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ON L_w^2 -QUASI-DERIVATIVES FOR SOLUTIONS OF PERTURBED
GENERAL QUASI-DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with square integrable quasi-derivatives for any solution of a general quasi-differential equation of n th order with complex coefficients $M[y] - \lambda w y = w f(t, y^{[0]}, \dots, y^{[n-1]})$, $t \in [a, b)$ provided that all r th quasi-derivatives of solutions of $M[y] - \lambda w y = 0$ and all solutions of its normal adjoint $M^+[z] - \bar{\lambda} w z = 0$ are in $L_w^2(a, b)$ and under suitable conditions on the function f .

Keywords: quasi-differential operators, regular, singular, bounded and square integrable solutions

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1. INTRODUCTION

In [16] Anton Zettl proved, under suitable conditions on f , that $y^{(j)} \in L^2[0, \infty)$, $j = 0, 1, \dots, n-1$ for any solution y of $M[y] = f(t, y)$ provided that all j th derivatives of solutions of $M[y] = 0$ and all solutions of $M^+[y] = 0$ are in $L^2[0, \infty)$, when M is a regular ordinary linear differential operator of order n with coefficients which are locally integrable on $[0, \infty)$ and M^+ is the formal adjoint of M . The case $j = 0$ was considered in [16] for a general n th order M and for $n = 2$ by Bradley [1]. Also, in [14] Wong proved that all solutions of a perturbed linear differential equation belong to $L^2[0, \infty)$ assuming that all solutions of the unperturbed equation possess the same property. These results were generalized by Ibrahim in [10] for general ordinary quasi-differential equations of n th order.

Our objective in this paper is to extend the results of Ibrahim, Wong and Zettl in [10], [14], [15] and [16] to a general ordinary quasi-differential expression M of order n with complex coefficients and to prove, under suitable conditions on f , that the

quasi-derivatives $y^{[r]} \in L_w^2(a, b)$, $r = 0, 1, \dots, n - 1$ for any solution y of

$$(1.1) \quad M[y] - \lambda wy = wf(t, y^{[0]}, \dots, y^{[n-1]}) \quad (\lambda \in \mathbb{C}) \quad \text{on } [a, b]$$

provided that all r th quasi-derivatives of solutions of the equation

$$(1.2) \quad M[y] - \lambda wy = 0$$

and all solutions of

$$(1.3) \quad M^+[y] - \bar{\lambda}wy = 0$$

are in $L_w^2(a, b)$. These results are extensions of those proved by Ibrahim in [10]. Our approach is based on an extension of Gronwall's inequality used by Bradley and due to Gollwitzer [6], on a technical lemma from Goldberg's book [5] and on an appropriate formulation of the variation of parameters formula.

We deal throughout with a quasi-differential expression M of an arbitrary order n defined by a Shin-Zettl matrix on the interval $I = (a, b)$. The left-hand end-point of I is assumed to be regular but the right-hand end-point may be either regular or singular.

2. NOTATION AND PRELIMINARIES

The set $Z_n(I)$ of Shin-Zettl matrices on the interval I consists of $n \times n$ -matrices $A = \{a_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$(2.1) \quad \begin{cases} a_{rs} \in L_{\text{loc}}^1(I), & (1 \leq r, s \leq n, n \geq 2), \\ a_{r,r+1} \neq 0 & \text{a.e. on } I, \quad (1 \leq r \leq n-1), \\ a_{rs} = 0 & \text{a.e. on } I, \quad (2 \leq r+1 < s \leq n). \end{cases}$$

For $A \in Z_n(I)$, the quasi-derivatives associated with A are defined by

$$(2.2) \quad \begin{cases} y^{[0]} := y, \\ y^{[r]} := a_{r,r+1}^{-1} \left\{ (y^{[r-1]})' - \sum_{s=1}^r a_{rs} y^{[s-1]} \right\}, \quad (1 \leq r \leq n-1), \\ y^{[n]} := (y^{[n-1]})' - \sum_{s=1}^n a_{ns} y^{[s-1]} \end{cases}$$

where the prime $'$ denotes differentiation.

The quasi-differential expression M associated with A is given by

$$(2.3) \quad M[y] = i^n y^{[n]}, \quad (n \geq 2),$$

this being defined on the set

$$(2.4) \quad V(M) := \{y: y^{[r-1]} \in AC_{\text{loc}}(I), r = 1, 2, \dots, n\},$$

where $AC_{\text{loc}}(I)$ denotes the set of all functions which are absolutely continuous on every compact subinterval of I .

The formal adjoint M^+ of M is defined by the matrix $A^+ \in Z_n(I)$ given by

$$(2.5) \quad A^+ = -L^{-1}A^*L,$$

where A^* is the conjugate transpose of A and L is the non-singular $n \times n$ -matrix,

$$(2.6) \quad L := \left\{ (-1)^r \delta_{r, n+1-s} \right\}, \quad (1 \leq r, s \leq n),$$

δ being the Kronecker delta. If $A^+ = \{a_{rs}^+\}$, then it follows that

$$(2.7) \quad a_{rs}^+ = (-1)^{r+s+1} \bar{a}_{n-s+1, n-r+1} \quad \text{for each } r \text{ and } s.$$

The quasi-derivatives associated with A^+ are therefore

$$(2.8) \quad \begin{cases} y_+^{[0]} := y, \\ y_+^{[r]} := (\bar{a}_{n-r, n-r+1})^{-1} \left\{ (y_+^{[r-1]})' - \sum_{s=1}^r (-1)^{r+s+1} \bar{a}_{n-s+1, n-r+1} y_+^{[s-1]} \right\}, \\ y_+^{[n]} := (y_+^{[n-1]})' - \sum_{s=1}^n (-1)^{n+s+1} \bar{a}_{n-s+1, 1} y_+^{[s-1]} \end{cases} \quad (1 \leq r \leq n-1),$$

and

$$(2.9) \quad M^+[y] = i^n y_+^{[n]} \quad (n \geq 2)$$

for all y in

$$(2.10) \quad V(M^+) := \{y: y_+^{[r-1]} \in AC_{\text{loc}}(I), r = 1, 2, \dots, n\}.$$

Note that $(A^+)^+ = A$ and so $(M^+)^+ = M$. We refer to [3], [9] and [17] for a full account of the above and the subsequent results on quasi-differential expressions.

Let the interval I have end-points a and b , $-\infty \leq a < b \leq \infty$, and let w be a function which satisfies

$$(2.11) \quad w > 0 \quad \text{almost everywhere on } I, \quad w \in L^1_{\text{loc}}(I).$$

The equation

$$(2.12) \quad M[y] = \lambda wy \quad (\lambda \in \mathbb{C})$$

on I is said to be *regular* at the left end-point a if for all $X \in (a, b)$,

$$(2.13) \quad a \in \mathbb{R}, \quad a_{rs} \in L^1[a, X], \quad (r, s = 1, 2, \dots, n).$$

Otherwise (2.12) is said to be *singular* at a . Similarly we define the terms regular and singular at b . If (2, 12) is regular at both end-points, then it is said to be regular; in this case we have

$$(2.14) \quad a, b \in \mathbb{R}, \quad a_{rs} \in L^1(a, b) \quad (r, s = 1, 2, \dots, n);$$

see [2], [8] and [9].

We shall be concerned with the case when a is a regular end-point for (2.12), the end-point b being allowed to be either regular or singular. Note that in view of (2.7), an end-point of the interval I is regular for (2, 12) if and only if it is regular for the equation

$$(2.15) \quad M^+[y] = \bar{\lambda}wy \quad (\lambda \in \mathbb{C}).$$

Let $L^2_w(a, b)$ denote the usual weighted L^2 -space with the inner-product

$$(2.16) \quad (f, g) = \int_a^b f(x)\overline{g(x)}w(x) \, dx$$

and the norm $\|f\| := (f, f)^{1/2}$; this is a Hilbert space provided we identify functions which differ only on a null space.

Denote by $S(M)$ and $S(M^+)$ the sets of all solutions of

$$M[y] - \lambda wy = 0 \quad \text{and} \quad M^+[y] - \bar{\lambda}wy = 0,$$

respectively, and let $S^r(M) = \{y^{[r]} | M[y] - \lambda wy = 0, r = 0, 1, \dots, n-1\}$ denote the set of all quasi-derivatives of solutions of $M[y] - \lambda wy = 0$, etc. Let $\varphi_k(t, \lambda)$ for

$k = 1, 2, \dots, n$ be the solutions of the homogeneous equation (1.2) determined by the initial conditions

$$(2.17) \quad \varphi_k^{[r]}(t_0, \lambda) = \delta_{k,r+1} \quad \text{for all } t_0 \in [a, b),$$

($k = 1, 2, \dots, n; r = 0, 1, \dots, n - 1$). Then $\varphi_k^{[r]}(t_0, \lambda)$ is continuous in (t, λ) for $a < t < b, |\lambda| < \infty$, and for fixed t it is entire in λ . Let $\varphi_k^+(t, \lambda)$ for $k = 1, 2, \dots, n$ be the solutions of the homogeneous equation (1.3) determined by the initial conditions

$$(2.18) \quad (\varphi_k^+)^{[r]}(t_0) = (-1)^{k+r} \delta_{k,n-r}, \quad \text{for all } t_0 \in [a, b)$$

($k = 1, 2, \dots, n; r = 0, 1, \dots, n - 1$).

Suppose $a < t_0 < b$. According to Gilbert [4, Section 3] and Ibrahim [11, Section 3], a solution of $M[\varphi] - \lambda w\varphi = wf, f \in L_w^1(a, b)$ satisfying $\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1}, r = 0, 1, \dots, n - 1$, is given by

$$(2.19) \quad \varphi(t, \lambda) = \sum_{j=1}^n \alpha_j(\lambda) \varphi_j(t, \lambda_0) + ((\lambda - \lambda_0)/(i^n)) \sum_{j,k=1}^n \xi^{jk} \varphi_j(t, \lambda_0) \\ \times \int_{t_0}^t \overline{\varphi_j^+(s, \lambda_0)} f(s) w(s) ds$$

where $\varphi_k^+(t, \lambda)$ stands for the complex conjugate of $\varphi_k(t, \lambda)$, and for each j, k, ξ^{jk} is a constant which is independent of t, λ (but does depend in general on t), also see [9, Corollary 3.10] and [17, Theorem 3].

Theorem 2.1 (Existence and Uniqueness Theorem). *Suppose $f \in L_w^1(a, b)$ and suppose that the conditions (2.1) are satisfied. Then given any complex numbers $\alpha_j \in \mathbb{C}, j = 0, \dots, n - 1$ and $t_0 \in (a, b)$ there exists a unique solution of $M[\varphi] - \lambda\varphi w = wf$ in (a, b) which satisfies*

$$\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1}, \quad r = 0, \dots, n - 1.$$

Proof. See [2, Theorem 3.10.1], [8, Theorem 16.2.2] and [9, Theorem 1.11]. \square

3. SOME TECHNICAL LEMMAS

Our first lemma is a form of the variation of parameters formula. However it is different from the form of this formula generally found in textbooks and literature. For the variation of parameters formula for general quasi-differential equations, see [11, Section 3] and [17, Theorem 3]. These contain Lemma 3.1 as a special case, see [16].

Lemma 3.1. *Suppose f is a locally $L_w^1(a, b)$ function and $\varphi(t, \lambda)$ is the solution of $M[y] - \lambda wy = wf$ satisfying*

$$\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1}(\lambda), \quad t_0 \in [a, b] \quad \text{for all } r = 0, 1, \dots, n-1.$$

Then

$$(3.1) \quad \begin{aligned} \varphi^{[r]}(t, \lambda) = & \sum_{j=1}^n \alpha_j(\lambda) \varphi_j^{[r]}(t, \lambda_0) + ((\lambda - \lambda_0)/(i^n)) \sum_{j,k=1}^n \xi^{jk} \varphi_j^{[r]}(t, \lambda_0) \\ & \times \int_a^t \overline{\varphi_j^+(s, \lambda_0)} f(s) w(s) ds \quad \text{for } r = 0, 1, \dots, n-1. \end{aligned}$$

Crucial in the study of boundedness of solutions of quasi-differential equations is the fundamental inequality of Gronwall, see [7]. Here, we will also need the following variant which may be found in [13].

Lemma 3.2. *Let $u(t), v(t)$ be two non-negative functions, locally integrable on $[a, b)$. Then the inequality*

$$u(t) \leq c_0 + \int_a^t v(s) u^p(s) ds, \quad c_0 > 0,$$

for $0 \leq p < 1$, implies that

$$(3.2) \quad u(t) \leq \left(c_0^{1-p} + (1-p) \int_a^t v(s) ds \right)^{1/(1-p)}.$$

In particular, if $v(t) \in L^1(a, b)$, then (3.2) implies that $u(t)$ is bounded.

The next lemma is a special case of an extension of the well-known Gronwall inequality due to Gollwitzer [6] (See also Willett [12] and Willet-Wong [13]).

Lemma 3.3. Let u, z, g, h be continuous non-negative functions on $[a, b)$ and suppose that

$$u(t) \leq z(t) + g(t) \left(\int_a^t u^2(s)h(s) \, ds \right)^{\frac{1}{2}} \quad \text{for } t \geq a.$$

Then

$$u(t) \leq z(t) + g(t) \left(\int_a^t 2z^2(s)h(s) \exp \left[\int_a^s 2g^2(x)h(x) \, dx \right] \, ds \right)^{\frac{1}{2}} \quad \text{for } t \geq a.$$

The next two lemmas have been proved in [9] for the general case.

Lemma 3.4 ([9, Proposition 3.23]). *If all solutions of $M[\varphi] = \lambda_0 w \varphi$ are bounded on $[a, b)$ for some $\lambda_0 \in \mathbb{C}$ and $\varphi_j^+ \in L_w^1(a, b)$ for $j = 1, 2, \dots, n$, then all solutions of $M[\varphi] = \lambda w \varphi$ are also bounded on $[a, b)$ for all $\lambda \in \mathbb{C}$.*

Lemma 3.5 ([9, Proposition 3.24]). *If all solutions of $M[\varphi] = \lambda_0 w \varphi$ and $M^+[\varphi] = \bar{\lambda}_0 w \varphi$ are in $L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$, then all solutions of $M[\varphi] = \lambda w \varphi$ and $M^+[\varphi] = \bar{\lambda} w \varphi$ are in $L_w^2(a, b)$ for all $\lambda \in \mathbb{C}$.*

4. MAIN RESULTS

Suppose there exist non-negative continuous functions $k(t)$ and $h_i(t)$, $i = 0, \dots, n - 1$ such that

$$(4.1) \quad |f(t, y^{[0]}, \dots, y^{[n-1]})| \leq k(t) + \sum_{i=0}^{n-1} h_i(t) |y^{[i]}|^\sigma$$

for $t \geq a$, $-\infty < y^{[i]} < \infty$, for each $i = 0, \dots, n - 1$; $0 \leq \sigma \leq 1$.

Theorem 4.1. *Suppose f satisfies (4.1) with $\sigma = 1$, $s^r(M) \cup S(M^+) \subset L^\infty(a, b)$ for some $r = 0, 1, \dots, n - 1$ and some $\lambda_0 \in \mathbb{C}$ and*

- (i) $k(t) \in L_w^1(a, b)$ for all $t \in [a, b)$,
- (ii) $h_i(t) \in L_w^1(a, b)$ for all $t \in [a, b)$, $i = 0, 1, \dots, n - 1$.

Then $\varphi^{[r]}(t, \lambda)$ is bounded on $[a, b)$ for any solution $\varphi(t, \lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$.

P r o o f. Note that (4.1) and Theorem 2.1 imply that all solutions exist on the entire interval $[a, b)$, see [2, Chapter 3], [10], [14] and [15].

Let $\{\varphi_1(t, \lambda_0), \dots, \varphi_n(t, \lambda_0)\}$ and $\{\varphi_1^+(t, \lambda_0), \dots, \varphi_n^+(t, \lambda_0)\}$ be two sets of linearly independent solutions of (1.2) and (1.3), respectively and let $\varphi(t, \lambda)$ be any solution of (1.1) on $[a, b)$. Then by Lemma 3.1 we have

$$(4.2) \quad \begin{aligned} \varphi^{[r]}(t, \lambda) &= \sum_{j=1}^n \alpha_j(\lambda) \varphi_j^{[r]}(t, \lambda_0) + ((\lambda - \lambda_0)/(i^n)) \sum_{j,k=1}^n \xi^{jk} \varphi_j^{[r]}(t, \lambda_0) \\ &\quad \times \int_a^t \overline{\varphi_j^+(s, \lambda_0)} f(s, \varphi^{[0]}, \dots, \varphi^{[n-1]}) w(s) \, ds \\ &\text{for } r = 0, \dots, n-1. \end{aligned}$$

Hence

$$(4.3) \quad \begin{aligned} |\varphi^{[r]}(t, \lambda)| &\leq \sum_{j=1}^n |\alpha_j(\lambda)| |\varphi_j^{[r]}(t, \lambda_0)| + |\lambda - \lambda_0| \sum_{j,k=1}^n |\xi^{jk}| |\varphi_j^{[r]}(t, \lambda_0)| \\ &\quad \times \int_a^t \overline{|\varphi_k^+(s, \lambda_0)|} \left(k(s) + \sum_{i=0}^{n-1} h_i(s) |\varphi^{[i]}(s, \lambda)| \right) w(s) \, ds, \\ &r = 0, 1, \dots, n-1. \end{aligned}$$

Since $k(t) \in L_w^1(a, b)$ and $\varphi_j^+(t, \lambda_0)$ is bounded on $[a, b)$ for some $\lambda_0 \in \mathbb{C}$, we have $\varphi_j^+ k \in L_w^1(a, b)$, $j = 1, 2, \dots, n$ for some $\lambda_0 \in \mathbb{C}$. Setting

$$(4.4) \quad c_j = |\lambda - \lambda_0| \sum_{j,k=1}^n |\xi^{jk}| \int_a^b \overline{|\varphi_k^+(s, \lambda_0)|} k(s) w(s) \, ds, \quad (j = 1, 2, \dots, n),$$

then

$$(4.5) \quad \begin{aligned} |\varphi^{[r]}(t, \lambda)| &\leq \sum_{j=1}^n (c_j + |\alpha_j(\lambda)|) |\varphi_j^{[r]}(t, \lambda_0)| \\ &\quad + |\lambda - \lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}| |\varphi_j^{[r]}(t, \lambda_0)| \\ &\quad \times \int_a^t \overline{|\varphi_k^+(s, \lambda_0)|} h_i(s) |\varphi^{[i]}(s, \lambda)| w(s) \, ds, \\ &r = 0, 1, \dots, n-1. \end{aligned}$$

By hypothesis, there exist positive constants K_0 and K_1 such that

$$|\varphi_j^{[r]}(t, \lambda_0)| \leq K_0 \quad \text{and} \quad |\varphi_j^+(t, \lambda_0)| \leq K_1 \quad \text{for all } t \in [a, b),$$

$j = 1, 2, \dots, n$ and some $r = 0, 1, \dots, n - 1$. Hence, by summing both sides of (4.5) from $r = 0, \dots, n - 1$ we get

$$(4.6) \quad \sum_{r=0}^{n-1} |\varphi^{[r]}(t, \lambda)| \leq (n-1)K_0 \sum_{j=1}^n (c_j + |\alpha_j(\lambda)|) + (n-1)K_0K_1|\lambda - \lambda_0| \\ \times \sum_{j,k=1}^n |\xi^{jk}| \int_a^t \left(\max_{0 \leq i \leq n-1} h_i(s) \right) \left(\sum_{i=0}^{n-1} |\varphi^{[i]}(s, \lambda)| \right) w(s) ds.$$

Applying Gronwall's inequality to (4.6) and using (ii), we deduce that $\sum_{r=0}^{n-1} |\varphi^{[r]}(t, \lambda)|$ is finite and hence the result. \square

Remark. From [16, Section 3] and [9, Lemma 3.3], φ and $\varphi^{[j]} \in L_w^2(a, b)$ implies that $\varphi^{[r]} \in L_w^2(a, b)$ for any solution φ of the equation (1.1) for all $r = 1, \dots, j - 1$, $1 \leq j \leq n - 1$.

Theorem 4.2. Suppose f satisfies (4.1) with $\sigma = 1$, $S^r(M) \cup S(M^+) \subset L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n - 1$ and that

- (i) $k(t) \in L_w^2(a, b)$,
- (ii) $h_i(t) \in L^\infty(a, b)$, $i = 0, 1, \dots, n - 1$, for all $t \in [a, b]$. Then $\varphi^{[r]}(t, \lambda) \in L_w^2(a, b)$ for any solution $\varphi(t, \lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$.

Proof. Applying the Cauchy Schwartz inequality to the integral in (4.5) we get

$$(4.7) \quad |\varphi^{[r]}(t, \lambda)| \leq \sum_{j=1}^n (c_j + |\alpha_j(\lambda)|) |\varphi_j^{[r]}(t, \lambda_0)| \\ + |\lambda - \lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}| |\varphi_j^{[r]}(t, \lambda_0)| \\ \times \left(\int_a^t \overline{|\varphi_j^+(s, \lambda_0)|^2} |h_i(s)| w ds \right)^{\frac{1}{2}} \left(\int_a^t |h_i(s)| |\varphi^{[i]}(s, \lambda)|^2 w ds \right)^{\frac{1}{2}}, \\ r = 0, 1, \dots, n - 1.$$

Since $\varphi_j^+(t, \lambda_0) \in L_w^2(a, b)$, $j = 1, 2, \dots, n$ for some $\lambda_0 \in \mathbb{C}$ and $h_i(t) \in L^\infty(a, b)$ by hypothesis, then $\varphi_j^+(t, \lambda_0) |h_i(t)|^{1/2} \in L_w^2(a, b)$, $j = 1, 2, \dots, n$, $i = 0, 1, \dots, n - 1$. Let

$$D_{ji} = \left(\int_a^t \overline{|\varphi_j^+(s, \lambda_0)|^2} |h_i(s)| w(s) ds \right)^{\frac{1}{2}}, \quad z(t) = \sum_{j=1}^n (c_j + |\alpha_j(\lambda)|) |\varphi_j^{[r]}(t, \lambda_0)|$$

and $G(t) = |\lambda - \lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} D_{ji} |\xi^{jk}| |\varphi_j^{[r]}(t, \lambda_0)|$.

From Lemma 3.3 we have

$$|\varphi^{[r]}(t, \lambda)| \leq Z(t) + G(t) \left(\int_a^t 2Z^2(s) |h_i(s)| \exp \left[\int_a^s 2G^2(x) |h_i(x)| w \, dx \right] w \, ds \right)^{\frac{1}{2}}.$$

Since $\int_a^b Z^2(s) |h_i(s)| w(s) \, ds$ and $\int_a^b G^2(x) |h_i(x)| w(x) \, dx$ are both finite, we conclude that $|\varphi^{[r]}(t, \lambda)|$ is bounded by a linear combination of $L_w^2(a, b)$ functions $Z(t)$ and $G(t)$. Therefore, by using Lemma 3.5, $\varphi^{[r]}(t, \lambda) \in L_w^2(a, b)$ for all $\lambda \in \mathbb{C}$. \square

Remark. If we use the Cauchy-Schwartz inequality for the integral in (4.5) as

$$\int_a^t \overline{|\varphi_j^+|} |h_i| |\varphi| w \, ds \leq \left(\int_a^t \overline{|\varphi_j^+|^2} |h_i|^2 w \, ds \right)^{\frac{1}{2}} \left(\int_a^t |\varphi^{[i]}|^2 w \, ds \right)^{\frac{1}{2}}, \quad i = 0, \dots, n-1,$$

we also get the result. We refer to [14] and [15] for more details.

Corollary 4.3. Suppose that $f(t, y^{[0]}, \dots, y^{[n-1]}) = \sum_{i=1}^{n-1} h_i(t) y^{[i]}$, $S^r(M) \cup S(M^+) \subset L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n-1$ and that $h_i(t) \in L_w^p(a, b)$ for some $p \geq 2$, $t \in [a, b]$; $i = 0, 1, \dots, n-1$. Then $\varphi^{[r]}(t, \lambda) \in L_w^1(a, b)$ for any solution $\varphi(t, \lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$ and all $r = 0, 1, \dots, n-1$.

Proof. The proof is similar to Theorem 4.2 and therefore omitted.

The special case $h_i(t) \equiv 0$, $i = 0, \dots, n-1$ and $k \in L_w^2(a, b)$ yields \square

Corollary 4.4. If all solutions of $M[\varphi] = \lambda_0 w \varphi$ and $M^+[\varphi] = \bar{\lambda}_0 w \varphi$ are in $L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and $k \in L_w^2(a, b)$, then all solutions of $M[\varphi] - \lambda w \varphi = w k$ are in $L_w^2(a, b)$ for all $\lambda \in \mathbb{C}$.

Next, we consider (4.1) with $0 \leq \sigma < 1$, and arrive at the following:

Theorem 4.5. Suppose f satisfies (4.1) with $0 \leq \sigma < 1$, $S^r(M) \cup S(M^+) \subset L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n-1$ and

- (i) $k(t) \in L_w^2(a, b)$ for all $t \in [a, b]$,
- (ii) $h_i(t) \in L_w^{2/(1-\sigma)}(a, b)$ $0 \leq \sigma < 1$, $i = 0, 1, \dots, n-1$.

Then $\varphi^{[r]}(t, \lambda) \in L_w^2(a, b)$ for any solution $\varphi(t, \lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$.

Proof. For $0 \leq \sigma < 1$, the proof is the same up to (4.5). In this case (4.5) becomes

$$(4.8) \quad |\varphi^{[r]}(t, \lambda)| \leq \sum_{j=1}^n (e_j + |\alpha_j(\lambda)|) |\varphi_j^{[r]}(t, \lambda_0)| + |\lambda - \lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}| \\ \times |\varphi_j^{[r]}(t, \lambda_0)| \int_a^t \overline{|\varphi_j^+(s, \lambda_0)|} |h_i(s)| |\varphi^{[i]}(s, \lambda)|^\sigma w(s) \, ds, \\ r = 0, 1, \dots, n-1.$$

Applying Cauchy-Schwartz inequality to the integral in (4.8), we find

$$(4.9) \quad \int_a^t \overline{|\varphi_j^+(s, \lambda_0)|} |h_i(s)| |\varphi(s, \lambda)|^\sigma w(s) \, ds \\ \leq \left(\int_a^t \overline{|\varphi_j^+(s, \lambda_0)|} h_i(s) |^\mu w(s) \, ds \right)^{\frac{1}{\mu}} \left(\int_a^t |\varphi^{[i]}(s, \lambda)|^2 w(s) \, ds \right)^{\frac{\sigma}{2}},$$

where $\mu = 2/(2 - \sigma)$. Since $\varphi_j^+(s, \lambda_0) \in L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$, $j = 1, 2, \dots, n$ and $h_i(s) \in L_w^{2/(1-\sigma)}(a, b)$ by hypothesis, then we have $\varphi_j^+(s, \lambda_0) h_i(s) \in L_w^\mu(a, b)$ for some $\lambda_0 \in \mathbb{C}$, $j = 1, 2, \dots, n$; $i = 0, 1, \dots, n - 1$. Using this fact and (4.9) in (4.8), we obtain

$$(4.10) \quad |\varphi^{[r]}(t, \lambda)| \leq \sum_{j=1}^n (c_j + |\alpha_j(\lambda)|) |\varphi_j^{[r]}(t, \lambda_0)| \\ + K_0 |\lambda - \lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}| |\varphi_j^{[r]}(t, \lambda_0)| \left(\int_a^t |\varphi^{[i]}(s, \lambda)|^2 w(s) \, ds \right)^{\frac{\sigma}{2}}, \\ r = 0, 1, \dots, n - 1,$$

where $K_0 = \|\varphi_j^+(t, \lambda_0) h_i(s)\|_\mu$, $\|\cdot\|_\mu$ denotes the norm in $L_w^\mu(a, b)$. The inequality,

$$(4.11) \quad (u + v)^2 \leq 2(u^2 + v^2),$$

implies that

$$(4.12) \quad |\varphi^{[r]}(t, \lambda)|^2 \leq 4 \sum_{j=1}^n (c_j^2 + |\alpha_j(\lambda)|^2) |\varphi_j^{[r]}(t, \lambda_0)|^2 \\ + 4K_0^2 |\lambda - \lambda_0|^2 \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}|^2 |\varphi_j^{[r]}(t, \lambda_0)|^2 \left(\int_a^t |\varphi^{[i]}(s, \lambda)|^2 w(s) \, ds \right)^\sigma, \\ r = 0, 1, \dots, n - 1.$$

Setting $K_1 = \int_a^b |\varphi_j^{[r]}(t, \lambda_0)|^2 w(s) \, ds$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, \dots, n - 1$; $j = 1, 2, \dots, n$ and integrating (4.12) we obtain

$$(4.13) \quad \int_a^t |\varphi^{[r]}(s, \lambda)|^2 w(s) \, ds \leq K_2 + 4K_0^2 |\lambda - \lambda_0|^2 \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}|^2 \int_a^t |\varphi_j^{[r]}(s, \lambda_0)|^2 \\ \times \left[\left(\int_a^s |\varphi^{[i]}(x, \lambda)|^2 w(x) \, dx \right)^\sigma \right] w(s) \, ds,$$

where

$$K_2 = 4 \sum_{j=1}^n (c_j^2 + \alpha_j(\lambda)^2) K_1.$$

An application of Lemma 3.2 to (4.13) for $0 \leq \sigma < 1$ and of Gronwall's inequality to (4.13) for $\sigma = 1$ yields the result. \square

Theorem 4.6. *Suppose f satisfies (4.1) with $0 \leq \sigma < 1$, $S^r(M) \cup S(M^+) \subset L_w^2(a, b) \cap L^\infty(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n-1$ and that*

(i) $k(t) \in L_w^2(a, b)$ for all $t \in [a, b]$,

(ii) $h_i(t) \in L_w^p(a, b)$ for any p , $1 \leq p \leq 2/(1 - \sigma)$, $i = 0, 1, \dots, n-1$.

Then $\varphi^{[r]}(t, \lambda) \in L_w^2(a, b) \cap L^\infty(a, b)$ for any solution $\varphi(t, \lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$.

Proof. Since $S^r(M) \cup S(M^+) \subset L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n-1$, then $\varphi_j^{[r]}(t, \lambda_0)$, $\varphi_j^+(t, \lambda_0) \in L_w^q(a, b)$, $j = 1, 2, \dots, n$ for every $q \geq 2$ and for some $\lambda_0 \in \mathbb{C}$, $r = 0, 1, \dots, n-1$.

First, suppose that $h_i(t) \in L_w^p(a, b)$ for some p , $1 \leq p \leq 2$. Setting

$$K_0 = \|\varphi^{[r]}(t, \lambda_0)\|_\infty \quad \text{and} \quad K_1 = \|\varphi_j^+(t, \lambda_0)\|_\infty, \quad j = 1, 2, \dots, n,$$

for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n-1$, we have from (4.8)

$$(4.14) \quad |\varphi^{[r]}(t, \lambda)| \leq K_0 \sum_{j=1}^n (c_j + |\alpha_j(\lambda)|) \\ + K_0 K_1 |\lambda - \lambda_0| \left(\sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}| \int_a^t h_i(s) |\varphi^{[i]}(s, \lambda)|^\sigma w(s) ds \right).$$

Since $h_i(t) \in L_w^p(a, b)$, $1 \leq p \leq 2$, then Lemma 3.2 together with Gronwall's inequality implies that $\varphi^{[r]}(t, \lambda) \in L^\infty(a, b)$ for all $\lambda \in \mathbb{C}$, i.e., there exists a positive constant K_2 such that

$$(4.15) \quad |\varphi^{[r]}(t, \lambda)| \leq K_2 \text{ for all } \lambda \in \mathbb{C}, \quad t \in [a, b], \quad r = 0, 1, \dots, n-1.$$

From (4.8) and (4.15) we obtain

$$|\varphi^{[r]}(t, \lambda)| \leq \sum_{j=1}^n (c_j + |\alpha_j(\lambda)| + K_3) |\varphi_j^{[r]}(t, \lambda_0)|$$

for an appropriate constant K_3 . Since $\varphi_j^{[r]}(t, \lambda_0) \in L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0, 1, \dots, n-1$, this proves $\varphi^{[r]}(t, \lambda) \in L_w^p(a, b)$ for all $\lambda \in \mathbb{C}$, $1 \leq p \leq 2$.

Next, suppose that $h_i(t) \in L_w^p(a, b)$, $2 < p \leq 2/(1 - \sigma)$; $i = 0, 1, \dots, n - 1$. Define $q \geq 2$ by

$$\frac{1}{q} = \frac{2 - \sigma}{2} - \frac{1}{p}$$

(which is possible because of the restriction on p). Thus $\varphi_j^{[r]}(t, \lambda_0)\varphi_j^+(t, \lambda_0) \in L_w^q(a, b)$ and $\varphi_j^+(t, \lambda_0)h_i(t) = L_w^\mu(a, b)$, $\mu = 2/(2 - \sigma)$.

Repeating the same argument in the proof of Theorem 4.5, from (4.8) to (4.13), we obtain that $\varphi^{[r]}(t, \lambda) \in L_w^2(a, b)$. Returning to (4.9), we find that the integral on the left-hand side is bounded, which implies, by (4.8), that

$$|\varphi^{[r]}(t, \lambda)| \leq \sum_{j=1}^n (c_j + |\alpha_j(\lambda)| + K_3)|\varphi_j^{[r]}(t, \lambda_0)|,$$

for an appropriate constant K_3 . Since $\varphi_j^{[r]}(t, \lambda_0) \in L^\infty(a, b)$, this completes the proof. We refer to [10], [14] and [16] for more details. \square

References

- [1] *J.S. Bradley*: Comparison theorems for the square integrability of solutions of $(r(t)y')' + q(t)y = f(t, y)$. Glasgow Math., Soc. 13 (1972), 75–79.
- [2] *D.E. Edmunds and W.D. Evans*: Spectral Theory and Differential Operators (Oxford University) Press. 1987.
- [3] *W.N. Everitt and D. Race*: Some remarks on linear ordinary quasi-differential expressions. Proc. London Math. Soc. (3) 54 (1987), 300–320.
- [4] *R.C. Gilbert*: Simplicity of linear ordinary differential operators. Journal of Differential Equations 11 (1972), 672–681.
- [5] *S. Goldberg*: Unbounded Linear Operators. McGraw. Hill, New York, 1966.
- [6] *H.E. Gollwitzer*: A note on functional inequality. Proc. Amer. Math. Soc. 23 (1969), 642–647.
- [7] *M.R. Mohana Rao*: Ordinary Differential Equations. (Theory and Applications); First Published in the United Kingdom in 1989 by Edward Arnold (Publishers) Limited, London.
- [8] *M.N. Naimark*: Linear Differential Operators. (G.I.T.T.L., Moscow, 1954), Ungar, New York, Vol. I (1967), Vol. II (1968).
- [9] *Sobhy El-sayed Ibrahim*: Problems associated with differential operators. Ph.D. thesis (1989), Faculty of Sciences, Department of Mathematics, Benha University, Egypt.
- [10] *Sobhy El-sayed Ibrahim*: Boundedness for solutions of general ordinary quasi-differential equations. Journal of the Egyptian Mathematical Society 2 (1994), 33–44.
- [11] *Sobhy El-sayed Ibrahim*: The spectra of well-posed operators. Proc. Royal Soc. of Edinburgh 124A (1995), 1331–1348.
- [12] *D. Willett*: Nonlinear vector integral equations as contraction mappings. Arch. Rational Mech. Anal. 15 (1964), 79–86.
- [13] *D. Willett and J.S.W. Wong*: On the discrete analogues of some generalizations of Gronwall's inequality. Monatsh. Math. 69 (1965), 362–367 MR 32 \neq 2644.
- [14] *J.S.W. Wong*: Square integrable solutions of perturbed linear differential equations. Proc. Royal Society of Edinburgh 73A, 16 (1974/75), 251–254.

- [15] *A. Zettl*: Square integrable solutions of $Ly = f(t, y)$. Proceedings of the American Mathematical Society *26* (1970), 635–639.
- [16] *A. Zettl*: Perturbation of the limit circle case. Quart. J. Math., Oxford (3) *26* (1975), 355–360.
- [17] *A. Zettl*: Formally self-adjoint quasi-differential operators. Rocky Mountain Journal of Mathematics (3) *5* (1975), 453–474.

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