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Horst Alzer

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ON OZEKI'S INEQUALITY FOR POWER SUMS

Horst Alzer, Waldbröl

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Abstract. Let $p \in (0,1)$ be a real number and let $n \geqslant 2$ be an even integer. We determine the largest value $c_n(p)$ such that the inequality

$$\sum_{i=1}^{n} |a_i|^p \geqslant c_n(p)$$

holds for all real numbers $a_1, \ldots, a_n$ which are pairwise distinct and satisfy $\min_{i \neq j} |a_i - a_j| = 1$. Our theorem completes results of Ozeki, Mitrinović-Kalajdžić, and Russell, who found the optimal value $c_n(p)$ in the case $p > 0$ and n odd, and in the case $p \geqslant 1$ and n even.

MSC 2000: 26D15

In 1968, N. Ozeki [2] published without proof the following inequality for power sums.

Let $p > 0$, and let $a_1, \ldots, a_n$ be different real numbers which satisfy the condition $\min_{i \neq j} |a_i - a_j| = 1$. Then

$$\sum_{i=1}^{n} |a_i|^p \geqslant \alpha_n(p), \tag{1}$$

where

$$\alpha_n(p) = \begin{cases} 2 \sum\limits_{i=1}^{(n-1)/2} i^p, & \text{if } n \text{ is odd,} \\ 2 \sum\limits_{i=1}^{n/2} (i - \frac{1}{2})^p, & \text{if } n \text{ is even.} \end{cases}$$

In 1980, D.S. Mitrinović and G. Kalajdžić [1] proved Ozeki's inequality for all positive real numbers p. However, their proof contains an error as was pointed out by D.C.

Russell [3] in 1984. He remarked that inequality (1) holds for $p \geqslant 1$, but it is in general not valid if $p \in (0,1)$. Indeed, if we choose, for instance, $n = 2$, $p \in (0,1)$, $a_1 = 0$, $a_2 = 1$, then inequality (1) is false.

In the same paper Russell established a new version of Ozeki's inequality which is valid for all $p > 0$.

Let $p > 0$ be a real number and let $e_p = \min\{1, 2^{1-p}\}$. If $a_1, \dots, a_n$ are different real numbers with $\min\limits_{i \neq j} |a_i - a_j| = 1$, then

$$\sum_{i=1}^{n} |a_i|^p \geqslant \beta_n(p), \tag{2}$$

where

$$\beta_n(p) = \begin{cases} 2 \sum\limits_{i=1}^{(n-1)/2} i^p, & \text{if } n \text{ is odd,} \\ e_p \sum\limits_{i=1}^{n/2} (2i-1)^p, & \text{if } n \text{ is even.} \end{cases}$$

Since the sign of equality holds in (2) for $n = 2m+1$, $p > 0$, $a_i = i - m - 1$ $(i = 1, \dots, 2m+1)$, and for $n = 2m$, $p \geqslant 1$, $a_i = i - m - \frac{1}{2}$ $(i = 1, \dots, 2m)$, we conclude that the value $\beta_n(p)$ provides the best possible lower bound for the sum $\sum\limits_{i=1}^{n} |a_i|^p$, if n is odd and $p > 0$, and if n is even and $p \geqslant 1$.

Thus, it remains to determine the largest lower bound for $\sum\limits_{i=1}^{n} |a_i|^p$ in the case that n is even and $p \in (0,1)$. It is the aim of this note to solve this problem. The following theorem reveals that Russell's bound $\sum\limits_{i=1}^{n/2} (2i-1)^p$ $(e_p = 1)$ can be replaced by a larger term.

Theorem. *Let $p > 0$ be a real number and let $n \geqslant 2$ be an integer. If $a_1, \dots, a_n$ are different real numbers which satisfy $\min\limits_{i \neq j} |a_i - a_j| = 1$, then*

$$\sum_{i=1}^{n} |a_i|^p \geqslant c_n(p),$$

where the best possible lower bound is given by

$$c_n(p) = \begin{cases} 2 \sum\limits_{i=1}^{(n-1)/2} i^p, & \text{if } n \text{ is odd,} \\ 2 \sum\limits_{i=1}^{(n/2)-1} i^p + (\tfrac{1}{2}n)^p, & \text{if } n \text{ is even and } \quad 0 < p < 1, \\ 2 \sum\limits_{i=1}^{n/2} (i - \tfrac{1}{2})^p, & \text{if } n \text{ is even and } \quad p \geqslant 1. \end{cases}$$

Proof. It remains to consider the case that n is even and $p \in (0,1)$. We set $n = 2m$ and define

$$S = \big\{a = (a_1, \ldots, a_{2m}) \in \mathbb{R}^{2m} \mid a_1 < \ldots < a_{2m},\ \min_{1 \leqslant i \leqslant 2m-1} (a_{i+1} - a_i) = 1\big\}.$$

Then we have to show that the inequality

$$f(a) := \sum_{i=1}^{2m} |a_i|^p \geqslant 2 \sum_{i=1}^{m-1} i^p + m^p \tag{3}$$

holds for all $a \in S$.

Let $a = (a_1, \ldots, a_{2m}) \in S$; we may assume that at most m of the values $a_1, \ldots, a_{2m}$ are negative. Hence, there exists an integer $k \in \{1, \ldots, m+1\}$ such that

$$a_1 < \ldots < a_{k-1} < 0 \leqslant a_k < \ldots < a_{2m}.$$

We consider two cases.

Case 1. $a_k \leqslant 1$.

Since $a_{i+1} - a_i \geqslant 1$ $(i = 1, \ldots, 2m-1)$, we get

$$-a_i \geqslant k - i - a_k \geqslant 0 \quad (i = 1, \ldots, k-1)$$

and

$$a_i \geqslant i - k + a_k \geqslant 0 \quad (i = k, \ldots, 2m).$$

This leads to

$$\begin{aligned} f(a) &= \sum_{i=1}^{k-1} (-a_i)^p + \sum_{i=k}^{2m} a_i^p \\ &\geqslant \sum_{i=1}^{k-1} (k - i - a_k)^p + \sum_{i=k}^{2m} (i - k + a_k)^p \\ &= \sum_{i=1}^{m} (i - a_k)^p + \sum_{i=0}^{m-1} (i + a_k)^p \\ &\qquad + \sum_{i=k}^{m} \big((i + m - k + a_k)^p - (i - a_k)^p\big). \end{aligned}$$

Since $0 \leqslant a_k \leqslant 1$ and $1 \leqslant k \leqslant i \leqslant m$ imply $i + m - k + a_k \geqslant i - a_k \geqslant 0$, we get

$$f(a) \geqslant \sum_{i=1}^{m} (i - a_k)^p + \sum_{i=0}^{m-1} (i + a_k)^p.$$

A simple calculation yields that the function

$$g(x) = \sum_{i=1}^{m} (i - x)^p + \sum_{i=0}^{m-1} (i + x)^p$$

is increasing on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$. Since $g(0) = g(1) = 2 \sum_{i=1}^{m-1} i^p + m^p$, we obtain

$$f(a) \geqslant g(a_k) \geqslant 2 \sum_{i=1}^{m-1} i^p + m^p.$$

Case 2. $a_k > 1$.

Let

$$a' = (a_1, \dots, a_{k-1}, 1, 2, a_{k+2}, \dots, a_{2m}).$$

Since $1 - a_{k-1} > 1$ and $a_{k+2} - 2 \geqslant a_{k+1} - 1 \geqslant a_k > 1$, we conclude that $a' \in S$. From

$$f(a) - f(a') = a_k^p + a_{k+1}^p - 1 - 2^p \geqslant a_k^p + (a_k + 1)^p - 1 - 2^p > 0$$

and the result we have proved in Case 1 we get

$$f(a) > f(a') \geqslant 2 \sum_{i=1}^{m-1} i^p + m^p.$$

This completes the proof of inequality (3). □

Finally, we note that the sign of equality holds in (3) if we set $a_i = i - m$ $(i = 1, \dots, 2m)$. Therefore, the given lower bound is the best possible.

References

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Author's address: Morsbacher Str. 10, 51545 Waldbröl, Germany.