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# CONSTRUCTION OF *po*-GROUPS WITH QUASI-DIVISORS THEORY

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*Abstract.* A method is presented making it possible to construct *po*-groups with a strong theory of quasi-divisors of finite character and with some prescribed properties as subgroups of restricted Hahn groups $H(\Delta, \mathbb{Z})$, where $\Delta$ are finitely atomic root systems. Some examples of these constructions are presented.

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*Keywords*: quasi-divisor theory, divisor class group

## 1. Introduction

In the algebraic number theory the notion of a *theory of divisors* was introduced by Borevic and Shafarevic [4] as a map $h$ from the group of divisibility $G$ of an integral domain $A$ into a free abelian group $\mathbb{Z}^{(P)}$ (considered as an $l$-group with pointwise ordering) satisfying some conditions. It is wellknown (see [4]) that a divisibility group of a domain $A$ admits a theory of divisors if and only if $A$ is a Krull domain.

L. Skula [21] introduced a notion of a theory of divisors for a partly ordered group (*po*-group) (or, equivalently, for a semigroup with a cancellation law) as a very natural generalization of a theory of divisors for rings, and he derived an extensive theory of these *po*-groups.

A step towards a further generalization was done by K. E. Aubert in [2], where for the first time the notion of a quasi-divisors theory was introduced. Recall that a directed *po*-group $(G, .)$ has a *theory of quasi-divisors* if there exists an $l$-group $(\Gamma, .)$ and a map $h\colon G \longrightarrow \Gamma$ such that

(i) $h$ is an order isomorphism from $G$ into $\Gamma$;
(ii) $(\forall \alpha \in \Gamma_+)(\exists g_1, \ldots, g_n \in G_+)\alpha = h(g_1) \wedge \ldots \wedge h(g_n)$.

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The principal tool for the investigation of these properties in *po*-groups seems to be the notion of an $r$-ideal. We recall here that by an $r$-system of ideals in a directed *po*-group $G$ we mean a map $X \mapsto X_r$ ($X_r$ is called an $r$-ideal) from the set of all lower bounded subsets $X$ of $G$ into the power set of $G$ which satisfies the following conditions:

(1) $X \subseteq X_r$;
(2) $X \subseteq Y_r \Longrightarrow X_r \subseteq Y_r$;
(3) $\{a\}_r = a \cdot G^+ = (a)$ for all $a \in G$;
(4) $a \cdot X_r = (a \cdot X)_r$ for all $a \in G$.

One of the first characterizations of *po*-groups with a theory of quasi-divisors was done by P. Jaffard [11]. He proved that a directed *po*-group $G$ has a theory of quasi-divisors if and only if the semigroup $(\mathcal{I}_t^{(f)}, \times)$ of finitely generated $t$-ideals is a group, i.e. if and only if $G$ is a $t$-Prüfer group. (For a comprehensive description see e.g. [2].)

In [14] we introduced a stronger version of *po*-groups with a theory of quasi-divisors. Recall that a theory of quasi-divisors $h\colon\ G \longrightarrow \Gamma$ is called a *strong theory of quasi-divisors*, if

$$(\forall \alpha, \beta \in \Gamma_+)(\exists \gamma \in \Gamma_+)\alpha \cdot \gamma \in h(G), \beta \wedge \gamma = 1.$$

It may be proved that any strong theory of quasi-divisors is a theory of quasi-divisors as well.

Moreover, in a classical divisor theory of *po*-groups an important role is played by a divisor class group. This notion was introduced by L. Skula [21] as a natural generalization of a class group known from the theory of Krull domains. This notion can be defined naturally for any $o$-isomorphism $h\colon\ G \longrightarrow \Gamma$ of a *po*-group $G$ into another *po*-group $\Gamma$. Such a definition was introduced in [15] and let us recall that a *divisor class group* $\mathcal{C}_h$ of $h$ is then the abstract group $\Gamma/h(G)$. The canonical map $\Gamma \longrightarrow \mathcal{C}_h$ is then denoted by $\varphi_h$.

It was again L. Skula who showed that $\mathcal{C}_h$ and $\varphi_h$ have great importance when deciding whether or not $h$ is a theory od divisors. We proved (see [19], [16]) that the divisor class group is of the same importance also for *po*-groups with a theory of quasi-divisors as it is for groups with the classical divisors theory. Namely, we proved that by using some properties of $\mathcal{C}_h$ it is possible to characterize *po*-groups $G$ with the strong theory of quasi-divisors of a finite character (see [16; Theorem 3.3], [19; Theorem 2.1]).

In this paper we present a general method which enables us to construct examples of *po*-groups with a strong theory of quasi-divisors of a finite character with some prescribed properties. Using this method we present several examples of *po*-groups with a quasi-divisors theory with some prescribed properties.

In [16]; Theorem 3.3, we proved the following theorem characterizing *po*-groups with a theory of quasi-divisors with a finitely atomic value group $\Gamma$. Recall that an $l$-group $\Gamma$ is *finitely atomic*, if for any element $\alpha \in \Gamma$, $\alpha > 1$, the set of all atoms $\sigma \in \Gamma_+$ such that $\sigma \leqslant \alpha$ is nonempty and finite. A trivial example of a finitely atomic $l$-group is the group $\mathbb{Z}^{(P)}$.

**Theorem 1** ([16]; 3.3). *Let $h$ be an o-isomorphism from a directed po-group $G$ into an l-group $\Gamma$, let $\mathcal{C}_h$ be a divisor class group of $h$ and let $\varphi\colon \Gamma \longrightarrow \mathcal{C}_h$ be a canonical map. Let us consider the following statements:*

(1) *$h$ is a strong theory of quasi-divisors.*
(2) *If $\alpha_1, \ldots, \alpha_n$ are elements of $\Gamma$ such that $\alpha_i > 1$ for all $i$, then $\varphi(\Gamma_+ \setminus \{\alpha_1, \ldots, \alpha_n\}_t) = \mathcal{C}_h$, where $\{\alpha_1, \ldots, \alpha_n\}_t = \{\alpha \in \Gamma\colon \exists_{1 \leqslant i \leqslant n} i, \alpha \geqslant \alpha_i\}$.*
(3) *If $\alpha_1, \ldots, \alpha_n$ are atoms in $\Gamma_+$, then $\varphi(\Gamma_+ \setminus \{\alpha_1, \ldots, \alpha_n\}_t) = \mathcal{C}_h$.*

*Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If $\Gamma$ is finitely atomic, then all the statements are equivalent.*

A method for constructing examples of *po*-groups with a strong theory of quasi-divisors of a finite character that we will present here is based on an application of Theorem 1 for a special $l$-group, the restricted Hahn group $H(\Delta, \mathbb{Z})$.

Recall that if $\Delta$ is a *root system* (i.e. $(\Delta, \leqslant)$ is a partly ordered set for which $\{\alpha \in \Delta\colon \alpha \geqslant \gamma\}$ is totally ordered for any $\gamma \in \Delta$), then the *restricted Hahn group* $H(\Delta, \mathbb{Z})$ on $\Delta$ is the group $\mathbb{Z}^{(\Delta)}$ ordered in the following way:

$$a \in H(\Delta, \mathbb{Z}), a \geqslant 0 \Leftrightarrow a_\alpha > 0 \text{ for all } \alpha \in \operatorname{ms}(a),$$

where $\operatorname{ms}(a)$ is the *maximal support of* $a$, i.e. the set of all maximal elements in $\operatorname{supp}(a) = \{\alpha \in \Delta\colon a_\alpha \neq 0\}$. Then $H(\Delta, \mathbb{Z})$ is an $l$-group (see e.g. [1]).

Now, let $\Delta_0$ be the set of all minimal elements of $\Delta$. We say that $\Delta$ is *atomic* if for any element $\alpha \in \Delta$ there exists $\beta \in \Delta_0$ such that $\alpha \geqslant \beta$. Moreover, we say that $\Delta$ is *finitely atomic* if for any $\alpha \in \Delta$, the set $\{\sigma \in \Delta_0\colon \sigma \leqslant \alpha\}$ is nonempty and finite. Finally, let $\alpha \in \Delta$. Then by $a^\alpha$ we denote the element of $H(\Delta, \mathbb{Z})$ such that

$$a^\alpha_\beta = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

In the following lemma we recall some properties of $H(\Delta, \mathbb{Z})$ which can be of interest for our examples of groups with a strong theory of quasi-divisors.

**Lemma 2** ([16]; 3.4). *Let $\Delta$ be a root system.*

(1) *Let $\alpha \in \Delta_0$, $b \in H(\Delta, \mathbb{Z})_+$. Then $b \geqslant a^\alpha$ if and only if there exists $\beta \in \mathrm{ms}(b)$ such that $\beta \geqslant \alpha$.*

(2) *If $\Delta$ is atomic, then $a \in H(\Delta, \mathbb{Z})$ is an atom if and only if $a = a^\alpha$ for some $\alpha \in \Delta_0$.*

(3) *If $\Delta$ is finitely atomic, then $H(\Delta, \mathbb{Z})$ is finitely atomic.*

Our examples of groups with a strong theory of quasi-divisors will be based on the application of Theorem 1 and Lemma 2 to a group $H(\Delta, \mathbb{Z})$. Hence, we will investigate homomorphisms $\varphi$ of $H(\Delta, \mathbb{Z})$ into an abelian group $\mathcal{C}$. We will be interested in homomorphisms $\overline{\varphi}$ which are determined by maps $\varphi\colon \Delta \longrightarrow \mathcal{C}$ in the following way.

We say that a homomorphism $\psi\colon H(\Delta, \mathbb{Z}) \longrightarrow \mathcal{C}$ is determined by a map $\varphi\colon \Delta \longrightarrow \mathcal{C}$ if for any $a \in H(\Delta, \mathbb{Z})$ we have

$$\psi(a) = \sum_{\alpha \in \Delta} \varphi(\alpha) a_\alpha.$$

In this case the following diagram commutes:

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi} & \mathcal{C} \\ {\scriptstyle \mathrm{nat}}\downarrow & & \| \\ H(\Delta, \mathbb{Z}) & \xrightarrow{\psi} & \mathcal{C} \end{array}$$

where $\mathrm{nat}(\alpha) = a^\alpha$. The homomorphism $\psi$ will be then denoted by $\overline{\varphi}$.

**Corollary (Examples generating method).** *Let $(\Delta, \leqslant)$ be a finitely atomic root system, $\mathcal{C}$ an abelian group and let $\varphi\colon \Delta \longrightarrow \mathcal{C}$ be a map such that for any finite set $\{\alpha_1, \ldots, \alpha_n\}$ of atoms in $\Delta$ the set $\varphi(\Delta \setminus \{\alpha_1, \ldots, \alpha_n\}_t)$ is a semigroup generator of $\mathcal{C}$, where $\{\alpha_1, \ldots, \alpha_n\}_t = \{\alpha \in \Delta\colon \exists i, \alpha \geqslant \alpha_i\}$. Then in the diagram*

$$\operatorname{Ker} \overline{\varphi} \xrightarrow{h} H(\Delta, \mathbb{Z}) \xrightarrow{\overline{\varphi}} \mathcal{C},$$

*the inclusion map $h$ is a strong theory of quasi-divisors of a finite character and $\mathcal{C}$ is a divisor class group of $h$.*

P r o o f. Since $\Delta$ is finitely atomic, according to Lemma 2, the $l$-group $H(\Delta, \mathbb{Z})$ is finitely atomic. Let $\{a_1, \ldots, a_n\}$ be a finite set of atoms in $H(\Delta, \mathbb{Z})$. Then according to Lemma 2, for any $i, 1 \leqslant i \leqslant n$, there exists an atom $\alpha_i \in \Delta$ such that $a_i = a^{\alpha_i}$. Let $\mathbf{a} \in \mathcal{C}$. Since $\varphi(\Delta \setminus \{\alpha_1, \ldots, \alpha_n\}_t)$ is a semigroup generator of $\mathcal{C}$, there exist $\beta_1, \ldots, \beta_m \in \Delta \setminus \{\alpha_1, \ldots, \alpha_n\}_t$ and natural numbers $k_1, \ldots, k_m$ such that $\mathbf{a} =$

$\sum_{j=1}^{m} \varphi(\beta_j)k_j$. If $a^{\beta_j}k_j \geqslant a_i = a^{\alpha_i}$ for some $i$, then we have $\beta_j \geqslant \alpha_i$ according to Lemma 2, a contradiction. Hence $b = \sum_{j=1}^{m} a^{\beta_j}k_j \in H(\Delta, \mathbb{Z})_+ \setminus \{a_1, \dots, a_n\}_t$ and

$$\bar{\varphi}(b) = \sum_{\alpha \in \Delta} \varphi(\alpha) \cdot b_\alpha = \sum_{j=1}^{m} \varphi(\beta_j)k_j = \mathbf{a}.$$

We will show that $\operatorname{Ker} \bar{\varphi} = G$ is a directed *po*-group. Let $a \in G$. We put

$$b(\alpha) = \begin{cases} a(\alpha) & \text{if } \alpha \in \operatorname{ms}(a), a(\alpha) > 0, \\ 0 & \text{if } \alpha \in \operatorname{ms}(a), a(\alpha) < 0, \\ |a(\alpha)| & \text{if } \alpha \in \operatorname{supp}(a) \setminus \operatorname{ms}(a), \\ 0 & \text{otherwise.} \end{cases}$$

Then $b \geqslant a, 0$ in $H(\Delta, \mathbb{Z})$, since $(b-a)(\alpha) > 0$ for all $\alpha \in \operatorname{ms}(b-a)$. Let $\mathbf{a} = \sum_{\alpha \in \operatorname{supp}(b)} b(\alpha) \cdot \varphi(\alpha)$ and let

$$\Phi^b = \{\alpha \in \Delta\colon \alpha \text{ is an atom in } \Delta, \alpha \leqslant \beta \text{ for some } \beta \in \operatorname{supp}(b)\}.$$

Then $\Phi^b$ is a finite set in $\Delta$ and $\varphi(\Delta \setminus \Phi^b_t)$ is a semigroup generator of $\mathcal{C}$. Hence, there exist $\beta_1, \dots, \beta_n \in \Delta \setminus \Phi^b_t$ and positive numbers $c_1, \dots, c_n \in \mathbb{Z}$ such that $-\mathbf{a} = \varphi(\beta_1)c_1 + \ldots + \varphi(\beta_n)c_n$. Then $\beta_i \notin \operatorname{supp}(b)$ for any $i$ and we put

$$c(\alpha) = \begin{cases} b(\alpha) & \text{if } \alpha \in \operatorname{supp}(b, ) \\ c_i & \text{if } \alpha = \beta_i, i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $c > 0$ in $H(\Delta, \mathbb{Z})$ and $c \geqslant b \geqslant a$. Moreover,

$$\sum_{\alpha \in \Delta} c(\alpha)\varphi(\alpha) = \sum_{\alpha \in \operatorname{supp}(b)} b(\alpha)\varphi(\alpha) + \sum_{i=1}^{n} c_i\varphi(\beta_i) = 0.$$

Hence, $c \in G$ and since $G$ is a convex subgroup, we obtain that $b \in G$ and $G$ is directed.

Now, according to [17]; 2.2, $\Gamma$ satisfies Conrad's (F)-condition and it follows that $h$ is of finite character. The result then follows from Theorem 1. □

**Corollary** (Skula, L. [21]). *Let $G$ be a po-group and let $h\colon G \longrightarrow \mathbb{Z}^{(P)}$ be an o-isomorphism into. Then the following conditions are equivalent.*

(1) *$h$ is a strong theory of divisors.*

(2) *For $p_1, \ldots, p_n \in P$ $(n \geqslant 1)$, the set $\varphi_h(P \setminus \{p_1, \ldots, p_n\})$ is a semigroup generator of a divisor class group $\mathbb{Z}^{(P)}/h(G)$.*

We investigate first some additional properties of the inclusion $h\colon G = \operatorname{Ker} \bar{\varphi} \longrightarrow \Gamma = H(\Delta, \mathbb{Z})$. In what follows we will always assume that $\Delta$ is a finitely atomic root system.

Let $\Delta_0$ be the set of all atoms in $\Delta$. Then $\Delta_0$ is the maximal disjoint set in $\Delta$ and it follows that $\{\operatorname{nat}(\alpha)\colon \alpha \in \Delta_0\}$ is a base of $\Gamma$. Moreover, the set of polars of $\operatorname{nat}(\alpha), \alpha \in \Delta_0$, then defines an $l$-realization of $\Gamma$, i.e. if we put

$$\Delta_\alpha^+ = (\operatorname{nat}(\alpha)')_+ = \{g \in \Gamma_+\colon g \wedge \operatorname{nat}(\alpha) = 0\}$$

for $\alpha \in \Delta_0$ then since $\Gamma$ satisfies the Conrad (F)-condition (see [17]; 2,2), the set

$$W = \{w_\alpha\colon \Gamma \xrightarrow{w_\alpha} \Gamma/\Delta_\alpha \text{ is a canonical } l\text{-homomorphism}, \alpha \in \Delta_0\}$$

is a defining family of a finite character of $\Gamma$ (see [5]; p. 3.29). Moreover, for any $\alpha \in \Delta_0$ we have

$$\begin{aligned} \Delta_\alpha^+ &= \{g \in \Gamma_+\colon \operatorname{supp}(g) \cap (\alpha)_t = \emptyset\} \\ &= \{g \in \Gamma_+\colon g(\beta) = 0 \text{ for all } \beta \in \Delta, \beta \geqslant \alpha\} \end{aligned}$$

according to [16]; 3.4.

In the following proposition we present some properties of this defining family $W$.

**Proposition 3.** *Let $W$ be a defining family of $\Gamma$ introduced above.*

(1) *$W$ is an independent defininig family if and only if for all $\alpha, \beta \in \Delta_0$, $\alpha \neq \beta$, $(\alpha)_t \cap (\beta)_t = \emptyset$ holds.*

(2) *If $\alpha \in \Delta_0$ is such that $\alpha < \beta$ for some $\beta \in \Delta$, then $\Gamma/\Delta_\alpha$ is not isomorphic to $\mathbb{Z}$.*

P r o o f. (1). Let $W$ be independent and let us assume that there are $\alpha \neq \beta$ in $W$ such that $\gamma \geqslant \alpha, \beta$ for some $\gamma$. Let $g = \operatorname{nat}(\gamma) \in \Gamma$. Then $g \notin \Delta_\alpha \cup \Delta_\beta$. If $g \in [\Delta_\alpha, \Delta_\beta]$, then there exist $a \in \Delta_\alpha$, $b \in \Delta_\beta$ such that $a + b \geqslant g$. But in this case $a(\gamma) = b(\gamma) = 0$ and it follows that $\gamma \in \operatorname{ms}((a+b) - g)$. Hence, $a + b \ngeqslant g$, a contradiction. Conversely, let $(\alpha)_t \cap (\beta)_t = \emptyset$ for all $\alpha \neq \beta \in \Delta_0$. Let $g \in \Gamma_+$ and $\alpha \neq \beta \in \Delta_0$. We put

$$a(\gamma) = \begin{cases} 0 & \gamma \geqslant \alpha, \\ g(\gamma) & \gamma \ngeqslant \alpha, \end{cases}$$

$$b(\gamma) = \begin{cases} 0 & \gamma \geqslant \beta, \\ g(\gamma) & \gamma \ngeqslant \beta. \end{cases}$$

Then $a \in \Delta_\alpha, b \in \Delta_\beta$ and it follows from $(\alpha)_t \cap (\beta)_t = \emptyset$ that $\gamma \in \ \mathrm{ms}((a+b)-g)$ iff $\gamma \in \ \mathrm{ms}(g)$. Hence, $a + b > g$ and $[\Delta_\alpha, \Delta_\beta] = G$.

(2) Let $a$, $b$, $c_n$; $n = 1, 2, \ldots$ be defined such that

$$a(\gamma) = 2 \text{ if } \gamma = \beta,$$
$$b(\gamma) = 1 \text{ if } \gamma = \beta,$$
$$c_n(\gamma) = 2 \text{ if } \gamma = \beta,$$
$$c_n(\gamma) = -n, \text{ if } \gamma = \alpha,$$
$$a(\gamma) = b(\gamma) = c_n(\gamma) = 0 \text{ otherwise.}$$

Then

$$a + \Delta_\alpha > c_1 + \Delta_\alpha > c_2 + \Delta_\alpha > \ldots > b + \Delta_\alpha > \Delta_\alpha.$$

Hence, $\Gamma/\Delta_\alpha$ cannot be order isomorphic to $\mathbb{Z}$. □

**Example 1.** For any infinite cardinal number **a** there exists a *po*-group $G$ with a strong theory of quasi-divisors of a finite character such that $G_+$ has at least **a** maximal prime $t$-ideals.

In fact, let $\Delta$ be a set with cardinality **a**. Let $\Delta$ be considered to be an antichain. Let $\Delta = \Delta_1 \cup \Delta_2$ be a partition such that any $\Delta_i$ is infinite and let us define a map $\varphi\colon\ \Delta \longrightarrow \mathbf{Z}$ such that

$$\varphi(\delta) = \begin{cases} 1 & \text{if } \delta \in \Delta_1, \\ -1 & \text{if } \delta \in \Delta_2. \end{cases}$$

Hence, according to the first Corollary, $G = \ker(\overline{\varphi}) \xrightarrow{h} \Gamma = H(\Delta, \mathbb{Z})$ is a *po*-group with a strong theory of quasi-divisors of a finite character. Moreover, since $\Delta$ is a set of all atoms in $(\Delta, \leqslant)$, then for any $\delta \in \Delta$, $\Delta_\delta = \{g \in \Gamma\colon\ g_\beta = 0, \forall \beta \geqslant \delta\}$ is a minimal prime $l$-ideal in $\Gamma$. Hence, acording to [14]; Prop. 2.4, $P_\delta = \Gamma_+ \setminus \Delta_\delta$ is a maximal prime $t$-ideal in $\Gamma$. Since an embedding $h$ is a $(t,t)$-morphism, it follows that $Q_\delta = h^{-1}(P_\delta)$ is a prime $t$-ideal in $G$ as well. Moreover, $Q_\delta$ is maximal. In fact, if $Q$ is a prime $t$-ideal in $G$ such that $Q_\delta \subset Q$ then since $G$ is a $t$-Prüfer *po*-group (see [17];

Theorem 2.1, for example), a canonical map $w_Q\colon\ G \longrightarrow G/H$ is a $t$-valuation, where $H$ is the quotient group of a semigroup $G_+ \setminus Q$. Let $\hat{w}_Q$ be an extension of $w_Q$ onto a $t$-valuation of $\Gamma$. The existence of this extension follows from the universal property of $\Gamma$ (see [2], e.g.). Then $\Gamma_+ \setminus \ker(\hat{w}_Q)$ is a prime $t$-ideal strictly containing $P_\delta$, a contradiction. Hence, $\{Q_\delta\colon\ \delta \in \Delta\}$ is a set of maximal prime $t$-ideals of $G$.

Our next two examples will concern $G$-dense $l$-ideals of $\Gamma$. Recall that an $l$-ideal $\Delta$ of $\Gamma$ is called *$G$-dense* (with respect to an $o$-isomorphism $h$ from $G$ into $\Gamma$) if for any $\alpha \in \Delta$ there exists $g \in G$ such that $\alpha \leqslant h(g) \in \Delta$. In [18]; 3.3, it was proved that there exists a bijection between the set of $o$-ideals of a ($t$-closed) $po$-group $G$ and the set of $G$-dense $l$-ideals of its Lorenzen $l$-group $\Lambda_t(G)$. In the first example we show that even in the case that an inclusion $G \longrightarrow \Lambda_t(G)$ is a strong theory of quasi-divisors of a finite character, in $\Lambda_t(G)$ there exist (in general) $l$-ideals which are not $G$-dense.

**Example 2.** There exists a $po$-group $G$ with a strong theory of quasi-divisors of a finite character $h\colon\ G \longrightarrow \Gamma$ such that in $\Gamma$ there exists an $l$-ideal which is not $G$-dense.

In fact, let $A$ be an infinite finitely atomic root system as in Fig. 1. We define a map $\varphi\colon\ A \longrightarrow \mathbb{Z}$ such that

$$\varphi(\alpha_i) = (-1)^i; \quad \varphi(\alpha_{ij}) = 0.$$

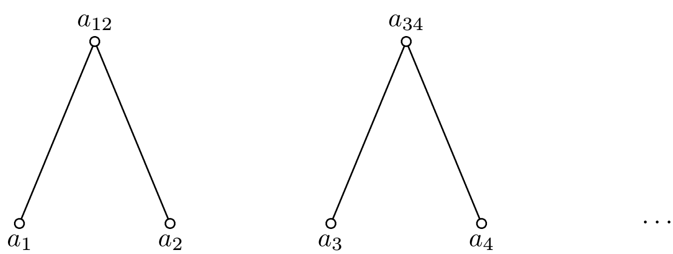


Fig. 1

Then for any finite set $F$ of atoms from $A$, $\varphi(A \setminus F_t)$ is a semigroup generator of $\mathbb{Z}$ and according to Corollary, $G = \operatorname{Ker}\overline{\varphi} \xrightarrow{h=\mathrm{id}} \Gamma = H(A, \mathbb{Z})$ is a strong theory of quasi-divisors of a finite character with $\mathbb{Z}$ as a divisor class group.

We set

$$\Delta = \bigcap_{i \text{ odd}} \Delta_{\alpha_i} = \{g \in \Gamma\colon\ g(\beta) = 0, \text{ for all } \beta \geqslant \alpha_i, i \text{ odd}\}$$

where $\Delta_{\alpha_i}$ is a polar of $\mathrm{nat}(\alpha_i)$. Then $\Delta$ is an $l$-ideal of $\Gamma$. Let $a = \mathrm{nat}(\alpha_2) \in \Gamma$. Then $a \in \Delta$. Now, if $\Delta$ is $G$-dense, there exists $g \in G$ such that $g \geqslant a$, $g \in \Delta$. Hence, $g(\alpha_i) = 0 = g(\alpha_{i,i+1})$ for all $i$ odd. Since $g \in G$ we then have $0 = \sum\limits_i g(\alpha_i)(-1)^i = \sum\limits_{i \text{ even}} g(\alpha_i)$. If $g(\alpha_2) < 1$, then $\alpha_2 \in \mathrm{ms}(g-a)$ and $(g-a)(\alpha_2) < 0$, a contradiction. Hence, $g(\alpha_2) \geqslant 1$ and there exists another $i$ such that $g(\alpha_i) < 0$. But then $i \in \mathrm{ms}(g)$ and $g \ngeqslant 0$, a contradiction. Therefore, $\Delta$ is not $G$-dense.

In [18]; 3.5, it was proved that any intersection of finitely many prime $l$-ideals of $\Gamma$ is $G$-dense. In Example 2 it was shown that there exists an intersection of infinitely many prime $l$-ideals from a base of $\Gamma$ which is not $G$-dense. In the next example we show that this is not a typical case.

**Example 3.** There exists a *po*-group $G$ with a strong theory of quasi-divisors of a finite character $h\colon G \longrightarrow \Gamma$ such that any intersection of infinitely many prime $l$-ideals from a base of $\Gamma$ is $G$-dense.

In fact, let $A$ be an infinite finitely atomic root system as in Fig. 2. We define a map $\varphi\colon A \longrightarrow \mathbb{Z}$ such that

$$\varphi(\alpha_i) = \varphi(\bar{\alpha}_i) = (-1)^i; \quad i = 1, 2, \ldots .$$

Then for any finite set $F$ of atoms from $A$, $\varphi(A \setminus F_t)$ is a semigroup generator of $\mathbb{Z}$ and according to Corollary, $G = \operatorname{Ker} \bar{\varphi} \xrightarrow{h=\mathrm{id}} \Gamma = H(A, \mathbb{Z})$ is a strong theory of quasi-divisors of a finite character.

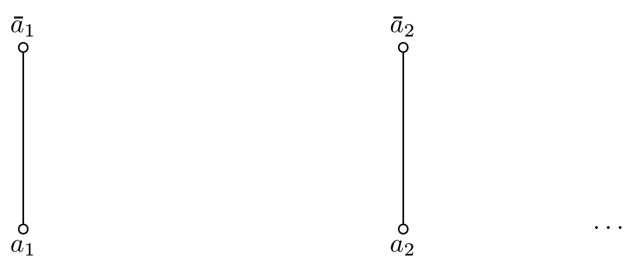


Fig. 2

Then $\{\Delta_{\alpha_i}\colon\ i = 1, 2, \ldots\}$ is a base of $\Gamma$, where $\Delta_\alpha = \{g \in \Gamma\colon\ g(\beta) = 0, \beta \in A, \beta \geqslant \alpha\}$. Let $B \subseteq \{\alpha_1, \ldots\}$ be an arbitrary infinite set and let $\Delta = \bigcap\limits_{\beta \in B} \Delta_\beta$. Let $a \in \Delta_+$ be arbitrary, i.e. $a(\alpha) = a(\bar{\alpha}) = 0$ for any $\alpha \in B$. We define an element $g \in \Gamma$ such that

$$g(\bar{\alpha}) = \begin{cases} 0 & \text{if } \alpha \in B, \\ a(\bar{\alpha}) + 1 & \text{if } a(\bar{\alpha}) \neq 0, \\ 1 & \text{if } a(\bar{\alpha}) = 0, a(\alpha) \neq 0, \\ 0 & \text{if } a(\bar{\alpha}) = a(\alpha) = 0, \end{cases}$$

$$g(\alpha) = -g(\bar{\alpha}).$$

Then $\sum_i (g(\alpha_i) + g(\bar{\alpha}_i))(-1)^i = 0$ and $g \in G$. Moreover, $g \in \Delta$ and $g \geqslant a$. Hence, $\Delta$ is $G$-dense.

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