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EXACT ASYMPTOTIC BEHAVIOR OF SINGULAR VALUES OF A CLASS OF INTEGRAL OPERATORS

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Abstract. We find an exact asymptotic formula for the singular values of the integral operator of the form $\int_\Omega T(x,y)k(x-y)\cdot \mathrm{d}y\colon\ L^2(\Omega)\to L^2(\Omega)$ ($\Omega\subset\mathbb{R}^m$, a Jordan measurable set) where $k(t)=k_0((t_1^2+t_2^2+\dots t_m^2)^{\frac{m}{2}})$, $k_0(x)=x^{\alpha-1}L(\frac{1}{x})$, $\frac12-\frac{1}{2m}<\alpha<\frac12$ and L is slowly varying function with some additional properties. The formula is an explicit expression in terms of L and T.

MSC 2000: 47B10

0. Introduction

The asymptotic properties of the spectrum of operators with a convolution kernel have been considered in many papers [1]–[6], [8]–[11], [14], [15]. The exact asymptotics have been obtained under the assumption that the Fourier transform of the kernel satisfies some conditions concerning the rate of growth.

M. Kac [5] obtained the exact asymptotic of the eigenvalues of the operators with the kernel $\varrho(y)|x-y|^{\alpha-1}$ ($0<\alpha<1$, $\varrho\in C[a,b]$, $\varrho>0$ on $[a,b]$). He used a probabilistic method and Karamata's Tauberian theorem.

M. Š. Birman and M. Z. Solomjak [1], G. P. Kostometov [6] and S. Y. Rotfeld [11] considered the asymptotics of the spectrum of operators with a kernel of the form

$$T(x,y)k(x,y). \tag{$*$}$$

They assumed that k is a homogeneous function from the class $C^\infty(\mathbb{R}\setminus\{0\})$ and that T is a function which is smooth of some order.

F. Cobos and T. Kühn [2] treated the problem of estimating the singular values of operators with a kernel of the form $(*)$ where

$$k(x)=\frac{(1+\ln\|x\|)^{\gamma}}{\|x\|^{m(1-\alpha)}},\qquad \gamma\in\mathbb{R},\ x\in\mathbb{R}^m,\ 0<\alpha\leqslant\frac{1}{2}.$$

They found an upper bound for singular values of such operators and proved its optimality (in the sense of growth order) in the case $m=1$, $\Omega=[-\frac{1}{2},\frac{1}{2}]$ and

$$T(x,y)=\begin{cases}|x-y|^{\alpha-1}(1-\ln|x-y|)^{\gamma}; & |x-y|\leqslant\frac{1}{2},\\ 0; & |x-y|>\frac{1}{2}.\end{cases}$$

In [3] we have proved a statement concerning the asymptotic order of singular values of the operator $\int_0^x k(x-y)\cdot\mathrm{d}y\colon\ L^2(0,1)\to L^2(0,1)$ in the case when $k(x)=x^{\alpha-1}L(\frac{1}{x})$, $0<\alpha<\frac{1}{2}$.

In this paper we give an exact asymptotic formula for singular values of integral with a kernel of the form

$$T(x,y)k(x,y)$$

acting on $L^2(\Omega)$ (Ω–a Jordan measurable set in $\mathbb{R}^m$). Here $k(x)=k_0\big((x_1^2+\ldots+x_m^2)^{\frac{m}{2}}\big)$, $k_0(t)=t^{\alpha-1}L(\frac{1}{t})$ $(t\in\mathbb{R})$, $\frac{1}{2}-\frac{1}{2m}<\alpha<\frac{1}{2}$, L is a slowly varying function satisfying some additional conditions and $T\in L^{\infty}(\Omega\times\Omega)$.

The asymptotic formula gives a direct expression in terms of the functions L and T.

1. Preliminaries

Suppose $\mathcal{H}$ is a complex Hilbert space and T is a compact operator on $\mathcal{H}$. The singular values of T $(s_n(T))$ are the eigenvalues of $(T^*T)^{1/2}$ (or $(TT^*)^{1/2}$).

The eigenvalues of $(T^*T)^{1/2}$ arranged in the decreasing order and repeated according to their multiplicity, form a sequence $s_1,s_2,s_3,\ldots$ tending to zero.

Denote the set of compact operators on $\mathcal{H}$ by C_∞.

An operator T is a Hilbert Schmidt operator ($T\in C_2$) if

$$\Big(\sum_{n\geqslant1}s_n^2(T)\Big)^{1/2}=|T|_2<\infty.$$

If $T\in C_2$ is an integral operator on $L^2(\Omega)$ defined by

$$Tf(x)=\int_{\Omega}M(x,y)f(y)\,\mathrm{d}y$$

then

$$|T|_2^2 = \int_\Omega \int_\Omega |M(x,y)|^2 \,\mathrm{d}x \,\mathrm{d}y.$$

Denote by $\int_\Omega K(x,y)\cdot \mathrm{d}y$ the integral operator on $L^2(\Omega)$ with a kernel $K(x,y)$. By $a_n \sim b_n$ $(f(x) \sim g(x),\ x \to x_0)$ we denote the fact that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \qquad \left(\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1\right).$$

Let $\mathcal{N}_t(T)$ be the singular value distribution function

$$\mathcal{N}_t(T) = \sum_{s_n(T)\geqslant t} 1 \qquad (t > 0).$$

A positive function L is a slowly varying function on $[a,+\infty)$ if it is measurable and for each $\lambda > 0$ the equality

$$\lim_{x\to+\infty} \frac{L(\lambda x)}{L(x)} = 1$$

holds. It is well known [13] that for every $\gamma > 0$ we have

$$\lim_{x\to+\infty} x^\gamma L(x) = +\infty,$$

$$\lim_{x\to+\infty} x^{-\gamma} L(x) = 0.$$

Denote by $|\Omega|$ the Lebesgue measure of the set $\Omega \subset \mathbb{R}^m$. In what follows we need some lemmas.

Lemma 1. *Let $\alpha > 0$ and suppose L is a slowly varying function such that $\varphi(x) = x^{-\alpha}L(x)$ and $\psi(x) = x^\alpha L(x)$ are monotone for $x \geqslant x_0$ and*

$$\lim_{x\to+\infty} \frac{L(x(L(x))^{\pm 1/\alpha})}{L(x)} = 1. \tag{0}$$

Then

$$\varphi^{-1}(y) \sim \left(\frac{L(y^{-1/\alpha})}{y}\right)^{1/\alpha}, \quad y \to 0+,$$

$$\psi^{-1}(y) \sim \left(\frac{y}{L(y^{1/\alpha})}\right)^{1/\alpha}, \quad y \to +\infty$$

where φ^{-1}, ψ^{-1} are the inverses of φ and ψ.

P r o o f. Follows directly from (0) by substitution. □

Observe that the functions $L(x) = \prod_{i=1}^{s} (\ln_{m_i} x)^{\alpha_i}$ $(\ln_m x = \underbrace{\ln\ln\ldots\ln}_{m} x)$ satisfy the conditions of Lemma 1.

Lemma 2. *Suppose the operator $H \in C_\infty$ is such that for every $\varepsilon > 0$ there exists a decomposition $H = H'_\varepsilon + H''_\varepsilon$ $(H'_\varepsilon, H''_\varepsilon \in C_\infty)$ with the following properties:*

1° *there exists* $\lim\limits_{t\to 0+} \left(\frac{t}{L(t^{-1/\alpha})}\right)^{1/\alpha} \mathcal{N}_t(H'_\varepsilon) = c(H'_\varepsilon)$ $(c(H'_\varepsilon)$ *is a bounded function in a neighborhood of the point* $\varepsilon = 0)$,

2°

$$\overline{\lim_{n\to\infty}} \frac{n^\alpha}{L(n)} s_n(H''_\varepsilon) < \varepsilon.$$

Then there exists $\lim\limits_{\varepsilon\to 0} c(H'_\varepsilon) = c(H)$ *and*

$$\lim_{t\to 0+} \left(\frac{t}{L(t^{-1/\alpha})}\right)^{1/\alpha} \mathcal{N}_t(H) = c(H)$$

(L *is a slowly varying function satisfying the conditions of Lemma* 1*).*

Lemma 3. *Let $H', H'' \in C_\infty$ and $H = H' + H''$. If*

$$\lim_{t\to 0+} \left(\frac{t}{L(t^{-1/\alpha})}\right)^{1/\alpha} \mathcal{N}_t(H') = c(H')$$

and

$$s_n(H'') = o\left(\frac{L(n)}{n^\alpha}\right)$$

then

$$\lim_{t\to 0+} \left(\frac{t}{L(t^{-1/\alpha})}\right)^{1/\alpha} \mathcal{N}_t(H) = c(H')$$

P r o o f. Lemmas 2 and 3 can be proved by a slight modification of the proof of the Ky-Fan theorem [1], [4]. □

2. Main result

Suppose $\Omega \subset \mathbb{R}^m$ is a bounded Jordan measurable set with a diameter d. Let L be a (positive, nondecreasing) slowly varying function, $L \in C^1[\frac{1}{d}, +\infty)$ such that $x \mapsto x\frac{L'(x)}{L(x)}$ is a decreasing function for x large enough and $\lim\limits_{x\to+\infty} x\frac{L'(x)}{L(x)} = 0$.

Consider integral operators

$$\begin{aligned} A\colon\ L^2(\Omega) &\to L^2(\Omega),\\ B\colon\ L^2(\Omega) &\to L^2(\Omega) \end{aligned}$$

define by

$$\begin{aligned} Af(x) &= \int_\Omega k(x-y)f(y)\,\mathrm{d}y,\\ Bf(x) &= \int_\Omega T(x,y)k(x-y)f(y)\,\mathrm{d}y \end{aligned}$$

where

$$\begin{aligned} k(t) &= k_0\big((t_1^2+t_2^2+\dots t_m^2)^{\frac{m}{2}}\big), \quad t\in\mathbb{R}^m,\\ k_0(x) &= k^{\alpha-1}L\Big(\frac{1}{x}\Big), \quad \alpha>0,\ x\in\mathbb{R},\ T\in L^\infty(\Omega\times\Omega). \end{aligned}$$

Let

$$d(m,\alpha) \stackrel{\text{def}}{=} \pi^{\frac{m}{2}(1-\alpha)}\frac{\Gamma(\frac{m\alpha}{2})}{\Gamma(\frac{m(1-\alpha)}{2})}\cdot\frac{1}{(\Gamma(1+\frac{m}{2}))^\alpha}.$$

Theorem 1. *If $\frac{1}{2}-\frac{1}{2m} < \alpha < \frac{1}{2}$ $(m \geqslant 2)$ and the function L satisfies the conditions of Lemma* 1, *then*

$$s_n(A) \sim d(m,\alpha)|\Omega|^\alpha\cdot\frac{L(n)}{n^\alpha}. \tag{1}$$

Theorem 2. *If $\frac{1}{2}-\frac{1}{2m} < \alpha < \frac{1}{2}$ $(m \geqslant 2)$ and the function $T\in L^\infty(\Omega\times\Omega)$ is such that it is continuous in a neighbourhood of the diagonal $y=x$, $T(x,x)>0$ on Ω and L satisfies the conditions of Lemma* 1, *then*

$$s_n(B) \sim d(m,\alpha)\bigg(\int_\Omega (T(x,x))^{1/\alpha}\,\mathrm{d}x\bigg)^\alpha\cdot\frac{L(n)}{n^\alpha}. \tag{2}$$

Observe that in [2] a special case of Theorem 2 is considered, namely $L(x) = (1+\frac{1}{m}|\ln|x||)^\gamma$.

Before proving Theorems 1 and 2 we prove a number of lemmas.

Lemma 4. [2] *For an integral operator* $\int_\Omega T(x,y)q(\|x-y\|^m)\cdot \mathrm{d}y$ $(x,y\in\Omega\subset\mathbb{R}^m$, Ω *a bounded domain) where* $T\in L^\infty(\Omega\times\Omega)$, $q\in L^1(0,\infty)$, $q\geqslant 0$ *and* $q\in L^2(a,\infty)$ *for every* $a>0$, *the following estimate holds:*

$$s_n\Big(\int_\Omega T(x,y)q(\|x-y\|^m)\cdot\mathrm{d}y\Big)\leqslant C\|T\|_\infty\Big[\int_0^a q(t)\,\mathrm{d}t+n^{-1/2}\Big(\int_a^\infty q^2(t)\,\mathrm{d}t\Big)^{1/2}\Big].$$

(The constant C depends only on Ω).

From the proof in [2] it can be concluded that one can take $C=\sigma_m+\sqrt{\sigma_m\cdot\operatorname{Vol}\Omega}$ where σ_m is the volume of the unit m-dimensional ball.

If we put $q(x)=x^{\alpha-1}L(\frac{1}{x})$ $(\equiv k_0(x))$, $\frac12-\frac{1}{2m}<\alpha<\frac12$ and $a=\frac1n$ in the previous lemma (L being positive, nondecreasing slowly varying function) we obtain

$$\begin{aligned}(3)\qquad & s_n\Big(\int_\Omega T(x,y)k_0(\|x-y\|^m)\cdot\mathrm{d}y\Big)\\ &\leqslant C\|T\|_\infty\Big(\int_0^{1/n}t^{\alpha-1}L\Big(\frac1t\Big)\,\mathrm{d}t+n^{-1/2}\Big(\int_{1/n}^\infty t^{2\alpha-2}L^2\Big(\frac1t\Big)\,\mathrm{d}t\Big).\end{aligned}$$

Having in mind

$$\begin{aligned}\int_0^{1/n}t^{\alpha-1}L\Big(\frac1t\Big)\,\mathrm{d}t&=\int_n^\infty\frac{L(x)}{x^{\alpha+1}}\,\mathrm{d}x\sim\frac1\alpha\,\frac{L(n)}{n^\alpha},\\ \int_{1/n}^{+\infty}t^{2\alpha-2}L^2\Big(\frac1t\Big)\,\mathrm{d}t&=\int_0^n\frac{L^2(x)}{x^{2\alpha}}\,\mathrm{d}x\sim\frac{1}{1-2\alpha}\,\frac{L^2(n)}{n^{2\alpha-1}}\qquad(n\to+\infty)\end{aligned}$$

from (3) we get

$$(4)\qquad s_n\Big(\int_\Omega T(x,y)k_0(\|x-y\|^m)\cdot\mathrm{d}y\Big)\leqslant C_1\|T\|_\infty\frac{L(n)}{n^\alpha}\qquad(n\to\infty)$$

(C_1 is a constant depending only on Ω).

Let $\xi\in\mathbb{R}^m$ and

$$K(\xi)\stackrel{\text{def}}{=}\int_{\mathbb{R}^m}\mathrm{e}^{\mathrm{i}t\cdot\xi}k(t)\,\mathrm{d}t=\int_{\mathbb{R}^m}\mathrm{e}^{\mathrm{i}t\cdot\xi}k_0(\|t\|^m)\,\mathrm{d}t$$

$=$ (according to [12], p. 358) $=$

$$\frac{(2\pi)^{\frac{m}{2}}}{\|\xi\|^{\frac{m-2}{2}}}\int_0^\infty k_0(\varrho^m)\cdot\varrho^{\frac{m}{2}}J_{\frac{m}{2}-1}(\varrho\|\xi\|)\,\mathrm{d}\varrho$$

(J_ν is the Bessel function with the index ν).

Let

$$\mathcal{K}(\lambda)\stackrel{\mathrm{def}}{=}\frac{(2\pi)^{\frac{m}{2}}}{\lambda^{\frac{m-2}{2}}}\int_0^\infty k_0(\varrho^m)\varrho^{\frac{m}{2}}J_{\frac{m}{2}-1}(\lambda\varrho)\,\mathrm{d}\varrho,\quad \lambda>0.$$

Then

$$K(\xi)=\mathcal{K}(\|\xi\|)\qquad \xi=(\xi_1,\xi_2,\ldots,\xi_m),\quad \|\xi\|^2=\xi_1^2+\xi_2^2+\ldots+\xi_m^2.$$

Lemma 5. *If* $\frac{1}{2}-\frac{1}{2m}<\alpha<\frac{1}{2}$ *then the asymptotic formula*

$$K(\lambda)\sim\pi^{\frac{m}{2}}2^{m\alpha}\frac{\Gamma(\frac{m\alpha}{2})}{\Gamma(\frac{m(1-\alpha)}{2})}\frac{L(\lambda^m)}{\lambda^{m\alpha}},\quad \lambda\to+\infty \tag{5}$$

holds.

Proof. Substituting $\lambda\varrho=\frac{1}{x}$ in the integral defining $\mathcal{K}$, after a simplification we get $\mathcal{K}(\lambda)=(2\pi)^{\frac{m}{2}}\lambda^{-m\alpha}\int_0^\infty x^{\frac{m}{2}-m\alpha-2}L((\lambda x)^m)J_{\frac{m}{2}-1}(\frac{1}{x})\,\mathrm{d}x$.

Put

$$\mathcal{K}(\lambda)=(2\pi)^{\frac{m}{2}}\lambda^{-m\alpha}(\mathcal{K}_1(\lambda)+\mathcal{K}_2(\lambda))$$

where

$$\mathcal{K}_1(\lambda)=\int_0^{x_1}x^{\frac{m}{2}-m\alpha-2}J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)L(\lambda^m x^m)\,\mathrm{d}x,$$
$$\mathcal{K}_2(\lambda)=\int_{x_1}^{+\infty}x^{\frac{m}{2}-m\alpha-2}J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)L(\lambda^m x^m)\,\mathrm{d}x$$

where x_1 is the reciprocal value of the smallest positive zero of the function $J_{\frac{m}{2}-2}$.

(It is known that for every $m\in\mathbb{N}$ the smallest positive zero of $J_{\frac{m}{2}-2}$ is greater than 1; so $0<x_1<1$).

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}x^\lambda J_\lambda\Big(\frac{1}{x}\Big)=x^{\lambda-2}J_{\lambda+1}\Big(\frac{1}{x}\Big) \tag{6}$$

we obtain (for $\lambda=\frac{m}{2}-2$)

$$\mathcal{K}_1(\lambda)=\int_0^{x_1}x^{2-m\alpha}L(\lambda^m x^m)\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{\frac{m}{2}-2}J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\Big).$$

Applying partial integration and having in mind that

$$\lim_{x\to 0} x^{\frac{m}{2}-m\alpha} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big) = 0 \qquad (0<\alpha<1/2)$$

and

$$J_{\frac{m}{2}-2}\Big(\frac{1}{x_1}\Big) = 0$$

we obtain

$$\mathcal{K}_1(\lambda) = -\int_0^{x_1} x^{\frac{m}{2}-2} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\big(x^{2-m\alpha}L(\lambda^m x^m)\big)'\,\mathrm{d}x.$$

So

$$\begin{aligned}\mathcal{K}_1(\lambda) = (m\alpha-2)&\int_0^{x_1} x^{\frac{m}{2}-2} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)x^{1-m\alpha}L(\lambda^m x^m)\,\mathrm{d}x\\ &-\int_0^{x_1} x^{\frac{m}{2}-2} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)m\lambda^m x^{1+m-m\alpha}L'(\lambda^m x^m)\,\mathrm{d}x\end{aligned}$$

and therefore

$$\begin{aligned}\text{(7)}\qquad \frac{\mathcal{K}_1(\lambda)}{L(\lambda^m)} &= (m\alpha-2)\int_0^{x_1} x^{\frac{m}{2}-m\alpha-1} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\frac{L(\lambda^m x^m)}{L(\lambda^m)}\,\mathrm{d}x\\ &\quad -m\int_0^{x_1} x^{\frac{m}{2}-m\alpha-1} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\frac{(\lambda x)^m L'((\lambda x)^m)}{L(\lambda^m x^m)}\cdot\frac{L(\lambda^m x^m)}{L(\lambda^m)}.\end{aligned}$$

By the asymptotic formula

$$J_\lambda(z) = \sqrt{\frac{2}{\pi z}}\Big(\cos\Big(z-\frac{\pi\lambda}{2}-\frac{\pi}{4}\Big)+O\Big(\frac{1}{z}\Big)\Big) \qquad [12]$$

we get

$$\int_0^{x_1} x^{\frac{m}{2}-m\alpha-1}\Big|J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\Big|\,\mathrm{d}x < \infty$$

and from (7), the Lebesgue Dominated Convergence Theorem, the fact that $x\frac{L'(x)}{L(x)} \downarrow 0$ and $0<x_1<1$ it follows that

$$\text{(8)}\qquad \mathcal{K}_1(\lambda) = L(\lambda^m)\Big((m\alpha-2)\int_0^{x_1} x^{\frac{m}{2}-m\alpha-1} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\,\mathrm{d}x + o(1)\Big),\ \lambda\to+\infty.$$

Applying (6) once more we obtain

$$(m\alpha-2)\int_0^{x_1} x^{\frac{m}{2}-m\alpha-1} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)\,\mathrm{d}x = \int_0^{x_1} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x$$

and from (8) we conclude

$$\mathcal{K}_1(\lambda) = L(\lambda^m)\Big[\int_0^{x_1} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x + o(1)\Big], \quad \lambda \to +\infty. \tag{9}$$

Let us now estimate the asymptotic behavior of the function $\mathcal{K}_2$. Since

$$J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{m}{2})} 2^{1-\frac{m}{2}-2k} x^{1-\frac{m}{2}-2k},$$

we obtain $\int_{x_1}^{\infty} x^{\frac{m}{2}-m\alpha-2} |J_{\frac{m}{2}-1}(\frac{1}{x})|\,\mathrm{d}x < \infty$ provided

$$\int_{x_1}^{\infty} x^{\frac{m}{2}-m\alpha-2} x^{1-\frac{1}{2}-0}\,\mathrm{d}x < \infty.$$

But this is true, because we have supposed that $\alpha > \frac{1}{2} - \frac{1}{2m}$.

Since

$$\frac{\mathcal{K}_2(\lambda)}{L(\lambda^m)} = \int_{x_1}^{\infty} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big) \cdot \frac{L(\lambda^m x^m)}{L(\lambda^m)},$$

Theorem 2.6 [13] yields

$$\mathcal{K}_2(\lambda) = L(\lambda^m)\Big[\int_{x_1}^{\infty} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x + o(1)\Big], \ \lambda \to \infty. \tag{10}$$

From (9) and (10) we obtain (after a simplification)

$$\mathcal{K}(\lambda) = (2\pi)^{\frac{m}{2}} \lambda^{-m\alpha} L(\lambda^m)\Big(\int_0^{\infty} x^{-\frac{m}{2}+m\alpha} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x + o(1)\Big), \quad \lambda \to +\infty. \tag{11}$$

Since

$$\int_0^{\infty} \varrho^{\beta} J_{\nu}(\varrho)\,\mathrm{d}\varrho = 2^{\beta}\Gamma\Big(\frac{\nu+\beta+1}{2}\Big)\Big/\Gamma\Big(\frac{\nu-\beta+1}{2}\Big) \quad \text{(Veber integral)}$$

we get

$$\int_0^{\infty} x^{-\frac{m}{2}+m\alpha} J_{\frac{m}{2}-1}(x)\,\mathrm{d}x = 2^{m\alpha-\frac{m}{2}} \frac{\Gamma(\frac{m\alpha}{2})}{\Gamma(\frac{m(1-\alpha)}{2})}$$

and (11) yields

$$\mathcal{K}(\lambda) = 2^{m\alpha}\pi^{\frac{m}{2}}\Gamma\Big(\frac{m\alpha}{2}\Big)\Big/\Gamma\Big(\frac{m(1-\alpha)}{2}\Big) \cdot \frac{L(\lambda^m)}{\lambda^{m\alpha}} \cdot (1+o(1)) \ \lambda \to +\infty.$$

$\square$

Lemma 6. *If L is a slowly varying nondecreasing function, $\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$, $\varepsilon > 0$ and $m \geqslant 2$ then*

$$\begin{aligned} S &= \int_{[0,\varepsilon]^m \times [0,\varepsilon]^m} \int \left| \frac{L\big(\big(\frac{1}{(x_1 \pm y_1)^2 + \ldots (x_m \pm y_m)^2}\big)^{m/2}\big)}{((x_1 \pm y_1)^2 + \ldots + (x_m \pm y_m)^2)^{\frac{m}{2}(1-\alpha)}} \right|^2 \mathrm{d}x \, \mathrm{d}y < \infty, \\ x &= (x_1, x_2, \ldots, x_m), \\ y &= (y_1, y_2, \ldots, y_m) \end{aligned}$$

where all combinations of $+$ and $-$ are possible, except the one with all $-$.

Proof. It is enough to prove the statement in the case $\varepsilon = 2$. As L is a nondecreasing, the expression under the integral sign is largest when one sign is $+$ and all the other signs are $-$. To be specific, let the sign $+$ be in the last term. We have

$$S = \int_0^2 \int_0^2 \mathrm{d}x_m \, \mathrm{d}y_m \int_0^2 \ldots \int_0^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{m-1} \int_0^2 \ldots \int_0^2 \left| \frac{L\big(\big(\frac{1}{(x_1 - y_1)^2 + \ldots (x_m + y_m)^2}\big)^{m/2}\big)}{((x_1 - y_1)^2 + \ldots + (x_m + y_m)^2)^{\frac{m}{2}(1-\alpha)}} \right|^2 \mathrm{d}y_1 \ldots \mathrm{d}y_{m-1}.$$

Let

$$\begin{aligned} &u_i - x_i = t_i, \quad i = 1, 2, \ldots, m-1, \\ &u = x_m + y_m \end{aligned}$$

and let

$$S_1(u) = \int_0^2 \ldots \int_0^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{m-1} \int_{\prod_{i=1}^{m-1} (-x_i, 2 - x_i)} \left| \frac{L\big(\big(\frac{1}{t_1^2 + \ldots + t_{m-1}^2 + u^2}\big)^{\frac{m}{2}}\big)}{(t_1^2 + \ldots + t_{m-1}^2 + u^2)^{\frac{m}{2}(1-\alpha)}} \right|^2 \mathrm{d}t_1 \ldots \mathrm{d}t_{m-1}.$$

It is enough to prove that

$$\int_0^2 \int_0^2 S_1(x_m + y_m) \, \mathrm{d}x_m \, \mathrm{d}y_m < \infty$$

and therefore it is enough to prove that

$$\int_0^2 \int_0^2 h(x_m + y_m) \, \mathrm{d}x_m \, \mathrm{d}y_m < \infty \tag{12}$$

where

$$h(u)=\int_0^2\dots\int_0^2 \mathrm{d}x_1\dots\mathrm{d}x_{m-1}\int_{\prod\limits_{i=1}^{m-1}(0,x_i)} \left|\frac{L\big((\frac{1}{t_1^2+\ldots+t_{m-1}^2+u^2})^{\frac{m}{2}}\big)}{(t_1^2+\ldots+t_{m-1}^2+u^2)^{\frac{m}{2}(1-\alpha)}}\right|^2 \mathrm{d}t_1\dots\mathrm{d}t_{m-1}.$$

Since

$$\prod_{i=1}^{m-1}(0,x_i)\subset\Big\{t\in\mathbb{R}^{m-1}\colon \sum_{i=1}^{m-1}t_i^2\leqslant\sum_{i=1}^{m-1}x_i^2=R^2\leqslant 4(m-1)\Big\}$$

we get

$$h(u)\leqslant\int_0^2\dots\int_0^2 \mathrm{d}x_1\dots\mathrm{d}x_{m-1}\int_{|t|\leqslant R}\left|\frac{L\big((\frac{1}{t_1^2+\ldots+t_{m-1}^2+u^2})^{m/2}\big)}{(t_1^2+\ldots+t_{m-1}^2+u^2)^{\frac{m}{2}(1-\alpha)}}\right|^2 \mathrm{d}t_1\dots\mathrm{d}t_{m-1}.$$

Let

$$\varphi_0(t)=\left|\frac{L(\frac{1}{(t^2+u^2)^{m/2}})}{(t^2+u^2)^{\frac{m}{2}(1-\alpha)}}\right|^2.$$

Then

$$h(u)\leqslant\int_0^2\dots\int_0^2 \mathrm{d}x_1\dots\mathrm{d}x_{m-1}\int_{\sum\limits_{i=1}^{m-1}t_i^2\leqslant\sum\limits_{i=1}^{m-1}x_i^2=R^2}\varphi_0(\|t\|)\,\mathrm{d}t.$$

According to the formula

$$\int_{\sum\limits_{i=1}^{m-1}t_i^2\leqslant\sum\limits_{i=1}^{m-1}x_i^2=R^2}\varphi_0(\|t\|)\,\mathrm{d}t =\frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}\int_0^{\sqrt{\sum\limits_{i=1}^{m-1}x_i^2}}\varrho^{m-2}\left|\frac{L(\frac{1}{(\varrho^2+u^2)^{m/2}})}{(\varrho^2+u^2)^{\frac{m}{2}(1-\alpha)}}\right|^2\mathrm{d}\varrho$$

[12] we obtain

$$h(u)\leqslant\int_0^2\dots\int_0^2 \mathrm{d}x_1\dots d_{x_{m-1}}\frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}\int_0^{\sqrt{\sum\limits_{i=1}^{m-1}x_i^2}}\varrho^{m-2}\left|\frac{L(\frac{1}{(\varrho^2+u^2)^{m/2}})}{(\varrho^2+u^2)^{\frac{m}{2}(1-\alpha)}}\right|^2\mathrm{d}\varrho.$$

Since $\sum\limits_{i=1}^{m-1}x_i^2\leqslant 4(m-1)\leqslant 4m$ we conclude that

$$h(u)\leqslant 2^m\frac{\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}\int_0^{2\sqrt{m}}\varrho^{m-2}\left|\frac{L(\frac{1}{(\varrho^2+u^2)^{m/2}})}{(\varrho^2+u^2)^{\frac{m}{2}(1-\alpha)}}\right|^2\mathrm{d}\varrho.$$

After the substitution $\varrho = uv$, $v \in (0, \frac{2\sqrt{m}}{u})$ we obtain

$$h(u) \leqslant 2^m \frac{\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})} \int_0^{\frac{2\sqrt{m}}{u}} u^{-m-1+2m\alpha} \cdot v^{m-2} \frac{L^2(\frac{1}{(u^2(1+v^2))^{m/2}})}{(1+v^2)^{m(1-\alpha)}} \,\mathrm{d}v. \tag{13}$$

Since the function L is nondecreasing and $\int_0^\infty v^{m-2}(1+v^2)^{-m(1-\alpha)}\,\mathrm{d}v < \infty$, we obtain for $m \geqslant 2$ and $\alpha < 1/2$ from (13) the inequality

$$h(u) \leqslant \text{const } u^{2m\alpha-m-1}\Big(L\Big(\frac{1}{u^m}\Big)\Big)^2$$

where const. does not depend on u.

To prove (12) it is enough to prove (by virtue of the previous inequality) that

$$\int_0^2\int_0^2 \frac{L^2(\frac{1}{(x+y)^m})}{(x+y)^{m+1-2m\alpha}}\,\mathrm{d}x\,\mathrm{d}y < \infty$$

($\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$, L is a slowly varying function).

By direct calculation we get that this integral is finite provided

$$\int_0^2 \frac{L^2(\frac{1}{y^m})}{y^{m-2m\alpha}}\,\mathrm{d}y < \infty. \tag{14}$$

Since

$$\int_0^2 \frac{L^2(\frac{1}{y^m})}{y^{m-2m\alpha}}\,\mathrm{d}y = \frac{1}{m}\int_{2^{-m}}^\infty \frac{L^2(x)}{x^{2\alpha+\frac{1}{m}}}\,\mathrm{d}x,$$

the integral (14) is finite if $2\alpha + \frac{1}{m} > 1$, i.e. $\alpha > \frac{1}{2} - \frac{1}{2m}$, which is true by the assumption. $\square$

Now, we perform a modification of the function L. Let

$$L_a(x) = \begin{cases} L(x); & x \geqslant a \quad (a > \frac{1}{a}), \\ L'(a)(x-a) + L(a); & 0 < x \leqslant a \end{cases}$$

and $k_a(x) = x^{\alpha-1}L_a(x)$, $\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$.

We introduce an operator

$$A_a\colon\ L^2(\Omega) \to L^2(\Omega)$$

(Ω being a bounded, Jordan measurable set in $\mathbb{R}^m$), defined by

$$A_a f(x) = \int_\Omega k_a(\|x-y\|^m) f(y)\,\mathrm{d}y.$$

Let

$$K_a(\xi) = \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}t\cdot\xi} k_a(t)\,\mathrm{d}t \quad (\xi, t \in \mathbb{R}^m)$$

and

$$\mathcal{K}_a(\lambda) = \frac{(2\pi)^{\frac{m}{2}}}{\lambda^{\frac{m-2}{2}}} \int_0^\infty k_a(\varrho^m)\varrho^{\frac{m}{2}} J_{\frac{m}{2}-1}(\lambda\varrho)\,\mathrm{d}\varrho.$$

Clearly $K_a(\xi) = \mathcal{K}_a(\|\xi\|)$, $\xi \in \mathbb{R}^m$ and so Lemma 5 implies $\mathcal{K}_a(\lambda) \sim \lambda^{-m\alpha} L(\lambda^m)\pi^{\frac{m}{2}} \cdot 2^{m\alpha} \cdot \Gamma(\frac{\alpha m}{2})/\Gamma(\frac{m(1-\alpha)}{2})$.

Lemma 7. *If a is a fixed number large enough, then the function $\mathcal{K}_a(\lambda)$ is monotonicaly decreasing for λ large enough.*

P r o o f. Differentiating the function $\mathcal{K}_a$ by λ, after a simplification we obtain

$$\begin{aligned}\mathcal{K}_a'(\lambda) = (2\pi)^{\frac{m}{2}} \lambda^{-m\alpha} L(\lambda^m)\Big[&-m\alpha \int_0^\infty x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x \\ &+ m\int_0^\infty x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big) \cdot \frac{(\lambda x)^m L_a'((\lambda x)^m)}{L_a(\lambda^m)}\,\mathrm{d}x\Big].\end{aligned}$$

Since

$$\int_0^\infty x^{\frac{m}{2}-m\alpha} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x = 2^{m\alpha-\frac{m}{2}}\Gamma\Big(\frac{m\alpha}{2}\Big)\Big/\Gamma\Big(\frac{m(1-\alpha)}{2}\Big),$$

it is enough to prove that if a is a fixed number large enough and λ is large enough then

$$\Big|\int_0^\infty x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big) \cdot \frac{(\lambda x)^m L_a'((\lambda x)^m)}{L_a(\lambda^m)}\,\mathrm{d}x\Big| \leqslant \alpha 2^{m\alpha-\frac{m}{2}} \frac{\Gamma(\frac{m\alpha}{2})}{\Gamma(\frac{m(1-\alpha)}{2})}. \tag{15}$$

Since (for $x \geqslant 1$)

$$\lim_{\lambda\to\infty} \frac{(\lambda x)^m L_a'((\lambda x)^m)}{L_a((\lambda x)^m)} = 0,$$

it follows from Theorem 2.6 [13] that

$$\lim_{\lambda\to\infty} \int_1^\infty x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)(\lambda x)^m \frac{L_a'((\lambda x)^m)}{L_a((\lambda x)^m)}\,\mathrm{d}x = 0. \tag{16}$$

Now, consider the integral $\int_0^1 x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}(\frac{1}{x}) \cdot (\lambda x)^m \frac{L_a'((\lambda x)^m)}{L_a((\lambda x)^m)}\,\mathrm{d}x$. If we suppose $\lambda > \sqrt[m]{a}$ then the integral can be splitted in the following way:

$$\int_0^1 = \int_0^{\sqrt[m]{a}/\lambda} = \int_{\sqrt[m]{a}/\lambda}^1.$$

Since $\lambda^m > a$ and $x \leqslant \frac{\sqrt[m]{a}}{\lambda} < 1$, we have $\lambda^m x^m < a$ and $L'_a((\lambda x)^m) = L'(a)$ and hence

$$\int_0^{\frac{\sqrt[m]{a}}{\lambda}} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big) \cdot (\lambda x)^m \frac{L'_a((\lambda x)^m)}{L_a(\lambda^m)}\,\mathrm{d}x = \frac{L'(a)}{L(\lambda^m)} \int_0^{\frac{\sqrt[m]{a}}{\lambda}} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)(\lambda x)^m\,\mathrm{d}x.$$

Therefore

$$\left|\int_0^{\frac{\sqrt[m]{a}}{\lambda}} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-2}\Big(\frac{1}{x}\Big)(\lambda x)^m \frac{L'_a((\lambda x)^m)}{L_a(\lambda^m)}\,\mathrm{d}x\right| \leqslant \frac{L'(a)}{L(a)}\left|\int_0^{\frac{\sqrt[m]{a}}{\lambda}} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)(\lambda x)^m\,\mathrm{d}x\right|.$$

From the asymptotic behavior of the function $J_{\frac{m}{2}-1}(t)$ $(t \to \infty)$, having in mind that $\lambda^m > a$ and $\alpha < \frac{1}{2}$, we obtain by direct calculation

$$\left|\int_0^{\frac{\sqrt[m]{a}}{\lambda}} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)(\lambda x)^m\,\mathrm{d}x\right| \leqslant \mathrm{const.}a$$

where const. does not depend on λ and a.

So

$$\left|\int_0^{\frac{\sqrt[m]{a}}{\lambda}} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big) \cdot (\lambda x)^m \frac{L'_a((\lambda x)^m)}{L_a(\lambda^m)}\,\mathrm{d}x\right| \leqslant \mathrm{const.}\frac{aL'(a)}{L(a)}. \tag{17}$$

Since the function $a \mapsto \frac{aL'(a)}{L(a)}$ tends to zero when $a \to +\infty$, the integral on the left hand side of (17) can be made arbitrary small for a large enough.

Now we estimate

$$R \stackrel{\mathrm{def}}{=} \int_{\frac{\sqrt[m]{a}}{\lambda}}^1 x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)(\lambda x)^m \frac{L'_a((\lambda x)^m)}{L_a(\lambda^m x^m)} \cdot \frac{L(\lambda^m x^m)}{L(\lambda^m)}\,\mathrm{d}x.$$

Applying the Bonnet Mean Value Theorem to the monotone increasing function $L_a((\lambda x)^m)$ we obtain

$$R = \frac{L_a(\lambda^m)}{L_a(\lambda^m)} \int_{\xi_1}^1 x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)(\lambda x)^m \frac{L'_a((\lambda x)^m)}{L_a((\lambda x)^m)}\,\mathrm{d}x$$

where $\frac{\sqrt[m]{a}}{\lambda} \leqslant \xi_1 < 1$.

Applying once more the Bonnet Mean Value Theorem to the nonincreasing function $x \mapsto (\lambda x)^m \frac{L_a'((\lambda x)^m)}{L_a((\lambda x)^m)}$ we obtain

$$R = (\lambda \xi_1)^m \frac{L_a'((\lambda \xi_1)^m)}{L_a((\lambda \xi_1)^m)} \int_{\xi_1}^{\xi_2} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x$$

where $\xi_1 \leqslant \xi_2 \leqslant 1$.

Since $(\lambda\xi_1)^m \geqslant a$ and the function $x \mapsto x^m \frac{L_a'(x^m)}{L_a(x^m)}$ is nonincreasing we get

$$|R| \leqslant a \frac{L_a'(a)}{L_a(a)} \cdot \bigg| \int_{\xi_1}^{\xi_2} x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}\Big(\frac{1}{x}\Big)\,\mathrm{d}x \bigg|.$$

Having in mind that $L_a'(a) = L'(a)$, $L_a(a) = L(a)$ and the fact that the integral $\int_0^\infty x^{\frac{m}{2}-m\alpha-2} J_{\frac{m}{2}-1}(\frac{1}{x})\,\mathrm{d}x$ is convergent we conclude that

$$|R| \leqslant \text{const.}\ a \frac{L'(a)}{L(a)} \tag{18}$$

where const. does not depend on a.

Since the function $a \mapsto a\frac{L'(a)}{L(a)}$ tends to zero (when $a \to +\infty$), R can be forced to be arbitrary small by choosing a large enough and $\lambda > \sqrt[m]{a}$.

The statement of Lemma 7 follows from (15), (16), (17) and (18). □

Lemma 8. *Consider all numbers* $\sum\limits_{k=1}^{m} n_k^2$, *where* $n_k \in \mathbb{N} \cup \{0\}$, $k = 1, 2, \ldots, m$. *If we arrange these numbers in the nondecreasing order* $\lambda_1' \leqslant \lambda_2' \leqslant \lambda_3' \leqslant$ *then* $\lambda_n' \sim C_m^{-2/m} \cdot n^{\frac{2}{m}}$ *where*

$$C_m = \pi^{\frac{m}{2}} / 2^m \Gamma\Big(1 + \frac{m}{2}\Big).$$

P r o o f. This is easily deduced from [7], p. 330. □

Let us now consider a special case of the domain Ω. Namely, we assume $\Omega = I^m$ where $I = (-1, 1)$. Then

$$A\colon\ L^2(I^m) \to L^2(I^m),$$
$$Af(x) = \int_{I^m} k_0(\|x-y\|^m) f(y)\,\mathrm{d}y \quad \Big(= \int_{I^m} k(x-y) f(y)\,\mathrm{d}y\Big).$$

Lemma 9. *If* $\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$, $m \geqslant 2$ *then*

$$s_n\Big(\int_{I^m} k_0(\|x-y\|^m) \cdot \mathrm{d}y\Big) \sim c(\alpha, m) \frac{L(n)}{n^\alpha} \quad (n \to \infty)$$

where $c(\alpha, m) = 2^{m\alpha}\pi^{\frac{m}{2}(1-\alpha)}\Gamma(\frac{\alpha m}{2})/\Gamma(\frac{m(1-\alpha)}{2})\cdot(\Gamma(1+\frac{m}{2}))^{\alpha}$.

P r o o f. As we do not know in advance whether the function $\mathcal{K}(\lambda)$ is monotone for λ large enough, we consider instead of A the asymptotics $s_n(A_a)$ where

$$\begin{aligned} &A_a\colon\ L^2(I^m) \to L^2(I^m), \\ &A_a f(x) = \int_{I^m} k_a(\|x-y\|^m) f(y)\,\mathrm{d}y. \end{aligned}$$

We shall show that

$$s_n(A_a) \sim c(\alpha, m)\frac{L(n)}{n^{\alpha}}$$

and

$$\lim_{n\to\infty} \frac{s_n(\int_{I^m} k_0(\|x-y\|^m)\cdot\,\mathrm{d}y)}{s_n(A_a)} = 1$$

for a fixed and large enough.

Let $h_a(t) = k_a(\|t\|^m)$, $t \in \mathbb{R}^m$.

Introduce functions $h_{a,1}, h_{a,2}, \ldots, h_{a,m-1}$, H_a is the following way:

$$\begin{aligned} &h_{a,1}(t_1,\ldots,t_{m-1}) \\ &\quad= \sum_{n_m\in\mathbb{Z}} \big[h_a(t_1,\ldots,t_{m-1},x_m-y_m+4n_m) - h_a(t_1,\ldots,t_{m-1},x_m+y_m+4n_m+2)\big], \\ &h_{a,2}(t_1,\ldots,t_{m-2}) \\ &\quad= \sum_{n_{m-1}\in\mathbb{Z}} \big[h_{a,1}(t_1,\ldots,t_{m-2},x_{m-1}-y_{m-1}+4n_{m-1}) \\ &\qquad - h_{a,1}(t_1,\ldots,t_{m-2},x_{m-1}+y_{m-1}+4n_{m-1}+2), \\ &\qquad \vdots \\ &\qquad h_{a,m-1}(t_1) \\ &\quad= \sum_{n_2\in\mathbb{Z}} \big[h_{a,m-2}(t_1,x_2-y_2+4n_2) - h_{a,m-2}(t_1,x_2+y_2+4n_2+2)\big], \\ &H_a(x,y) \\ &\quad= \sum_{n_1\in\mathbb{Z}} \big[h_{a,m-1}(x_1-y_1+4n_1) - h_{a,m-1}(x_1+y_1+4n_1+2)\big]. \end{aligned}$$

By direct calculation we obtain

$$\int_{I^m} H_a(x,y)\varphi_{n_1n_2\ldots n_m}(y)\,\mathrm{d}y = K_a\Big(\frac{n_1\pi}{2}, \frac{n_2\pi}{2}, \ldots \frac{n_m\pi}{2}\Big)\varphi_{n_1n_2\ldots n_m}(x)$$

where $\varphi_{n_1n_2\ldots n_m}(x) = \prod\limits_{i=1}^{m} \sin\frac{n_i\pi(1+x_i)}{2}$ is an orthonormal base of $L^2(I^m)$. According to Lemma 6 the operator

$$\int_{I^m} (H_a(x,y) - k_a(\|x-y\|^m))\cdot\,\mathrm{d}y\colon\ L^2(I^m) \to L^2(I^m)$$

is a Hilbert Schmidt operator; hence

$$s_n\Big(\int_{I^m}(H_a(x,y)-k_a(\|x-y\|^m))\cdot\mathrm{d}y\Big)=o(n^{-1/2}) \tag{19}$$
$$=o\Big(\frac{L(n)}{n^\alpha}\Big)\quad\Big(0<\alpha<\frac12\Big).$$

The singular values of the operator $\int_{I^m}H_a(x,y)\cdot\mathrm{d}y$ are

$$s_{n_1n_2\dots n_m}=K_a\Big(\frac{n_1\pi}{2},\frac{n_2\pi}{2},\dots\frac{n_m\pi}{2}\Big)=\mathcal{K}_a\Big(\frac{\pi}{2}\sqrt{n_1^2+\dots+n_m^2}\Big).$$

Arrange the sequence $s_{n_1n_2\dots n_m}$ to the nonincreasing sequence $s'_1\geqslant s'_2\geqslant\dots$

According to Lemma 7 the function $\mathcal{K}_a(\lambda)$ is decreasing for a fixed and large enough and for λ large enough. Hence

$$\Big(\frac{2}{\pi}\mathcal{K}_a^{-1}(s_{n_1n_2\dots n_m})\Big)^2=n_1^2+n_2^2+\dots+n_m^2$$

($\mathcal{K}_a^{-1}$ is inverse function of $\mathcal{K}_a$), i.e. $\big(\frac{2}{\pi}\mathcal{K}_a^{-1}(s'_n)\big)^2=n_1^2+\dots+n_m^2$ (for $n_1\dots n_m$, n large enough).

By Lemma 8 we obtain $(\frac{2}{\pi}\mathcal{K}_a^{-1}(s'_n))^2\sim C_m^{-2/m}n^{2/m}$ and therefore

$$\mathcal{K}_a^{-1}(s'_n)\sim\frac{\pi}{2}C_m^{-1/m}\cdot n^{\frac1m}.$$

The function $\mathcal{K}_a$ behaves (when $\lambda\to+\infty$) as a regularly varying function (Lemma 7) and so

$$s'_n\sim K_a\Big(\frac{\pi}{2}C_m^{-1/m}\cdot n^{\frac1m}\Big).$$

Having in mind the asymptotic behavior of $\mathcal{K}_a(\lambda)$ when $\lambda\to+\infty$ we get from this asymptotic relation

$$s'_n\sim c(\alpha,m)\frac{L(n)}{n^\alpha}$$

and

$$s_n\Big(\int_{I^m}Ha(x,y)\cdot\mathrm{d}y\Big)\sim c(\alpha,m)\cdot\frac{L(n)}{n^\alpha}. \tag{20}$$

From (19), (20) and the Ky-Fan Theorem [4] we obtain

$$s_n(A_a)\sim c(\alpha,m)\frac{L(n)}{n^\alpha}.$$

Let $S_a = \{x\colon \|x\| < \frac{1}{2\sqrt[m]{a}}\}$, $\varrho\colon L^2(I^m) \to L^2(I^m)$, $Pf(x) = \chi_{S_a} f(x)$, $Q = J - P$ (J—the identical operator).

Then

$$A_a = (P+Q)A_a(P+Q) = PA_aP + QA_aP + PA_aQ + QA_aQ$$

and similarly

$$A = (P+Q)A(P+Q) = PAP + QAP + PAQ + QAQ.$$

Since $PA_aP = PAP$, we have

$$A = A_a + Q(A - A_a)P + P(A - A_a)Q + Q(A - A_a)Q. \tag{21}$$

Having in mind that $A - A_a \in C_2$ (Hilbert Schmidt) we get

$$s_n(Q(A - A_a)P + P(A - A_a)Q + Q(A - A_a)Q) = o(n^{-1/2}) = o\Big(\frac{L(n)}{n^\alpha}\Big) \quad \Big(\alpha < \frac{1}{2}\Big). \tag{22}$$

Since $s_n(A_a) \sim c(\alpha, m)\frac{L(n)}{n^\alpha}$, the statement of Lemma 9 follows from (21), (22) and the Ky-Fan Theorem [4]. $\square$

Remark. From the previous lemma (by substituting) we get the following result:

If Δ is a cube with edges parallel to the coordinate axes, then

$$s_n\bigg(\int_\Delta k(x-y)\cdot \mathrm{d}y\bigg) \sim |\Delta|^\alpha d(m,\alpha)\frac{L(n)}{n^\alpha} \quad \Big(\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}\Big). \tag{23}$$

Lemma 10. *Suppose Δ_1 and Δ_2 are two cubes of the same size in $\mathbb{R}^m$ having no common internal points and with the edges parallel to the coordinate axes. Then for $\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$*

$$\int_{\Delta_1}\int_{\Delta_2} |k(x-y)|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty$$

holds.

Proof. If Δ_1 and Δ_2 have no common boundary points, then $\inf\limits_{(x,y)\in\Delta_1\times\Delta_2} \|x - y\| > 0$ and the statement is trivial.

If Δ_1 and Δ_2 have some common boundary points, then repeating the procedure as in Lemma 6, the statement of Lemma 10 is obtained under the condition $\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}$. $\square$

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^m$ be bounded measurable sets and let $\Omega_1 \subset \Omega_i$. Let $F_i\colon L^2(\Omega_i) \to L^2(\Omega_i)$, $i = 1, 2$ be compact operators defined by

$$F_i f(x) = \int_{\Omega_i} M(x,y) f(y)\,\mathrm{d}y.$$

Lemma 11. *The singular value distribution functions of the operators F_i $(i = 1, 2)$ satisfy the inequality*

$$\mathcal{N}_t(F_1) \leqslant \mathcal{N}_t(F_2) \qquad (t > 0).$$

P r o o f. Let $P\colon L^2(\Omega_2) \to L^2(\Omega_1)$ be the orthoprojector $(Pf(x) = \chi_{\Omega_1}(x) f(x))$. Since $F_1 = PF_2P$, we have

$$s_n(F_1) \leqslant s_n(F_2)$$

and hence

$$\mathcal{N}_t(F_1) \leqslant \mathcal{N}_t(F_2).$$

$\square$

Lemma 12. *Let $\Omega = \bigcup\limits_{i=1}^{s} Q_i$ where Q_i are cubes such that $Q_1^0 \cap Q_j^0 = \emptyset$, $i \neq j$ (V^0-the interior of the set V) and with the edges parallel to the coordinate axes. Then*

$$s_n\Bigl(\int_\Omega k(x-y)\cdot \mathrm{d}y\Bigr) \sim d(m,\alpha)|\Omega|^\alpha \frac{L(n)}{n^\alpha} \qquad \Bigl(\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}\Bigr).$$

P r o o f. $A = \int_\Omega k(x-y)\cdot \mathrm{d}y\colon L^2(\Omega) \to L^2(\Omega)$,

$$P_i\colon L^2(\Omega) \to L^2(\Omega_i);\ P_i f(x) = \chi_{Q_i}(x) f(x),\ i = 1, 2, \dots, s.$$

Hence

$$A = \Bigl(\sum_{i=1}^{s} P_i\Bigr) A \Bigl(\sum_{i=1}^{s} P_i\Bigr) = \sum_{i=1}^{s} P_i A P_i + \sum_{i\neq j}^{s} P_i A P_j.$$

Since, according to Lemma 10, $P_i A P_j \in C_2$ for $i \neq j$, we have $\sum\limits_{i\neq j} P_i A P_j \in C_2$ and hence

$$s_n\Bigl(\sum_{i\neq j}^{s} P_i A P_j\Bigr) = o(n^{-1/2}) = o\Bigl(\frac{L(n)}{n^\alpha}\Bigr) \quad \Bigl(\alpha < \frac{1}{2}\Bigr). \tag{24}$$

By (23) we have

$$s_n(P_iAP_i) \sim |Q_i|^\alpha d(m,\alpha)\frac{L(n)}{n^\alpha} \quad (n \to \infty)$$

and hence

$$\mathcal{N}_t(P_iAP_i) \sim \Big(\frac{L(t^{-1/\alpha})}{t}\Big)^{1/\alpha}|Q_i|(d(m,\alpha))^{1/\alpha},\ t \to 0+\,.$$

Having in mind $\mathcal{N}_t(\sum\limits_{i=1}^{s} P_iAP_i) = \sum\limits_{i=1}^{s}\mathcal{N}_t(P_iAP_i)$, we obtain

$$\lim_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t\Big(\sum_{i=1}^{s} P_iAP_i\Big) = (d(m,\alpha))^{1/\alpha}|\Omega|. \tag{25}$$

From (24), (25) and Lemma 3 we obtain

$$\lim_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(A) = (d(m,\alpha))^{1/\alpha}|\Omega|.$$

Putting $t = s_n(A)$ and $\mu_n \stackrel{\text{def}}{=} (s_n(A))^{-1/\alpha}$ in the previous equality, after a simplification we get

$$\mu_n^\alpha L(\mu_n) \sim \frac{n^\alpha}{d(m,\alpha)|\Omega|^\alpha} \quad (n \to \infty).$$

Applying Lemma 1 to this asymptotic relation we conclude that

$$\mu_n^\alpha \sim \frac{1}{d(m,\alpha)|\Omega|^\alpha}\cdot\frac{n^\alpha}{L(n)} \quad (n \to \infty),$$

i.e.

$$s_n(A) \sim d(m,\alpha)|\Omega|^\alpha\frac{L(n)}{n^\alpha}.$$

$\square$

P r o o f o f T h e o r e m 1. Let Ω be a bounded Jordan measurable set. Let $\underline{\Omega}_N \subset \Omega \subset \overline{\Omega}_N$ where the sets $\underline{\Omega}_N$ and $\overline{\Omega}_N$ are the unions of equal cubes (with disjoint interiors) such that

$$\begin{aligned} &m(\underline{\Omega}_N) \to m(\Omega) = |\Omega|, \\ &m(\overline{\Omega}_N) \to m(\Omega) = |\Omega|, \qquad N \to +\infty \qquad (m \text{ is the Lebesgue measure}). \end{aligned}$$

Let $\underline{A}_N$ and $\overline{A}_N$ be linear operators acting on $L^2(\underline{\Omega}_N)$ and $L^2(\overline{\Omega}_N)$ defined by

$$\underline{A}_N f(x) = \int_{\underline{\Omega}_N} k(x-y) f(y)\,\mathrm{d}y,$$
$$\overline{A}_N f(x) = \int_{\overline{\Omega}_N} k(x-y) f(y)\,\mathrm{d}y,$$

respectively.

According to Lemma 11 we get

$$\mathcal{N}_t(\underline{A}_N) \leqslant \mathcal{N}_t(A) \leqslant \mathcal{N}_t(\overline{A}_n)$$

and

$$\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(\underline{A}_N) \leqslant \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(A) \leqslant \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(\overline{A}_N), \quad t > 0.$$

Next, we get

$$\begin{aligned} \lim_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(\underline{A}_N) &\leqslant \varliminf_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(A) \\ &\leqslant \varlimsup_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(A) \\ &\leqslant \varlimsup_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(\overline{A}_N). \end{aligned}$$

Since there exist $\lim\limits_{t\to 0+} (\frac{t}{L(t^{-1/\alpha})})^{1/\alpha} \mathcal{N}_t(\underline{A}_N)$ and $\lim\limits_{t\to 0+} (\frac{t}{L(t^{-1/\alpha})})^{1/\alpha} \mathcal{N}_t(\overline{A}_n)$ and as they are equal (according to Lemma 12) to $(d(m,\alpha))^{1/\alpha} |\underline{\Omega}_N|$ and $(d(m,\alpha))^{1/\alpha} |\overline{\Omega}_N|$, respectively, (26) implies

$$\begin{aligned} (d(m,\alpha))^{1/\alpha} |\underline{\Omega}_N| &\leqslant \varliminf_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(A) \\ &\leqslant \varlimsup_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(A) \\ &\leqslant (d(m,\alpha))^{1/\alpha} |\overline{\Omega}_N|. \end{aligned}$$

If $N \to +\infty$ then we obtain

$$\lim_{t\to 0+} \Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha} \mathcal{N}_t(A) = (d(m,\alpha))^{1/\alpha} |\Omega|.$$

Putting here $t = s_n(A)$, by Lemma 1 we obtain

$$s_n(A) \sim d(m,\alpha) \cdot |\Omega|^\alpha \frac{L(n)}{n^\alpha},$$

which proves (1). $\square$

Lemma 13. *Let* $\Omega = \bigcup\limits_{i=1}^{s} Q_i$ *where* Q_i *are equal cubes in* $\mathbb{R}^m$ *such that* $Q_i^0 \cap Q_j^0 = \emptyset$ $(i \neq j)$ *and with the edges parallel to the coordinate axes. Let a function* $T \in L^\infty(\Omega\times\Omega)$ *be continuous in a neighborhood of the diagonal* $y = x$ *and let* $T(x,x) > 0$ *on* Ω. *If a function* L *satisfies the condition* (0), *then for the operator* B *defined on* $L^2(\Omega)$ *by*

$$Bf(x) = \int_\Omega T(x,y)k(x-y)f(y)\,\mathrm{d}y$$

the asymptotic formula

$$s_n(B) \sim d(m,\alpha)\Bigl(\int_\Omega (T(x,x))^{1/\alpha}\,\mathrm{d}x\Bigr)^\alpha \frac{L(n)}{n^\alpha} \qquad \Bigl(\frac{1}{2} - \frac{1}{2m} < \alpha < \frac{1}{2}\Bigr)$$

holds.

P r o o f. Divide the cubes Q_i $(1 \leqslant i \leqslant s)$ in equal smaller cubes so that their number in Ω in N. Denote these cubes by Δ_i $(1 \leqslant i \leqslant N)$ and denote by x_i the midpoint of Δ_i. Let operators

$$\begin{aligned} A_i^N &\colon\ L^2(\Omega) \to L^2(\Omega), &\qquad i &= 1,2,\ldots,N,\\ A_{ij} &\colon\ L^2(\Omega) \to L^2(\Omega), &\qquad i,j &= 1,2,\ldots,N \end{aligned}$$

be defined by

$$\begin{aligned} A_i^N f(x) &= \int_\Omega k(x-y)\chi_{\Delta_i}(x)\chi_{\Delta i}(y)T(x_i,x_i)f(y)\,\mathrm{d}y,\\ A_{ij} f(x) &= \int_\Omega k(x-y)\chi_{\Delta_i}(x)\chi_{\Delta_j}(y)T(x,y)f(y)\,\mathrm{d}y, \quad i \neq j. \end{aligned}$$

Let

$$A_N = \sum_{i=1}^{N} A_i^N.$$

Then

$$B = A_N + \sum_{i\neq j}^{N} A_{ij} + B_N,$$

where B_N is the operator defined by

$$B_N f(x) = \int_\Omega k(x-y)G_N(x,y)f(y)\,\mathrm{d}y.$$

Here

$$G_N(x,y) = \sum_{i=1}^{N} \chi_{\Delta i}(x)\chi_{\Delta_i}(y)(T(x,y) - T(x_i,x_i)).$$

It follows from the continuity of the function T in a neighborhood of the diagonal $y = x$ that for an arbitrary $\varepsilon > 0$ and for N large enough we have $|T(x,y) - T(x_i,x_i)| < \varepsilon$ for $(x,y) \in \Delta_i \times \Delta_i$, $i = 1,2,\dots,N$.

Hence for $(x,y) \in \Omega \times \Omega$ we have

$$|G_N(x,y)| < \varepsilon.$$

This inequality and Remark after Lemma 4 give

$$s_n(B_N) < C_1 \cdot \varepsilon \cdot \frac{L(n)}{n^\alpha} \tag{27}$$

where the constant C_1 does not depend on n and ε. Since for $i \neq j$, $A_{ij} \in C_2$ (by Lemma 10 in the case $\frac{1}{2} - \frac{1}{2\alpha} < \alpha < \frac{1}{2}$), we have

$$\sum_{i\neq j}^{N} A_{ij} \in C_2$$

and

$$\lim_{n\to\infty} n^{1/2} s_n\left(\sum_{i\neq j} A_{ij}\right) = 0.$$

Combining this with (27) and using the properties of the singular values of the sum of two operators we obtain that for every $\varepsilon > 0$ there exists a positive integer N such that

$$\overline{\lim_{t\to 0+}} \frac{n^\alpha}{L(n)} s_n\left(\sum_{i\neq j}^{N} A_{ij} + B_N\right) < \varepsilon. \tag{28}$$

Since the operator A_N is the direct sum of the operators A_i^N we obtain

$$\mathcal{N}_t(A_N) = \sum_{i=1}^{N} \mathcal{N}_t(A_i^N). \tag{29}$$

Theorem 1 implies

$$\lim_{t\to 0+} \left(\frac{t}{L(t^{-1/\alpha})}\right)^{1/\alpha} \mathcal{N}_t(A_i^N) = (T(x_i,x_i))^{1/\alpha} (d(m,\alpha))^{1/\alpha} |\Delta_i|$$

and (29) gives

$$\lim_{t\to 0+} \left(\frac{t}{L(t^{-1/\alpha})}\right)^{1/\alpha} \mathcal{N}_t(A_N) = (d(m,\alpha))^{1/\alpha} \sum_{i=1}^{N} (T(x_i,x_i))^{1/\alpha} |\Delta_i|. \tag{30}$$

From (28), (30) and Lemma 2 we conclude

$$\lim_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(B)=\lim_{N\to\infty}(d(m,\alpha))^{1/\alpha}\sum_{i=1}^{N}(T(x_i,x_i))^{1/\alpha}|\Delta_i|$$
$$=(d(m,\alpha))^{1/\alpha}\cdot\Big(\int_\Omega (T(x,x))^{1/\alpha}\,\mathrm{d}x\Big).$$

Putting here $t=s_n(B)$ and $(s_n(B))^{-1/\alpha}=\mu_n$, after a simplification we obtain

$$\mu_n^\alpha L(\mu_n)\sim\frac{n^\alpha}{\ell_0}\quad\text{where}\quad \ell_0=d(m,\alpha)\Big(\int_\Omega (T(x,x))^{1/\alpha}\,\mathrm{d}x\Big)^\alpha. \tag{31}$$

Applying Lemma 1 to (31) we conclude

$$\mu_n^\alpha\sim\frac{n^\alpha}{\ell_0L(n)}\quad\text{and}\quad s_n(B)\sim d(m,\alpha)\Big(\int_\Omega (T(x,x))^{1/\alpha}\,\mathrm{d}x\Big)^\alpha\cdot\frac{L(n)}{n^\alpha}.$$

□

P r o o f o f T h e o r e m 2. Let us extend the function T to a bounded function $\widetilde{T}$ in some neighborhood $\Omega\times\Omega$ so that it is continuous in a neighborhood of the diagonal $y=x$.

Let Ω be a bounded Jordan measurable set in $\mathbb{R}^m$.

Let $\underline{\Omega}_N\subset\Omega\subset\overline{\Omega}_N$ where the sets $\underline{\Omega}_N$ and $\overline{\Omega}_N$ are the unions of equal cubes (with disjoint interiors) such that

$$m(\underline{\Omega}_N)\to|\Omega|,$$
$$m(\overline{\Omega}_N)\to|\Omega|,\qquad N\to\infty.$$

Let $\underline{B}_N$ and $\overline{B}_N$ be operators acting on $L^2(\underline{\Omega}_N)$ and $L^2(\overline{\Omega}_N)$ defined by

$$\underline{B}_Nf(x)=\int_{\underline{\Omega}_N}T(x,y)k(x-y)f(y)\,\mathrm{d}y,$$
$$\overline{B}_Nf(x)=\int_{\overline{\Omega}_N}\widetilde{T}(x,y)k(x-y)f(y)\,\mathrm{d}y,$$

respectively.

It follows from Lemma 11 that $\mathcal{N}_t(\underline{B}_N)\leqslant\mathcal{N}_t(B)\leqslant\mathcal{N}_t(\overline{B}_N)$ and $(\frac{t}{L(t^{-1/\alpha})})^{1/\alpha}\mathcal{N}_t$ $(\underline{B}_N)\leqslant(\frac{t}{L(t^{-1/\alpha})})^{1/\alpha}\mathcal{N}_t(B)\leqslant(\frac{t}{L(t^{-1/\alpha})})^{1/\alpha}\mathcal{N}_t(\overline{B}_N)$, $t>0$ and therefore

$$\begin{aligned}\varliminf_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(\underline{B}_N)&\leqslant\varliminf_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(B)\\&\leqslant\varlimsup_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(B)\\&\leqslant\varlimsup_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(\overline{B}_N).\end{aligned}\tag{32}$$

Since $\lim\limits_{t\to 0+}(\frac{t}{L(t^{-1/\alpha})})^{1/\alpha}\mathcal{N}_t(\underline{B}_N)$ and $\lim\limits_{t\to 0+}(\frac{t}{L(t^{-1/\alpha})})^{1/\alpha}\mathcal{N}_t(\overline{B}_N)$ exist and as they are equal (according to Lemma 13) to

$$(d(m,\alpha))^{1/\alpha}\int_{\underline{\Omega}_N}(T(x,x))^{1/\alpha}\,\mathrm{d}x \qquad \text{and} \qquad (d(m,\alpha))^{1/\alpha}\int_{\overline{\Omega}_N}(\widetilde{T}(x,x))^{1/\alpha}\,\mathrm{d}x,$$

respectively, (32) implies

$$\begin{aligned}(d(m,\alpha))^{1/\alpha}\int_{\underline{\Omega}_N}(T(x,x))^{1/\alpha} &\leqslant \varliminf_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(B)\\ &\leqslant \varlimsup_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(B)\\ &\leqslant (d(m,\alpha))^{1/\alpha}\int_{\overline{\Omega}_N}(\widetilde{T}(x,x))^{1/\alpha}\,\mathrm{d}x.\end{aligned}$$

Letting $N\to+\infty$ we get

$$\lim_{t\to 0+}\Big(\frac{t}{L(t^{-1/\alpha})}\Big)^{1/\alpha}\mathcal{N}_t(B)=(d(m,\alpha))^{1/\alpha}\int_{\overline{\Omega}}(T(x,x))^{1/\alpha}\,\mathrm{d}x.$$

Putting here $t=s_n(B)$, it follows by Lemma 1 that

$$s_n(B)\sim d(m,\alpha)\Big(\int_{\Omega}(T(x,x))^{1/\alpha}\,\mathrm{d}x\Big)^{\alpha}\frac{L(n)}{n^{\alpha}},$$

which proves (2). □

Remark. The condition $\frac{1}{2}-\frac{1}{2m}<\alpha<\frac{1}{2}$ is used indirectly in the proof of Lemma 13. It is an open problem whether Theorems 1 and 2 hold in the case $0<\alpha<1/2$. But if $m=1$ it appears that Theorem 1 and 2 are also true, i.e. they hold in the case $0<\alpha<1/2$ (their proofs have to be slightly modified).

References

[1] *M. Š. Birman, M. Z. Solomyak*: Asymptotic behavior of the spectrum of weakly polar integral operators. Izv. Akad. Nauk. SSSR, Ser. Mat. Tom *34* (1970), N^{0}5, 1151–1168.
[2] *F. Cobos, T. Kühn*: Eigenvalues of weakly singular integral operators. J. London Math. Soc. *(2) 41* (1990), 323–335.
[3] *M. Dostanić*: An estimation of singular of convolution operators. Proc. Amer. Math. Soc. *123* (1995), N^{0}5, 1399–1409.
[4] *I. C. Gohberg, M. G. Krein*: Introduction to the Theory of Linear Nonselfadjoint Operators, in "Translation of Math. monographs" Vol. 18. Amer. Math. Soc., Providence, R.I., 1969.
[5] *M. Kac*: Distribution of eigenvalues of certain integral operators. Mich. Math. J. *3* (1955/56), 141–148.

[6] *G. P. Kostometov*: Asymptotic behavior of the spectrum of integral operators with a singularity on the diagonal. Math. USSR Sb. T 94 (136) $N^0 3$ (7), 1974, pp. 445–451.
[7] *S. G. Mihlin*: Lectures on Mathematics Physics. Moscow, 1968.
[8] *C. Oehring*: Asymptotics of singular numbers of smooth kernels via trigonometric transforms. J. of Math. Analysis and Applications *145* (1990), 573–605.
[9] *J. B. Reade*: Asymptotic behavior of eigenvalues of certain integral equations. Proceeding of the Edinburgh Math. Soc. *22* (1979), 137–144.
[10] *M. Rosenblat*: Some results on the asymptotic behavior of eigenvalues for a class of integral equations with translations kernels. J. Math. Mech. *12* (1963), 619–628.
[11] *S. Y. Rotfeld*: Asymptotic of the spectrum of abstract integral operators. Trudy. Moscow Mat. Obšč. T. *34* (1977), 105–128.
[12] *S. G. Samko, A. A. Kilbas, O. I. Maricev*: Fractional Integrals and Derivative and Some Applications. Minsk, 1987.
[13] *E. Seneta*: Regularly Varying Functions. Springer Verlag, 1976.
[14] *H. Widom*: Asymptotic behavior of the eigenvalues of certain integral equations. Arch. Rational Mech. Analys. *17* (1964), 215–229.
[15] *H. Widom*: Asymptotic behavior of the eigenvalues of certain integral equations. Trans. Amer. Math. Soc. *109* (1963), 278–295.

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