R})$, $\text{supp } u = \langle -1; 1 \rangle$, $u(x) > 0$ for $x \in (-1; 1)$, $\|u^{(n)}\|_\infty \leq \frac{1}{2} C (C+1)^2 (2C)^n M_n$, $\hat{u}(x) = \frac{1}{2C} \hat{g}^2(\frac{x}{2C}) \geq 0$, $x \in \mathbb{R}$. Since $g \in L^2(\mathbb{R})$ we can use Plancherel's theorem to prove $\int_{-\infty}^{\infty} \hat{u}(x) dx = \frac{1}{2C} \int_{-\infty}^{\infty} \hat{g}^2(\frac{x}{2C}) dx = \int_{-\infty}^{\infty} \hat{g}^2(t) dt = 2\pi \int_{-\infty}^{\infty} g^2(t) dt = 2\pi \int_{-C}^C g^2(t) dt \geq \frac{2\pi}{2C} \left(\int_{-C}^C g(t) dt \right)^2 = \frac{\pi}{C}$ by (11) and the Hölder inequality. \square

Theorem 0.1. *Let $m, n \in \mathbb{N}$, $a_k \in \mathbb{R}$, $k = 1, \dots, m$. Let further $A(x)$ be a nonnegative nondecreasing function defined for $x \in \langle 0; +\infty \rangle$ and let the integral $f(s) = \int_0^\infty A(x)e^{-xs} dx$, $s = \sigma + it$, converge for $\sigma > 1$. Let the function $g(s) = f(s) - \sum_{k=1}^m \frac{a_k}{(s-1)^k}$ be continuous in the halfplane $\sigma \geq 1$. Define $g_0(t) = g(1+it)$, $t \in \mathbb{R}$. If the function g_0 satisfies the conditions: $g_0^{(i)}$ is absolutely continuous on \mathbb{R} , $\lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda^{n-i}} \int_0^\lambda |g_0^{(i)}(t)| dt = 0$ for $i = 0, 1, \dots, n-1$ and $g_0^{(n)} \in L^1(\mathbb{R})$, then*

$$\left| e^{-x} A(x) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} x^k \right| \leq \frac{2 \|g_0^{(n)}\|_1}{\pi x^n}, \quad x > 0.$$

Proof. We will prove the theorem in two steps. First, we shall estimate the expression $e^x A(x) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} x^k$ from above (cf. (18)), and, secondly, from below (cf. (23)). We have $f(s) = \int_0^\infty A(x)e^{-xs} dx$, $\frac{(k-1)!}{(s-1)^k} = \int_0^\infty x^{k-1} e^{-(s-1)x} dx$, $k \in \mathbb{N}$, for $\sigma > 1$. Hence $g(s) = f(s) - \sum_{k=1}^m \frac{a_k}{(s-1)^k} = \int_0^\infty \left(B(x) - \sum_{k=1}^m \frac{a_k x^{k-1}}{(k-1)!} \right) e^{-(s-1)x} dx = \int_0^\infty \left(B(x) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} x^k \right) e^{-(s-1)x} dx$, where $B(x) = e^{-x} A(x)$. Take any sequence $\{M_i\}_{i=1}^\infty$ of positive numbers, $M_{-1} = M_0 = 1$, such that $C = \sum_{i=0}^\infty \frac{M_{i-1}}{M_i} < \infty$. By the preceding lemma there exists a kernel $u \in C^\infty(\mathbb{R})$ with $\text{supp } u = \langle -1; 1 \rangle$ such

that u and \hat{u} are even, $u(x) > 0$, $x \in (-1; 1)$, $\hat{u}(x) \geq 0$, $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} \hat{u}(x) dx \geq \frac{\pi}{C}$, $\|u^{(n)}\|_{\infty} \leq D\beta^n M_n$, $n \in \mathbb{N} \cup \{0\}$, where $D = \frac{1}{2}C(C+1)^2$, $\beta = 2C$. Put $g_{\varepsilon}(t) = g(1 + \varepsilon + it)$, $t \in \mathbb{R}$, $\varepsilon > 0$. For $\lambda > 0$, $y > 0$ we have

$$\int_{-\lambda}^{\lambda} g_{\varepsilon}(t) u\left(\frac{t}{\lambda}\right) e^{iyt} dt = \int_{-\lambda}^{\lambda} u\left(\frac{t}{\lambda}\right) e^{iyt} \left(\int_0^{\infty} \left(B(x) - \sum_{k=0}^{m-1} a_{k+1} \frac{x^k}{k!} \right) e^{-(\varepsilon+it)x} dx \right) dt.$$

We want to change the order of integration in the last integral: this is possible as for $[t, x] \in \Omega = (-\lambda; \lambda) \times (0; +\infty)$ we have $|u(\frac{t}{\lambda})e^{iyt}x^k e^{-(\varepsilon+it)x}| \leq D x^k e^{-\varepsilon x} \in L^1(\Omega)$, $k = 0, 1, \dots, m-1$. For $s > 1$, $x > 0$, the relation $f(s) = \int_0^{\infty} A(u)e^{-us} du \geq A(x) \int_x^{\infty} e^{-us} du = s^{-1}A(x)e^{-xs}$ holds. It means $A(x) \leq sf(s)e^{sx}$, $B(x) \leq sf(s)e^{(s-1)x}$, $s > 1$, $x > 0$. In particular, $B(x) \leq (1 + \frac{\varepsilon}{2})f(1 + \frac{\varepsilon}{2})e^{\frac{\varepsilon}{2}x}$, $x > 0$. It follows that $|u(\frac{t}{\lambda})e^{iyt}B(x)e^{-(\varepsilon+it)x}| \leq (1 + \frac{\varepsilon}{2})f(1 + \frac{\varepsilon}{2})e^{-\frac{\varepsilon}{2}x} \cdot D \in L^1(\Omega)$. Consequently, $u(\frac{t}{\lambda})e^{iyt} \left(B(x) - \sum_{k=0}^{m-1} a_{k+1} \frac{x^k}{k!} \right) e^{-(\varepsilon+it)x}$ is in $L^1(\Omega)$. Hence we can change the order of integration to obtain

$$\begin{aligned} \int_{-\lambda}^{\lambda} g_{\varepsilon}(t) u\left(\frac{t}{\lambda}\right) e^{iyt} dt &= \int_0^{\infty} \left(B(x) - \sum_{k=0}^{m-1} a_{k+1} \frac{x^k}{k!} \right) e^{-\varepsilon x} \left(\int_{-\lambda}^{\lambda} e^{i(y-x)t} u\left(\frac{t}{\lambda}\right) dt \right) dx \\ &= \int_0^{\infty} \left(B(x) - \sum_{k=0}^{m-1} a_{k+1} \frac{x^k}{k!} \right) e^{-\varepsilon x} \cdot \lambda \hat{u}(\lambda(y-x)) dx. \end{aligned}$$

We need to let $\varepsilon \rightarrow 0+$ in this relation. As g_{ε} tends uniformly to g_0 in $(-\lambda; \lambda)$, we have $\lim_{\varepsilon \rightarrow 0+} \int_{-\lambda}^{\lambda} g_{\varepsilon}(t) u\left(\frac{t}{\lambda}\right) e^{iyt} dt = \int_{-\lambda}^{\lambda} g_0(t) u\left(\frac{t}{\lambda}\right) e^{iyt} dt \in \mathbb{R}$. Levi's theorem yields

$$\lim_{\varepsilon \rightarrow 0+} \int_0^{\infty} B(x) \lambda \hat{u}(\lambda(y-x)) e^{-\varepsilon x} dx = \lambda \int_0^{\infty} B(x) \hat{u}(\lambda(y-x)) dx \in \mathbb{R} \cup \{+\infty\},$$

and Lebesgue's theorem implies

$$\lim_{\varepsilon \rightarrow 0+} \int_0^{\infty} x^k e^{-\varepsilon x} \lambda \hat{u}(\lambda(y-x)) dx = \lambda \int_0^{\infty} x^k \hat{u}(\lambda(y-x)) dx \in \mathbb{R},$$

because

$$(12) \quad \hat{u}(t) \leq 2D\beta^n M_n |t|^{-n}, \quad t \neq 0, \quad n \in \mathbb{N}.$$

This fact implies that $\int_0^{\infty} B(x) \hat{u}(\lambda(y-x)) dx \in \mathbb{R}$, too, and

$$\begin{aligned} (13) \quad \int_{-\lambda}^{\lambda} g_0(t) u\left(\frac{t}{\lambda}\right) e^{iyt} dt &= \lambda \int_0^{\infty} \left(B(x) - \sum_{k=0}^{m-1} a_{k+1} \frac{x^k}{k!} \right) \hat{u}(\lambda(y-x)) dx \\ &= \int_{-\infty}^{\lambda y} \left(B\left(y - \frac{v}{\lambda}\right) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} \left(y - \frac{v}{\lambda}\right)^k \right) \hat{u}(v) dv. \end{aligned}$$

Integration by parts yields $\int_{-\lambda}^{\lambda} g_0(t)u(\frac{t}{\lambda})e^{iyt} dt = (\frac{i}{y})^n \int_{-\lambda}^{\lambda} [g_0(t)u(\frac{t}{\lambda})]^{(n)} e^{iyt} dt = (\frac{i}{y})^n \int_{-\lambda}^{\lambda} \left[\sum_{k=0}^n \binom{n}{k} u^{(n-k)}(\frac{t}{\lambda}) \cdot \frac{g_0^{(k)}(t)}{\lambda^{n-k}} \right] e^{iyt} dt$. Then, by the properties of the kernel u , we obtain

$$(14) \quad \left| \int_{-\lambda}^{\lambda} g_0(t)u(\frac{t}{\lambda})e^{iyt} dt \right| \leq 2Dy^{-n} \sum_{k=0}^n \binom{n}{k} \frac{\beta^{n-k} M_{n-k}}{\lambda^{n-k}} \int_0^{\lambda} |g_0^{(k)}(t)| dt, \quad y > 0.$$

Let $0 < a \leq \lambda y$. For $v \in \langle -a; a \rangle \subset (-\infty; \lambda y)$ we have by monotonicity of the function A

$$\begin{aligned} A\left(y - \frac{a}{\lambda}\right) &\leq A\left(y - \frac{v}{\lambda}\right) \leq A\left(y + \frac{a}{\lambda}\right), e^{y - \frac{a}{\lambda}} B\left(y - \frac{a}{\lambda}\right) \leq e^{y - \frac{v}{\lambda}} B\left(y - \frac{v}{\lambda}\right) \\ &\leq e^{y + \frac{a}{\lambda}} B\left(y + \frac{a}{\lambda}\right), \text{ i.e. } e^{-\frac{2a}{\lambda}} B\left(y - \frac{a}{\lambda}\right) \leq B\left(y - \frac{v}{\lambda}\right), \quad v \in \langle -a; a \rangle. \end{aligned}$$

From this inequality we deduce

$$e^{-\frac{2a}{\lambda}} B\left(y - \frac{a}{\lambda}\right) \int_{-a}^a \hat{u}(v) dv \leq \int_{-a}^a B\left(y - \frac{v}{\lambda}\right) \hat{u}(v) dv \leq \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \hat{u}(v) dv.$$

It follows easily from these inequalities that

$$\begin{aligned} e^{-\frac{2a}{\lambda}} B\left(y - \frac{a}{\lambda}\right) \int_{-a}^a \hat{u}(v) dv - \int_{-\infty}^{\lambda y} \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} \left(y - \frac{v}{\lambda}\right)^k \hat{u}(v) dv \\ \leq \int_{-\infty}^{\lambda y} \left(B\left(y - \frac{v}{\lambda}\right) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} \left(y - \frac{v}{\lambda}\right)^k \right) \hat{u}(v) dv. \end{aligned}$$

Using the relation (13) and the estimate (14) we finally obtain

$$(15) \quad \begin{aligned} e^{-\frac{2a}{\lambda}} B\left(y - \frac{a}{\lambda}\right) \int_{-a}^a \hat{u}(v) dv - \int_{-\infty}^{\lambda y} \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} \left(y - \frac{v}{\lambda}\right)^k \hat{u}(v) dv \\ \leq 2Dy^{-n} \sum_{k=0}^n \binom{n}{k} \beta^{n-k} M_{n-k} \cdot \frac{1}{\lambda^{n-k}} \int_0^{\lambda} |g_0^{(k)}(t)| dt. \end{aligned}$$

We shall show that for every $k \in \mathbb{N} \cup \{0\}$

$$(16) \quad \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\lambda y} \left(y - \frac{v}{\lambda}\right)^k \hat{u}(v) dv = y^k \int_{-\infty}^{\infty} \hat{u}(v) dv.$$

Note that the relation (12) implies $|v|^n \hat{u}(v) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\lambda y} \left(y - \frac{v}{\lambda}\right)^k \hat{u}(v) dv = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^0 \left(y - \frac{v}{\lambda}\right)^k \hat{u}(v) dv + \lim_{\lambda \rightarrow +\infty} \int_0^{\infty} f_{\lambda}(v) dv$, where

$f_\lambda(v) = [\max(y - \frac{v}{\lambda}, 0)]^k \hat{u}(v)$. Since $0 \leq f_\lambda(v) \leq y^k \hat{u}(v) \in L^1((0, +\infty))$, we have $\lim_{\lambda \rightarrow +\infty} \int_0^\infty f_\lambda(v) dv = y^k \int_0^\infty \hat{u}(v) dv$ by Lebesgue's theorem. Levi's theorem gives $\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^0 (y - \frac{v}{\lambda})^k \hat{u}(v) dv = y^k \int_{-\infty}^0 \hat{u}(v) dv$. The relation (16) is proved.

Let $\lambda \rightarrow +\infty$, $a \rightarrow +\infty$, in such a way that $\frac{a}{\lambda} \rightarrow 0+$ in the relation (15). From the assumptions of the theorem we obtain by (16)

$$(17) \quad \lim_{x \rightarrow y-} B(x) \cdot \int_{-\infty}^\infty \hat{u}(v) dv - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} y^k \int_{-\infty}^\infty \hat{u}(v) dv \leq Dy^{-n} \|g_0^{(n)}\|_1.$$

Since the function A is nondecreasing on $\langle 0; +\infty \rangle$, the proper limits $\lim_{x \rightarrow y-} A(x)$, $\lim_{x \rightarrow y+} A(x)$ exist for $y > 0$ and $\lim_{x \rightarrow y-} A(x) \leq A(y) \leq \lim_{x \rightarrow y+} A(x)$, $y > 0$. It follows that $\lim_{x \rightarrow y-} B(x) \leq B(y) \leq \lim_{x \rightarrow y+} B(x)$, $y > 0$. The functions A and B are continuous on $\langle 0; +\infty \rangle$ except for a countable set of points. The inequality

$$(18) \quad \left[B(y) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} y^k \right] \int_{-\infty}^\infty \hat{u}(v) dv \leq Dy^{-n} \|g_0^{(n)}\|_1$$

holds, however, at the points of discontinuity of the function B as well. We can verify this fact letting $y \rightarrow y_1+$, where y_1 is a point of discontinuity and y 's are points of continuity of the function B .

We know by (18) that there exists a polynomial \tilde{P} with nonnegative coefficients, $\deg \tilde{P} \leq m - 1$, such that $0 \leq B(y) \leq \tilde{P}(y)$ for $y > 0$. So we have for $\lambda > 0$, $y > 0$, $1 \leq a \leq \lambda y$

$$(19) \quad \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \hat{u}(v) dv \leq \int_{-\infty}^{-a} \tilde{P}\left(y - \frac{v}{\lambda}\right) \hat{u}(v) dv + \int_a^{\lambda y} \tilde{P}\left(y - \frac{v}{\lambda}\right) \hat{u}(v) dv + \int_{-a}^a B\left(y - \frac{v}{\lambda}\right) \hat{u}(v) dv.$$

There exists $K = K(y) > 0$ such that

$$(20) \quad 0 \leq \tilde{P}\left(y - \frac{v}{\lambda}\right) \hat{u}(v) \leq \frac{K}{v^2}$$

for $v \in (-\infty; -a) \cup \langle a; \lambda y \rangle$. For $v \in \langle -a; a \rangle$ we have

$$(21) \quad B\left(y - \frac{v}{\lambda}\right) \leq e^{\frac{2a}{\lambda}} B\left(y + \frac{a}{\lambda}\right).$$

The relations (19), (20) and (21) yield

$$\begin{aligned} e^{\frac{2a}{\lambda}} B\left(y + \frac{a}{\lambda}\right) \int_{-a}^a \hat{u}(v) \, dv &\geq \int_{-a}^a B\left(y - \frac{v}{\lambda}\right) \hat{u}(v) \, dv \\ &\geq \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \hat{u}(v) \, dv - 2 \int_a^{\infty} \frac{K}{v^2} \, dv. \end{aligned}$$

Using (13) a (14) we can deduce from this fact that

$$\begin{aligned} (22) \quad e^{\frac{2a}{\lambda}} B\left(y + \frac{a}{\lambda}\right) \int_{-a}^a \hat{u}(v) \, dv - \int_{-\infty}^{\lambda y} \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} \left(y - \frac{v}{\lambda}\right)^k \hat{u}(v) \, dv + \frac{2K}{a} \\ \geq -2Dy^{-n} \sum_{k=0}^n \binom{n}{k} \beta^{n-k} \frac{M_{n-k}}{\lambda^{n-k}} \int_0^{\lambda} |g_0^{(k)}(t)| \, dt. \end{aligned}$$

Letting in (22) $\lambda \rightarrow +\infty$, $a \rightarrow +\infty$, in such a way that $\frac{a}{\lambda} \rightarrow 0+$, we obtain

$$(23) \quad \left[B(y) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} y^k \right] \int_{-\infty}^{\infty} \hat{u}(v) \, dv \geq -Dy^{-n} \|g_0^{(n)}\|_1,$$

provided $y > 0$ is a point of continuity of the function B . Letting in (23) $y \rightarrow y_1-$, where y_1 is a point of discontinuity and y 's are points of continuity of the function B , we verify validity of (23) at the points of discontinuity of $B(x)$. Finally, $D(\int_{-\infty}^{\infty} \hat{u}(v) \, dv)^{-1} \leq \frac{C}{2}(C+1)^2 \cdot \frac{C}{\pi} = \frac{C^2(C+1)^2}{2\pi}$. Since $C = \sum_{n=0}^{\infty} \frac{M_{n-1}}{M_n} > 1$ can be chosen arbitrarily, the theorem is proved. \square

There exist constants $K > 0$ and $\beta > 1$ such that

$$(24) \quad \left| \left[\frac{\xi'}{\xi}(1+it) \right]^{(j)} \right| \leq K\beta^j (j!)^2 \log^{9(j+1)} t$$

for all $t \geq 3$ and $j = 0, 1, \dots$ (cf. [5]). This relation implies

$$(25) \quad \|h_1^{(j)}(1+it)\|_1 \leq \beta^j (j!)^2 [9(j+1)]!$$

for all $j = 0, 1, \dots$ and a suitable $\beta > 1$. Theorem 0.1 now yields

$$(26) \quad |e^{-x} \psi_1(e^x) - 1| \leq \frac{2}{\pi x^n} \beta^n (n!)^2 [9(n+1)]! \leq x^{-n} \beta_1^n n^{11n}$$

for all $x > 0$ and $n \in \mathbb{N}$ with suitable $\beta_1 > 1$. Minimizing $x^{-n} \beta_1^n n^{11n}$ over $n \in \mathbb{N}$ for a fixed $x > 0$ we obtain

$$\psi_1(x) = x + O(x \exp(-2c \log^{\frac{1}{11}} x)), \quad x \rightarrow +\infty$$

and

$$\pi(x) = \int_2^{\infty} \frac{du}{\log u} + O(x \exp(-c \log^{\frac{1}{11}} x)), \quad x \rightarrow +\infty$$

with some $c > 0$. By analytic methods we can obtain an essentially better estimation of the error term in the P.N.T. (cf. [5]).

This theorem generalizes Theorem 2 of [2] in two directions: both the main and the remainder terms are more general. The main term can be found in [4], Chapter 5. The proof of the present theorem is a modification of the proof of Theorem 2 from the book [1], p. 124.

References

- [1] *Chandrasekharan K.*: Introduction to Analytic Number Theory. Springer Verlag, 1968.
- [2] *Čížek J.*: On the proof of the prime number theorem. Čas. pěst. mat. 106 (1981), 395–401.
- [3] *Rudin W.*: Real and Complex Analysis. New York, 1974.
- [4] *Subčankulov M. A.*: Tauberovy teoremy s ostatkom. Nauka, Moskva, 1976.
- [5] *Walfisz A.*: Weyl'sche Exponentialsummen in der neuere Zahlentheorie. Berlin, 1963.

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