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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 4, 811–815

Persistent URL: <http://dml.cz/dmlcz/127529>

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THE STRUCTURE OF TRANSITIVE ORDERED
PERMUTATION GROUPS

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(Received November 18, 1996)

Abstract. We give some necessary and sufficient conditions for transitive l -permutation groups to be 2-transitive. We also discuss primitive components and give necessary and sufficient conditions for transitive l -permutation groups to be normal-valued.

MSC 2000: 06F15

Keywords: transitive l -permutation group, stabilizer subgroup, primitive component, normal-valued l -group.

1. INTRODUCTION

We continue the research ([1], [2], [3]) using stabilizer subgroups for transitive l -permutation groups. We will prove that a transitive l -permutation group is 2-transitive if and only if the stabilizer group G_α of a point α acts transitively on $\{w \mid w < \alpha\}$, if and only if for each $\gamma < \beta < \alpha$, there exists $g \in G_\alpha$ such that $\beta \leq \gamma g$. We will also discuss primitive components and will obtain that a transitive l -permutation group is normal-valued if and only if every primitive component is regular.

Let Ω be a chain and (G, Ω) an l -permutation group on Ω . If $\alpha G = \Omega$ for $\alpha \in \Omega$, then (G, Ω) is called transitive. Let Δ be a subset of Ω . The stabilizer of G on Δ , $G_\Delta = \{g \in G \mid \delta g = \delta \text{ for all } \delta \in \Delta\}$, G_α and $G_{(\Delta)} = \{g \in G \mid \Delta g = \Delta\}$ are prime subgroups of G ([2]). If Δ is a G_α -orbit, Δ is said to be positive if $\Delta > \{\alpha\}$ (negative if $\Delta < \{\alpha\}$). If Δ is an orbit of G_α and $|\Delta| > 1$, Δ is said to be a long orbit of G_α .

Let (G, Ω) be a transitive l -permutation group, $\alpha \in \Omega$, and let $(\mathcal{C}_k, \mathcal{C}^k)$ be a covering pair of convex congruences ([2]). Then $\alpha\mathcal{C}^k$ is a block of (G, Ω) , and G induces an action on Ω/\mathcal{C}^k . Let $\Omega_k = \alpha\mathcal{C}^k/\mathcal{C}_k$ be the chain of \mathcal{C}_k -classes within $\alpha\mathcal{C}^k$.

So the stabilizer $G_{(\alpha C^k)}$ induces an order-permutation of Ω_k by G_k . (G_k, Ω_k) is called the k^{th} primitive component of (G, Ω) ([2]).

2. TRANSITIVITY

We first discuss the transitivity for l -permutation groups.

Lemma 1. *Let (G, Ω) be a transitive l -permutation group, and let Δ be a block. Then $\{\Delta g \mid g \in G\}$ is a partition of Ω , and the convex congruence associated with the partition is denoted by \mathcal{C}_Δ . Furthermore each $g \in G$ induces an order-preserving permutation on the chain $\{\Delta g \mid g \in G\}$, i.e. $\Delta \rightarrow \Delta g$.*

Proof. The set $\{\Delta g \mid g \in G\}$ is a partition of Ω because of transitivity of G and the definition of the block ([2], Theorem 1.6.1). □

Theorem 2. *Let (G, Ω) be a transitive l -permutation group, and let Δ be a block. The following conditions are equivalent:*

- (i) $G_{(\Delta)}$ is a normal subgroup of G for each block Δ of (G, Ω) .
- (ii) $G_{(\Delta g)} = G_{(\Delta)}$ for every $g \in G$.
- (iii) $G_{(\Delta)} = e$ where e is the identity.
- (iv) $(G, \Omega/\mathcal{C}_\Delta)$ is regular, where \mathcal{C}_Δ is the congruence associated with the partition $\{\Delta g \mid g \in G\}$.

Proof. The subgroup $G_{(\Delta)}$ is normal in G if and only if $G_{(\Delta)} = G_{(\Delta g)}$ by transitivity and the fundamental identity ([3]). But (ii) \Rightarrow (iii), (iv) \Rightarrow (iii) and (iii) \Rightarrow (i) are obvious. Now we only prove that (iii) \Rightarrow (iv). If $(\Delta f)h_i = \Delta g$, $i = 1, 2$, for any $f, g \in G$, then $\Delta f h_1 g^{-1} = \Delta f h_2 g^{-1} = \Delta$, i.e., $f h_i g^{-1} \in G_{(\Delta)} = e$. Hence $f h_i = g$, and $h_1 = h_2$. So $(G, \Omega/\mathcal{C}_\Delta)$ is regular. □

Lemma 3. *Let (G, Ω) be a transitive l -permutation group. If the stabilizer subgroup G_α of a point α is transitive on $\{w \mid w < \alpha\}$, then the stabilizer subgroup G_β is also transitive on $\{w \mid w < \beta\}$ for every $\beta \in \Omega$.*

Proof. Let $\alpha = \beta f$ by the transitivity condition. If $\gamma, \delta < \beta$, then $\alpha f^{-1} > \gamma, \delta$ and $\alpha > \gamma f, \delta f$. There exists $g \in G_\alpha$ such that $(\delta f)g = \gamma f$ by hypothesis. So $\delta(f g f^{-1}) = \gamma$ and $f g f^{-1} \in G_\beta$ by the fundamental identity. Hence $\gamma \in \delta G_\beta$. □

Theorem 4. *Let (G, Ω) be a transitive l -permutation group. Then the following conditions are equivalent:*

- (i) (G, Ω) is 2-transitive.
- (ii) The stabilizer subgroup G_α is transitive on $\{w \mid w < \alpha\}$.

(iii) For every $\gamma < \beta < \alpha$, there exists $g \in G_\alpha$ such that $\beta \leq \gamma g$.

Proof. (i) \Rightarrow (ii). If $\beta, \gamma \in \{w \mid w < \alpha\}$, i.e., $\beta < \alpha$, $\gamma < \alpha$, then there exists $g \in G$ such that $\beta = \gamma g$ and $\alpha g = \alpha$. So $\beta \leq \gamma g$ and $g \in G_\alpha$.

(ii) \Rightarrow (iii). If $\gamma < \beta < \alpha$, i.e., $\gamma, \beta \in \{w \mid w < \alpha\}$, there is $g \in G_\alpha$ such that $\beta = \gamma g$, i.e. $\beta \leq \gamma g$.

(iii) \Rightarrow (ii). If $\beta, \gamma \in \{w \mid w < \alpha\}$, let $\gamma < \beta < \alpha$. There is $g \in G_\alpha$ such that $\beta \leq \gamma g$. Let $\beta = \gamma f$ for some $f \in G$ by transitivity. Since $\alpha((f \vee e) \wedge g) = \alpha$, we have $(f \vee e) \wedge g \in G_\alpha$ and $\gamma((f \vee e) \wedge g) = \beta$.

(ii) \Rightarrow (i). If $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, let $\alpha_2 h = \beta_2$ for some $h \in G$. Then $\alpha_1 h < \alpha_2 h = \beta_2$. For $\alpha_1 h$ and β_1 , there exists $g \in G_{\beta_2}$ such that $(\alpha_1 h)g = \beta_1$. But we have also $\alpha_2 h g = \beta_2$. Hence G is 2-transitive. \square

3. PRIMITIVITY

We now return to primitivity of l -permutation groups. Let α, β be distinct points of Ω . Let Δ be the union of blocks which contain α but not β . Then Δ is a block. Let Λ be the intersection of blocks containing both Δ and β . Then Λ is also a block, and Λ covers Δ in the chain of blocks containing α under inclusion. Let \mathcal{C}^k and \mathcal{C}_k be the convex congruences corresponding to Δ and Λ , respectively. Thus $(\mathcal{C}_k, \mathcal{C}^k)$ is a covering pair, and k is called the value $Val(\alpha, \beta)$ ([2]).

Theorem 5. *Let (G, Ω) be a transitive l -permutation group. Then the set $K(G, \Omega) = \{(G_k, \Omega_k) \mid k \in K\}$ of primitive components is a chain under inclusion. Moreover, every primitive component must be 2-transitive, regular or periodically primitive.*

Proof. Every block of (G, Ω) containing α is a chain because G is transitive, so $K(G, \Omega)$ is a chain where $(\mathcal{C}_k, \mathcal{C}^k) < (\mathcal{C}_{k'}, \mathcal{C}^{k'})$ if $\mathcal{C}^k \subsetneq \mathcal{C}^{k'}$ ([2], Theorem 3A). For the second part, every primitive component (G_k, Ω_k) is primitive ([2], Theorem 3E). Then G_k is 2-transitive, regular or periodically primitive ([2], Theorem 4.3.1). \square

We have the following applications for the above Structure Theory.

Theorem 6. *Let (G, Ω) be a transitive l -permutation group. Then the following conditions are equivalent:*

- (i) G is normal-valued.
- (ii) $fg \leq g^2 f^2$ for all $f, g \in G^+$.
- (iii) All primitive components of (G, Ω) are regular.

Proof. (i) \Rightarrow (iii). Suppose that a primitive component (G_k, Ω_k) is not regular, then it must be 2-transitive or periodic. For every $\Delta \in \Omega_k$, $G_{(\Delta)}$ is not a normal subgroup of G_k . There is $g \in G$ such that $\Delta \neq \Delta g \in \alpha \mathcal{C}^k$, i.e., $g \notin G_{(\Delta)}$. Thus $G_{(\Delta)}$ is a value of G . By primitivity of G_k , $G_{(\Delta)}$ is a maximal prime subgroup of G_k . Hence G_k is a cover of $G_{(\Delta)}$. So G is not normal valued.

(iii) \Rightarrow (ii). Suppose that $fg \not\leq g^2 f^2$ for some $f, g \in G^+$, then $\alpha fg > \alpha g^2 f^2$, where $\alpha \in \Omega$. Hence $\alpha fg > \alpha$. Let $k = \text{Val}(\alpha fg, \alpha)$. Then the primitive component (G_k, Ω_k) is regular. By primitivity of G_k , (G_k, Ω_k) is the regular representation of a subgroup of the set of real numbers \mathcal{R} . Let \bar{f} and \bar{g} be positive real translations induced respectively by f and g on $\alpha \mathcal{C}^k / \mathcal{C}_k$. Then we have $\Delta \bar{f} \bar{g} > \Delta \bar{g}^2 \bar{f}^2$ where $\Delta = \alpha \mathcal{C}_k \in \alpha \mathcal{C}^k / \mathcal{C}_k$, a contradiction.

(ii) \Rightarrow (i). By the Holland Representation Theorem, let $V(g)$ be a value. Then G is an l -subgroup of $A(\cup G/V(g))$ ([4], Theorem 5.4), and G is the transitive action on each individual space $G/V(g)$ for each $g \in G$. Let $V(g)^*$ be a cover of $V(g)$, and $G_{(V(g))} = V(g)$. Then $V(g)^*$ is the smallest prime subgroup containing $G_{(V(g))}$. Hence $G/V(g)$ has the smallest nontrivial convex congruence, and it has the smallest nonsingleton block of $(G, G/V(g))$. Suppose that a value $V(g)$ of G is not normal in its cover $V(g)^*$, and Δ is the smallest nonsingleton block containing the point $V(g)$. Since $V(g)$ is not normal in $G_{(\Delta)}$ and $G_{(\Delta)} = V(g)^*$, the primitive component $(G_{(\Delta)} \mid \Delta, \Delta)$ is not regular. So it must be 2-transitive or periodic, and it can not satisfy the identity “ $fg \leq g^2 f^2$ for all $f, g \in G^+$ ”. Let z be periodic and $\Delta = (\alpha, \alpha z)$. Then $(G_{(\Delta)} \mid \Delta, \Delta)$ is 2-transitive ([2], Theorem 4.3.1). Then this identity must also fail in G . Suppose this identity holds for G . Then it must be true in $G_{(\Delta)} \mid \Delta$, which is an l -homomorphic image of an l -subgroup of an l -homomorphic image of G . \square

References

- [1] *W. C. Holland*: Transitive lattice-ordered permutation groups. *Math. Zeit.* 87 (1965), 420–433; *MR31* (1966), # 2310.
- [2] *A. M. W. Glass*: Ordered Permutation Groups. Cambridge University Press, 1981, pp. 76–116.
- [3] *Z. T. Zhu and J. M. Huang*: Stability of l -permutation groups. *J. of Nanjing Uni. Math. Biquarterly* 11, No. 1 (1994), 18–21.
- [4] *M. Anderson and T. Feil*: Lattice-Ordered Groups. D. Reidel Publishing Company, Dordrecht, Holland, 1988, pp. 29–31.
- [5] *Z. T. Zhu and J. M. Huang*: Congruent pairs on a set. *Chinese Quarterly Journal of Math.* 9, No. 3 (1994), 37–41.
- [6] *S. H. McCleary*: The structure of intransitive ordered permutation groups. *Algebra Universalis* 6 (1976), 229–255.
- [7] *A. M. W. Glass*: Elementary types of automorphisms of linearly ordered sets—a survey. *Algebra*, Carbondale 1980 (R.K. Amayo, ed.). Springer, Lecture Notes No. 848, pp. 218–229.

- [8] *S. H. McCleary*: The structure of ordered permutation groups applied to lattice-ordered groups. *Notices Amer. Math. Soc.*, 21 (1974), February, # 712–714, PA336.
- [9] *Z. T. Zhu and Q. Chen*: The universal mapping problems of the l -group category. *Chinese Journal of Math.* 23, No. 2 (1995), 131–140.
- [10] *A. M. W. Glass and W. C. Holland (Eds)*: *Lattice-Ordered Groups*. Kluwer Academic Publishers, 1989, pp. 23–40.

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