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M-IDEALS OF COMPACT OPERATORS INTO $\ell_p$

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Abstract. We show for $2 \leq p < \infty$ and subspaces $X$ of quotients of $L_p$ with a 1-unconditional finite-dimensional Schauder decomposition that $K(X, \ell_p)$ is an $M$-ideal in $L(X, \ell_p)$.

1. Introduction

A closed subspace $J$ of a Banach space $X$ is called an $M$-ideal if the dual space $X^*$ decomposes into an $\ell_1$-direct sum $X^* = J^\perp \oplus_1 V$, where $J^\perp = \{x^* \in X^*: x^*|_J = 0\}$ is the annihilator of $J$ and $V$ is some closed subspace of $X^*$. This notion is due to Alfsen and Effros [1], and it is studied in detail in [4].

It has long been known that the space of compact operators $K(\ell_p)$ is an $M$-ideal in the space of bounded operators $L(\ell_p)$ for $1 < p < \infty$ whereas this property fails for $L_p = L_p[0, 1]$ unless $p = 2$; cf. Section VI.4 in [4]. More recently, it was shown in [6] that $K(L_p, \ell_p)$ is an $M$-ideal if $1 < p \leq 2$, and it is not an $M$-ideal if $p > 2$.

In this paper we wish to examine the $M$-ideal character of $K(X, \ell_p)$ for subspaces $X$ of quotients of $L_p$ and $2 \leq p < \infty$. Our idea is to exploit the fact that those $X$ have Rademacher cotype $p$ with constant 1. This leads to the result mentioned in the abstract.

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2. Results

Here is our main result.

**Theorem 2.1.** Let $1 < p < \infty$ and suppose that the Banach space $X$ admits a sequence of operators $K_n \in K(X)$ satisfying

(a) $K_n x \to x$ for all $x \in X$,
(b) $K_n^* x^* \to x^*$ for all $x^* \in X^*$,
(c) $\|Id_X - 2K_n\| \to 1$.

Then $K(X, \ell_p)$ is an $M$-ideal in $L(X, \ell_p)$ if

\[ \limsup_n (\|x\|^p + \|x_n\|^p)^{1/p} \leq \limsup_n \left( \frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} \]

for all $x, x_n \in X$ such that $x_n \to 0$ weakly.

**Proof.** Let $T: X \to \ell_p$ be a contraction. We shall show that $T$ has property $(M)$, i.e.,

\[ \limsup_n \|y + Tx_n\| \leq \limsup_n \|x + x_n\| \]

whenever $x \in X$, $y \in \ell_p$, $\|y\| \leq \|x\|$, and $x_n \to 0$ weakly in $X$. This implies our claim by [6, Th. 6.3].

In fact, we have

\[ \limsup_n \|y + Tx_n\| = \limsup_n (\|y\|^p + \|Tx_n\|^p)^{1/p} \leq \limsup_n (\|x\|^p + \|x_n\|^p)^{1/p} \leq \limsup_n \left( \frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} \]

so it is enough to show that

\[ \limsup_n \|x + x_n\| = \limsup_n \|x - x_n\|. \]

Let $\varepsilon > 0$. Pick $m \in \mathbb{N}$ so that

\[ \|K_m x - x\| \leq \varepsilon, \quad \|Id - 2K_m\| \leq 1 + \varepsilon. \]

Then pick $n_0 \in \mathbb{N}$ so that

\[ \|K_m x_n\| \leq \varepsilon \quad \forall n_0; \]

\[ \|K_m x_n - x\| \leq \varepsilon, \quad \|Id - 2K_m\| \leq 1 + \varepsilon. \]
this is possible since \( x_n \to 0 \) weakly and \( K_m \) is compact. We now have for \( n \geq n_0 \)

\[
(1 + \varepsilon)\|x_n + x\| \geq \| (Id - 2K_m) (x_n + x) \|
= \| x_n - x - 2K_m x_n + 2x - 2K_m x \|
\geq \| x_n - x \| - 2\varepsilon - 2\varepsilon
\]

so that

\[
\limsup_n \|x_n + x\| \geq \limsup_n \|x_n - x\|
\]

and by symmetry equality holds. \( \square \)

We note that (2.1) is not a necessary condition, for essentially trivial reasons: e.g., if \( p < 2 \) and \( X = \ell_2 \), then every operator from \( X \) to \( \ell_p \) is compact and, therefore, \( K(X, \ell_p) \) is an \( M \)-ideal, but (2.1) fails.

As the proof shows, one can as well consider all the Banach spaces sharing the property

\[
\limsup_n \|y + y_n\| \leq \limsup_n (\|y\|^p + \|y_n\|^p)^{1/p}
\]

whenever \( y_n \to 0 \) weakly, e.g., \( \ell_q \) or the Lorentz spaces \( d(w, q) \) for \( p \leq q < \infty \).

So our theorem is closely related to [10, Th. 3] and [11, Prop. 4.2]. Actually, we needed assumptions (a)–(c) only to ensure (2.2), a condition that could be called property \((wM)\) in accordance with Lima’s property \((wM^*)\) [7].

Now we wish to give more concrete examples where Theorem 2.1 applies. There is a natural class of Banach spaces in which inequality (2.1) is valid. Recall that a Banach space \( X \) has Rademacher type \( p \) with constant \( C \) if for all finite families \( \{x_1, \ldots, x_n\} \subset X \), with \( r_1, r_2, \ldots \) denoting the Rademacher functions,

\[
\left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \right)^{1/p} \leq C \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p};
\]

it has Rademacher cotype \( p \) with constant \( C \) if

\[
\left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p} \leq C \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \right)^{1/p}
\]

instead. Thus we see that the inequality (2.1) is always satisfied when \( X \) has Rademacher cotype \( p \) with constant 1, which is the case if \( X \) is a subspace of a quotient of \( L_p \) for \( 2 \leq p < \infty \). As for assumptions (a)–(c) from Theorem 2.1, these conditions are obviously fulfilled if \( X \) has a shrinking 1-unconditional finite-dimensional Schauder decomposition or merely the shrinking unconditional metric
compact approximation property of [2] and [3]. Let us mention that the “shrinking” character of these properties holds, by a well-known convex combinations argument (cf. [4, Lemma VI.4.9]), for reflexive spaces automatically. These observations yield the next corollary.

**Corollary 2.2.** Let $X$ be a subspace of a quotient of $L_p$, $2 \leq p < \infty$, and let $X$ have a 1-unconditional finite-dimensional Schauder decomposition or merely the unconditional metric compact approximation property. Then $K(X, \ell_p)$ is an $M$-ideal in $L(X, \ell_p)$.

More explicitly, we note that for instance $\ell_p$, $\ell_p \oplus_p \ell_r$ and $\ell_p(\ell_r)$, where $2 \leq r \leq \infty$, satisfy these assumptions; but for these spaces the result of Corollary 2.2 has already been known from [11] or [4, p. 327]. Yet there are other examples. In fact, Li [8] has exhibited spaces of $\Lambda$-spectral functions $L_{\Lambda}^p(\mathbb{T})$ for certain $\Lambda \subset \mathbb{Z}$ that enjoy the unconditional metric compact approximation property. Moreover, since for $2 \leq q \leq p < \infty$ the space $L_q$ is isometric to a quotient of $L_p$, one can substitute $q$ for $p$ in the above list of examples.

Another way to see that (2.1) holds for $L_p$, $2 \leq p < \infty$, is to observe that (2.1) follows immediately from Clarkson’s inequality in $L_p$, that is

\[
\|f\|^p + \|g\|^p \leq \frac{\|f + g\|^p + \|f - g\|^p}{2}
\]

for $p \geq 2$. Now, Clarkson’s inequalities are valid in the Schatten classes as well [9]. Therefore we obtain a noncommutative version of the previous corollary. (Actually, this argument is not that different, because the Clarkson inequality entails the desired cotype property.)

**Corollary 2.3.** Let $X$ be a subspace of a quotient of the Schatten class $c_p$, $2 \leq p < \infty$, and let $X$ have a 1-unconditional finite-dimensional Schauder decomposition or merely the unconditional metric compact approximation property. Then $K(X, \ell_p)$ is an $M$-ideal in $L(X, \ell_p)$.

There is a dual version of Theorem 2.1 which we state for completeness.

**Theorem 2.4.** Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Suppose that the Banach space $Y$ admits a sequence of operators $K_n \in K(Y)$ satisfying

(a) $K_n y \to y$ for all $y \in Y$,

(b) $K_n^* y^* \to y^*$ for all $y^* \in Y^*$,

(c) $\|Id_Y - 2K_n\| \to 1$. 

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Then $K(\ell_p, Y)$ is an $M$-ideal in $L(\ell_p, Y)$ if

$$
\limsup_n (\|y^*\|^{p'} + \|y^*_n\|^{p'})^{1/p'} \leq \limsup_n \left( \frac{\|y^* + y^*_n\|^{p'} + \|y^* - y^*_n\|^{p'}}{2} \right)^{1/p'}
$$

for all $y^*, y^*_n \in Y^*$ such that $y^*_n \to 0$ weak$^*$. 

The proof of Theorem 2.4 can be accomplished along the same lines as above using property $(M^*)$ of a contraction (cf. [6, p. 171] instead.

Again, inequality (2.3) is always satisfied when $Y^*$ has Rademacher cotype $p'$ with constant 1, which is the case if $Y$ has Rademacher type $p$ with constant 1. The latter holds if $Y$ is a subspace of a quotient of $L_p$ or $c_p$ for $1 < p \leq 2$.

3. Concluding remarks

The conditions (2.1) and (2.3) can be understood as averaging conditions. In an earlier draft of this manuscript we used these conditions to establish what we call $p$-averaged versions of the properties $(M)$ and $(M^*)$ of contractions $T$, that is

$$
\limsup_n \|y + Tx_n\| \leq \begin{cases} 
\limsup_n \left( \frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} & \text{for } p < \infty \\
\limsup_n \max(\|x + x_n\|, \|x - x_n\|) & \text{for } p = \infty 
\end{cases}
$$

whenever $x \in X$, $y \in Y$ with $\|y\| \leq \|x\|$ and $x_n \to 0$ weakly in $X$; respectively,

$$
\limsup_n \|x^* + T^* y^*_n\| \leq \begin{cases} 
\limsup_n \left( \frac{\|y^* + y^*_n\|^p + \|y^* - y^*_n\|^p}{2} \right)^{1/p} & \text{for } p < \infty \\
\limsup_n \max(\|y^* + y^*_n\|, \|y^* - y^*_n\|) & \text{for } p = \infty 
\end{cases}
$$

for all $x^* \in X^*$, $y^*_n \in Y^*$ such that $\|x^*\| \leq \|y^*\|$ and for all weak$^*$ null sequences $(y^*_n) \subset Y^*$. (As a matter of fact, (2.3) implies the $p'$-averaged property $(M^*)$ for a contraction $T$: $\ell_p \to Y$.) Using techniques from [6] (which in turn depend on those from [5]) one can prove the following results.

**Proposition 3.1.** Let $1 \leq p < \infty$ and suppose that the Banach space $X$ admits a sequence of operators $K_n \in K(X)$ satisfying

(a) $K_n x \to x$ for all $x \in X$,

(b) $K^*_n x^* \to x^*$ for all $x^* \in X^*$,

(c) $\|Id_X - 2K_n\| \to 1$.

Let $Y$ be a Banach space. Then $K(X, Y)$ is an $M$-ideal in $L(X, Y)$ if and only if every contraction $T$: $X \to Y$ has $p$-averaged $(M)$. 

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Proposition 3.2. Let $1 \leq p \leq \infty$ and suppose that the Banach space $Y$ admits a sequence of operators $K_n \in K(Y)$ satisfying

(a) $K_n y \to y$ for all $y \in Y$,
(b) $K_n^* y^* \to y^*$ for all $y^* \in Y^*$,
(c) $\|I y - 2K_n\| \to 1$.

Let $X$ be a Banach space. Then $K(X, Y)$ is an $M$-ideal in $L(X, Y)$ if and only if every contraction $T : X \to Y$ has $p$-averaged $(M^*)$.

It is well known (cf. [4, Th. I.2.2]) that a closed subspace $J$ of a Banach space $X$ is an $M$-ideal in $X$ if and only if the following 3-ball property holds: For all $y_1, y_2, y_3 \in B_J$, all $x \in B_X$ and all $\varepsilon > 0$ there is $y \in J$ such that $\|x + y_i - y\| \leq 1 + \varepsilon$ for $i = 1, 2, 3$. (Here $B_X$ denotes the closed unit ball of $X$.) Upon replacing the number 3 by some $n \in \mathbb{N}$ we obtain the $n$-ball property, which is equivalent to the 3-ball property provided $n \geq 3$. One may "average" this condition as well and obtain the following characterisation of $M$-ideals by means of an averaged 3-ball property.

Proposition 3.3. A closed subspace $J$ of a Banach space $X$ is an $M$-ideal in $X$ if and only if

\[(A)\quad \text{For all } y_1, y_2, y_3 \in B_J, x \in B_X \text{ and } \varepsilon > 0 \text{ there is } y \in J \text{ such that } \|x + y_i - y\| + \|x - y_i - y\| \leq 2(1 + \varepsilon) \text{ for } i = 1, 2, 3\]

holds.

Proof. Evidently the 6-ball property implies (A). Conversely, suppose (A). In order to show that $J$ is an $M$-ideal in $X$ we will verify the ordinary 3-ball property (see above). Now an inspection of the proof of [4, Theorem I.2.2] shows that one may additionally assume that $\text{dist}(x, J) \geq 1 - \varepsilon$, in which case (A) implies that

\[\|x + y_i - y\| \leq 2(1 + \varepsilon) - \|x - y_i - y\| \leq 1 + 3\varepsilon, \quad i = 1, 2, 3,\]

and we are done. \qed

References


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