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CHARACTERIZATION OF LATTICES OF CONVEX
SUBSETS OF POSETS

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Dedicated to Professor Ján Jakubík on the occasion of his seventy-fifth birthday

Systems of convex subsets of partially ordered sets, particularly those of convex sublattices of lattices, have been considered by many authors (see e.g. [1]–[6]). In this note we give necessary and sufficient conditions for a lattice to be isomorphic to the lattice of all convex subsets of a nonempty partially ordered set (Theorem 1.6). Such a lattice will be called a c -lattice. Further, we describe directly irreducible c -lattices and prove that each c -lattice is a direct product of directly irreducible c -lattices (Theorem 2.3).

Let $\mathbb{A} = (A, \leq)$ be a partially ordered set. A subset X of A is called convex if $x_1 \leq a \leq x_2$, $x_1, x_2 \in X$, $a \in A$ imply $a \in X$. Let $\text{Conv } \mathbb{A}$ denote the system of all convex subsets of \mathbb{A} . The system $\text{Conv } \mathbb{A}$, ordered by set-inclusion, is a complete lattice. Moreover, it is atomic in the sense that each element of $\text{Conv } \mathbb{A}$ different from the empty set is the join of some atoms. If $X \subseteq A$, the symbol $[X]$ will be used for the least convex subset of \mathbb{A} containing X . The set of all minimal and maximal elements of \mathbb{A} is denoted by $\text{Min } \mathbb{A}$ and $\text{Max } \mathbb{A}$, respectively.

1. CHARACTERIZATION OF $\text{Conv } \mathbb{A}$

In this section we give necessary and sufficient conditions for a lattice to be isomorphic to $\text{Conv } \mathbb{A}$ for a nonempty partially ordered set.

We start with some definitions.

Let $\mathbb{L} = (L, \leq)$ be a complete atomic lattice. An element $p \in L$ will be called *totally irreducible* if $p \leq \sup M$, $M \subseteq L$ imply $p \leq m$ for some $m \in M$.

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A complete lattice \mathbb{L} will be said to be a *z-lattice* if each $a \in L$ is a join of totally irreducible elements of L .

By a *complete sublattice* of a complete lattice, a sublattice closed under arbitrary joins and meets will be meant.

Let $\mathbb{C} = (C, \leq)$ be a complete lattice and let 0 and 1 denote the least and greatest element, respectively. Suppose that \mathbb{C} has a complete sublattice Z which is a *z-lattice* and contains 0 and 1. Denote by P the set of all totally irreducible elements of Z different from 0. Since $1 \in Z$, it is obvious that for any $c \in C$ the set $\{z \in Z: z \geq c\}$ has a least element. We denote it by $\downarrow c$.

Consider the following conditions:

- (i) if $c \in C$, $\{p_i: i \in I\} \subseteq P$, then $c \wedge \sup\{p_i: i \in I\} = \sup\{c \wedge p_i: i \in I\}$;
- (ii) if $p \in P$, $\{c_j: j \in J\} \subseteq C$ and $p \wedge c_j = 0$ for each $j \in J$, then $p \wedge \sup\{c_j: j \in J\} = 0$;
- (iii) if $c, c' \in C$, $\downarrow c \leq \downarrow c'$ and the relations $p \in P$, $p \leq \downarrow c$, $p \wedge c' = 0$ imply $p \wedge c = 0$, then $c \leq c'$;
- (iv) if $z_1, z_2 \in Z$, $z_1 \geq z_2$, $p \in P$, $p \leq z_1$ and c_0 is the greatest element of the set $\{c \in C: c \leq z_1, c \wedge z_2 = 0\}$, then $p \wedge c_0 = 0$ implies $p \leq z_2$ and $p \wedge c_0 > 0$ implies $p \leq \downarrow c_0$.

These conditions are not satisfied, in general. Let, e.g., \mathbb{C} be as in Fig. 1, $Z = \{0, p, q, 1\}$. Then $P = \{p, q\}$ and (i) does not hold, while (ii) holds. If \mathbb{C} is as in Fig. 2, $Z = \{0, p, q, 1\}$, then neither (i) nor (ii) is satisfied. On the other hand, if \mathbb{C} is any infinitely distributive complete lattice and Z is any of its complete sublattices which is a *z-lattice*, both (i) and (ii) are satisfied. So, e.g., a three-element chain with $Z = \{0, 1\}$ satisfies (i), (ii), (iv), while (iii) does not hold. Let \mathbb{C} be as in Fig. 3 with $Z = \{0, 1, p, q, r\}$. Then (i), (ii), (iii) hold but (iv) is not satisfied.

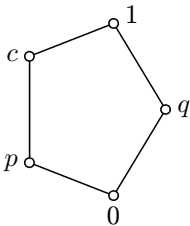


Fig. 1

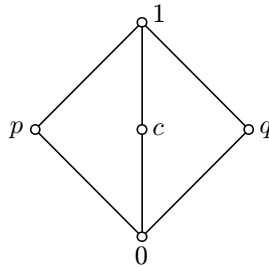


Fig. 2

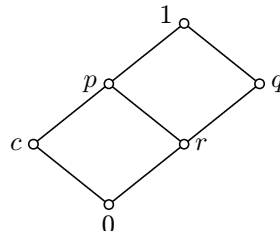


Fig. 3

Lemma 1.1. *Let \mathbb{C} , Z , P be as above and let the conditions (i), (ii) be satisfied. Then for any $z_1, z_2 \in Z$, $z_1 \geq z_2$, the set $\{c \in C: c \leq z_1, c \wedge z_2 = 0\}$ has a greatest element.*

Proof. Evidently $0 \in \{c \in C : c \leq z_1, c \wedge z_2 = 0\}$. Take $c_0 = \sup\{c \in C : c \leq z_1, c \wedge z_2 = 0\}$. Evidently $c_0 \leq z_1$. If $z_2 = 0$, then $c_0 \wedge z_2 = 0$ holds trivially. Let $z_2 > 0$. Then $z_2 = \sup\{p_i : i \in I\}$ for a nonempty subset $\{p_i : i \in I\}$ of P . The relation $c \wedge z_2 = 0$ implies $c \wedge p_i = 0$ for each $i \in I$. Thus $p_i \wedge c_0 = 0$ for each $i \in I$ by (ii). Using (i) we obtain $c_0 \wedge z_2 = \sup\{c_0 \wedge p_i : i \in I\} = 0$. \square

Under the assumptions as in 1.1 let us denote the greatest element of the set $\{c \in C : c \leq z_1, c \wedge z_2 = 0\}$ by $z_1 - z_2$.

Lemma 1.2. *Let the assumptions of 1.1 be satisfied and let, moreover, (iii) hold. Then each $c \in C$ can be expressed as $c = z_1 - z_2$ for some $z_1, z_2 \in Z$, $z_1 \geq z_2$.*

Proof. Let $c \in C$. Denote $z_1 = \downarrow c$, $z_2 = \sup\{p \in P : p \leq z_1, p \wedge c = 0\}$ (by $\sup \emptyset$ the element 0 is meant). Evidently $z_2 \leq z_1, c \leq z_1, c \wedge z_2 = 0$, so that $z_1 - z_2 \geq c$. Now we are going to show, using (iii), that $z_1 - z_2 \leq c$ holds, too. The inequalities $c \leq z_1 - z_2 \leq z_1$ imply $\downarrow c \leq \downarrow(z_1 - z_2) \leq \downarrow z_1 = z_1 = \downarrow c$, so that $\downarrow(z_1 - z_2) = z_1 = \downarrow c$. Let $p \in P, p \leq \downarrow(z_1 - z_2) = z_1$ and $p \wedge c = 0$. Then $p \leq z_2$ and consequently $p \wedge (z_1 - z_2) = 0$, since $z_2 \wedge (z_1 - z_2) = 0$. The condition (iii) yields $z_1 - z_2 \leq c$. \square

Notice that the elements z_1, z_2 in 1.2 are not determined uniquely. E.g., $1 - 1 = 0 - 0 = 0$.

Lemma 1.3. *Let the assumptions of 1.2 be satisfied and let, moreover, (iv) hold. Then the lattice $\mathbb{C} = (C, \leq)$ is isomorphic to $(\text{Conv}(P, \leq), \subseteq)$ (the partial order in P being inherited from that in C).*

Proof. Let us define a mapping φ from C into the system of subsets of P by $c \in C, c = z_1 - z_2, z_1, z_2 \in Z, z_1 \geq z_2 \implies \varphi(c) = \{p \in P : p \leq z_1, p \not\leq z_2\}$. First we will show that this definition is correct. Let $c \in C, c = z_1 - z_2 = z'_1 - z'_2$ for some $z_1, z_2, z'_1, z'_2 \in Z, z_1 \geq z_2, z'_1 \geq z'_2$. Let $p \in P, p \leq z_1, p \not\leq z_2$. Using (iv) we obtain $p \wedge c > 0, p \leq \downarrow c$. Obviously $\downarrow c \leq z'_1$, hence $p \leq z'_1$. If $p \leq z'_2$ held, we would have $p \wedge c = 0$, since $z'_2 \wedge c = 0$, a contradiction. We have proved $\{p \in P : p \leq z_1, p \not\leq z_2\} \subseteq \{p \in P : p \leq z'_1, p \not\leq z'_2\}$. The converse inclusion can be proved analogously.

Notice that if $c = z_1 - z_2$ for some $z_1, z_2 \in Z, z_1 \geq z_2$, then $\sup\{p \in P : p \leq z_1, p \wedge c > 0\} = \downarrow c$. Namely, we have $z_1 = \sup\{p \in P : p \leq z_1, p \wedge c = 0\} \vee \sup\{p \in P : p \leq z_1, p \wedge c > 0\}$, which implies $c = c \wedge z_1 = c \wedge \sup\{p \in P : p \leq z_1, p \wedge c > 0\} \leq \sup\{p \in P : p \leq z_1, p \wedge c > 0\}$ by (i). Now using (iv) we obtain $\sup\{p \in P : p \leq z_1, p \wedge c > 0\} \leq \downarrow c$ and consequently $\sup\{p \in P : p \leq z_1, p \wedge c > 0\} = \downarrow c$.

It is easy to see that $\varphi(c)$ is a convex subset of P . We are going to show that φ is onto. Let Q be any convex subset of P . Set $X = \{x \in P : x \leq q \text{ for some } q \in$

$Q\}, Y = X - Q$. Further, let $z_1 = \sup X$, $z_2 = \sup Y$. Obviously $z_1, z_2 \in Z$, $z_1 \geq z_2$. We are going to show that $\varphi(z_1 - z_2) = Q$. First, let $p \leq z_1$, $p \notin Q$. The relation $p \leq z_1$ yields $p \in X$, since p is totally irreducible, so that $p \in Y$. But then $p \leq z_2$. Thus $\{p \in P: p \leq z_1, p \not\leq z_2\} \subseteq Q$. Now let $p \in Q$. Then $p \in X$, which implies $p \leq z_1$. Assume that $p \leq z_2$. Then $p \leq y$ for some $y \in Y$. But as $Y \subseteq X$, there exists $q \in Q$ with $y \leq q$. We have $p \leq y \leq q, p, q \in Q$, which implies $y \in Q$, a contradiction.

It remains to prove that if $c, c' \in C$, then

$$c \leq c' \text{ if and only if } \varphi(c) \subseteq \varphi(c').$$

Let $c, c' \in C$. Take $z_1 = \downarrow c$, $z'_1 = \downarrow c'$, $z_2 = \sup\{p \in P: p \leq z_1, p \wedge c = 0\}$, $z'_2 = \sup\{p \in P: p \leq z'_1, p \wedge c' = 0\}$. We know that $c = z_1 - z_2$, $c' = z'_1 - z'_2$. Now suppose that $c \leq c'$. Then evidently $z_1 \leq z'_1$. Take any $p \in P$ with $p \leq z_1$, $p \not\leq z_2$. We have $p \leq z'_1$, $p \wedge c > 0$ and consequently $p \wedge c' > 0$, which implies $p \not\leq z'_2$. We have proved $\varphi(c) \subseteq \varphi(c')$. Conversely, let $\varphi(c) \subseteq \varphi(c')$. First we will show that $z_1 \leq z'_1$. As we have noticed, we have $\sup\{p \in P: p \leq z_1, p \wedge c > 0\} = \downarrow c = z_1$, $\sup\{p \in P: p \leq z'_1, p \wedge c' > 0\} = \downarrow c' = z'_1$. Since $\{p \in P: p \leq z_1, p \wedge c > 0\} = \{p \in P: p \leq z_1, p \not\leq z_2\} \subseteq \{p \in P: p \leq z'_1, p \not\leq z'_2\} = \{p \in P: p \leq z'_1, p \wedge c' > 0\}$, we have $z_1 \leq z'_1$. Further, $\varphi(c) \subseteq \varphi(c')$ implies also that if $p \leq z_1$, $p \wedge c' = 0$, then $p \wedge c = 0$. Using (iii) we infer $c \leq c'$. The proof is complete. \square

Now we are going to prove the converse.

Let $\mathbb{A} = (A, \leq)$ be any partially ordered set. Let us recall that $\mathcal{C} = (\text{Conv } \mathbb{A}, \subseteq)$ is a complete lattice, \emptyset is its least, A the greatest element. If $\{C_i: i \in I\} \subseteq \text{Conv } \mathbb{A}$, then $\bigwedge\{C_i: i \in I\} = \bigcap\{C_i: i \in I\}$, $\bigvee\{C_i: i \in I\} = \bigcup\{C_i: i \in I\}$. Consider the system \mathcal{Z} of all $Z \subseteq A$ which are down-closed, i. e. fulfil the condition

$$x \leq y, y \in Z \implies x \in Z.$$

It is easy to see that $\mathcal{Z} \subseteq \text{Conv } \mathbb{A}$ and that (\mathcal{Z}, \subseteq) is a complete sublattice of \mathcal{C} containing \emptyset and A . By the way, if $\{Z_i: i \in I\} \subseteq \mathcal{Z}$, then $\bigvee\{Z_i: i \in I\} = \bigcup\{Z_i: i \in I\}$. It is also easy to verify that nonempty totally irreducible elements of (\mathcal{Z}, \subseteq) are just the sets $(a) = \{x \in A: x \leq a\}$ for all possible $a \in A$ and that each $Z \in \mathcal{Z}$ is the join of all (z) , $z \in Z$. So we have proved

Lemma 1.4. *The complete sublattice (\mathcal{Z}, \subseteq) of $(\text{Conv } \mathbb{A}, \subseteq)$ is a z -lattice.*

Denote by \mathcal{P} the system of all (a) , $a \in A$.

Now it is clear that

- (1) if $C \in \text{Conv } \mathbb{A}$, $\{a_i: i \in I\} \subseteq A$, then $C \cap (\bigcup\{(a_i): i \in I\}) = \bigvee\{C \cap (a_i): i \in I\}$; and

- (2) if $a \in A$, $\{C_j : j \in J\} \subseteq \text{Conv } \mathbb{A}$, $(a) \cap C_j = \emptyset$ for each $j \in J$, then $(a) \cap (\vee \{C_j : j \in J\}) = \emptyset$.

If $C \in \text{Conv } \mathbb{A}$, then evidently $\downarrow C = \{x \in A : \text{there exists } c \in C \text{ with } x \leq c\}$. If $Z_1, Z_2 \in \mathcal{Z}$, $Z_1 \supseteq Z_2$, then the greatest element of the system $\{C \in \text{Conv } \mathbb{A} : C \subseteq Z_1, C \cap Z_2 = \emptyset\}$ is $Z_1 - Z_2$ (in the set theoretical meaning). The following can be proved easily:

- (3) if $C, C' \in \text{Conv } \mathbb{A}$, $\downarrow C \subseteq \downarrow C'$ and $a \in A$, $(a) \subseteq \downarrow C$, $(a) \cap C' = \emptyset$ imply $(a) \cap C = \emptyset$, then $C \subseteq C'$; and
(4) if $Z_1, Z_2 \in \mathcal{Z}$, $Z_1 \supseteq Z_2$, $a \in Z_1$, then $(a) \cap (Z_1 - Z_2) = \emptyset$ implies $(a) \subseteq Z_2$ and $(a) \cap (Z_1 - Z_2) \neq \emptyset$ implies $(a) \subseteq \downarrow(Z_1 - Z_2)$.

The above results can be summarized as follows:

Lemma 1.5. *If $\mathbb{A} = (A, \leq)$ is a partially ordered set, $\mathcal{C} = (\text{Conv } \mathbb{A}, \subseteq)$, \mathcal{Z} and \mathcal{P} are as above, then the conditions (i)–(iv) are satisfied.*

Combining 1.3 and 1.5 we obtain the following theorem.

Theorem 1.6. *Let $\mathbb{C} = (C, \leq)$ be a complete lattice, $\text{card } C \geq 2$. Then \mathbb{C} is isomorphic to $(\text{Conv } \mathbb{A}, \subseteq)$ for a partially ordered set \mathbb{A} if and only if \mathbb{C} has a complete sublattice Z containing the least and the greatest elements of C , which is a z -lattice, with the conditions (i)–(iv) being satisfied.*

2. DIRECT DECOMPOSITION

If a lattice $\mathbb{L} = (L, \wedge, \vee, \leq)$ is isomorphic to $\text{Conv } \mathbb{A}$ for a nonempty partially ordered set \mathbb{A} , we will refer to it as a *c-lattice*.

Theorem 2.1. *The direct product of any nonempty system of c-lattices is a c-lattice.*

Proof. Let $\{\mathbb{A}_i : i \in I\}$ be any nonempty system of partially ordered sets. Let \mathbb{A} be their cardinal sum. It is easy to see that the mapping $X \in \text{Conv } \mathbb{A} \mapsto (X \cap A_i)_{i \in I}$ is an isomorphism of the lattice $\text{Conv } \mathbb{A}$ onto the direct product of the lattices $\text{Conv } \mathbb{A}_i$ ($i \in I$). □

Let $\mathbb{A} = (A, \preceq)$ be any partially ordered set. Denoting by S the set of all couples $(u, v) \in A \times A$ such that $u \in \text{Min } \mathbb{A}$, $v \in \text{Max } \mathbb{A}$ and v covers u , define

$$a \preceq_c b (a, b \in A) \Leftrightarrow a \preceq b, \quad (a, b) \notin S.$$

It is easy to see that \preceq_c is a partial order in A and $\text{Conv}(A, \preceq) = \text{Conv}(A, \preceq_c)$. The order \preceq_c will be said to be the c -order corresponding to \preceq .

Theorem 2.2. *Let $\mathbb{A} = (A, \preceq)$ be any partially ordered set. The lattice $\text{Conv } \mathbb{A}$ is directly irreducible if and only if the partially ordered set (A, \preceq_c) is connected.*

Proof. If (A, \preceq_c) is disconnected, then there exist nonempty subsets B, C of A such that (A, \preceq_c) is the cardinal sum of (B, \preceq_c) and (C, \preceq_c) . But then $\text{Conv } \mathbb{A} = \text{Conv}(A, \preceq_c)$ is isomorphic to $\text{Conv}(B, \preceq_c) \times \text{Conv}(C, \preceq_c)$, so that $\text{Conv } \mathbb{A}$ is directly reducible.

Conversely, let $\text{Conv } \mathbb{A}$ be directly reducible, i.e. there exist lattices $\mathbb{L}_1, \mathbb{L}_2$, each containing at least two elements, and an isomorphism $\varphi: \text{Conv } \mathbb{A} \rightarrow \mathbb{L}_1 \times \mathbb{L}_2$. Evidently $\mathbb{L}_1, \mathbb{L}_2$ are complete atomic lattices. φ maps atoms of the lattice $\text{Conv } \mathbb{A}$ into atoms of the direct product $\mathbb{L}_1 \times \mathbb{L}_2$. Set $A_1 = \{a \in A: \varphi(\{a\}) = (p, 0) \text{ for an atom } p \text{ of } \mathbb{L}_1\}$, $A_2 = \{a \in A: \varphi(\{a\}) = (0, q) \text{ for an atom } q \text{ of } \mathbb{L}_2\}$. Evidently $A_1, A_2 \neq \emptyset$, $A_1 \cup A_2 = A$. Let \mathbb{A}_1 and \mathbb{A}_2 be A_1 and A_2 , respectively, with the order inherited from (A, \preceq_c) . The aim is to show that (A, \preceq_c) is the cardinal sum of \mathbb{A}_1 and \mathbb{A}_2 , which will imply that (A, \preceq_c) is disconnected. We have to prove that if $a_1 \in A_1$, $a_2 \in A_2$, then a_1, a_2 are incomparable in (A, \preceq_c) . Let $a_1 \in A_1$, $a_2 \in A_2$, $\varphi(\{a_1\}) = (p_1, 0)$, $\varphi(\{a_2\}) = (0, q_1)$. As $\varphi(\{\{a_1, a_2\}\}) = \varphi(\{a_1\} \vee \{a_2\}) = \varphi(\{a_1\}) \vee \varphi(\{a_2\}) = (p_1, 0) \vee (0, q_1) = (p_1, q_1)$ and $(p_1, 0), (0, q_1)$ are the only atoms in $\mathbb{L}_1 \times \mathbb{L}_2$ which are less than (p_1, q_1) , the elements a_1, a_2 are incomparable or one of them covers the other in (A, \preceq_c) . Assume, e.g., that a_2 covers a_1 . By the definition of \preceq_c there exists $a \in A$ such that either $a \preceq_c a_1$, $a \neq a_1$, or $a_2 \preceq_c a$, $a \neq a_2$. Let, e.g., the first possibility occur. Then $\{a_1\} \subset \{\{a, a_2\}\}$, which implies $(p_1, 0) = \varphi(\{a_1\}) < \varphi(\{\{a, a_2\}\}) = \varphi(\{a\}) \vee \varphi(\{a_2\})$. Now $\varphi(\{a\})$ is of the form $(p, 0)$ or $(0, q)$, so that $(p_1, 0) < (p, q_1)$ or $(p_1, 0) < (0, q_1 \vee q_2)$, respectively. The first inequality implies $p_1 = p$, which contradicts $a \neq a_1$. The latter case is also impossible. So a_1, a_2 are incomparable and the proof is complete. \square

Theorem 2.3. *Every c -lattice is the direct product of directly irreducible c -lattices.*

Proof. Let $\mathbb{A} = (A, \preceq)$ be any partially ordered set, \preceq_c the c -order corresponding to \preceq . Let $\mathbb{A}_i = (A_i, \preceq_c)$ ($i \in I$) be maximal connected subsets of (A, \preceq_c) . Then the lattice $\text{Conv } \mathbb{A} = \text{Conv}(A, \preceq_c)$ is isomorphic to the direct product of $\text{Conv } \mathbb{A}_i$ and all $\text{Conv } \mathbb{A}_i$ are directly irreducible c -lattices by 2.2. \square

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References

- [1] *C. C. Chen, M. K. Koh*: On the lattice of convex sublattices of a finite lattice. *Nanta Math.* 5 (1972), 93–95.
- [2] *V. I. Igošin*: Lattices of intervals and lattices of convex sublattices of lattices. *Mežvuzovskij naučnyj sbornik 6 – Uporjadočennyje množestva i rešetki, Saratov (1980)*, 69–76. (In Russian.)
- [3] *M. Kolibiar*: Intervals, convex sublattices and subdirect representations of lattices. *Universal Algebra and Applications, Banach Center Publications 9, Warsaw (1982)*, 335–339.
- [4] *J. Lihová*: On convexly isomorphic posets. *Czechoslovak Math. J.* 49 (1999), 135–148.
- [5] *V. I. Marmazejev*: The lattice of convex sublattices of a lattice. *Mežvuzovskij naučnyj sbornik 6 – Uporjadočennyje množestva i rešetki, Saratov (1986)*, 50–58. (In Russian.)
- [6] *M. Zelina*: Selfduality of the system of convex subsets of a partially ordered set. *CMUC AA 34, 3 (1990)*, 593–595.

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