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SOME REMARKS ON THE PRODUCT OF TWO C_α-COMPACT SUBSETS

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Abstract. For a cardinal α, we say that a subset B of a space X is C_α-compact in X if for every continuous function $f\colon X \to \mathbb{R}^\alpha$, $f[B]$ is a compact subset of $\mathbb{R}^\alpha$. If B is a C-compact subset of a space X, then $\varrho(B, X)$ denotes the degree of C_α-compactness of B in X. A space X is called α-pseudocompact if X is C_α-compact into itself. For each cardinal α, we give an example of an α-pseudocompact space X such that $X \times X$ is not pseudocompact: this answers a question posed by T. Retta in "Some cardinal generalizations of pseudocompactness" Czechoslovak Math. J. 43 (1993), 385–390. The boundedness of the product of two bounded subsets is studied in some particular cases. A version of the classical Glicksberg's Theorem on the pseudocompactness of the product of two spaces is given in the context of boundedness. This theorem is applied to several particular cases.

Keywords: bounded subset, C_α-compact, α-pseudocompact, degree of C_α-pseudocompactness, α_r-space

MSC 2000: 54C50, 54D35

0. Introduction

All topological spaces considered in this paper will be Tychonoff. The Greek letters α, γ and κ will stand for infinite cardinal numbers. For an ordinal number θ, $[0, \theta)$ will denote the space that consists of the underlying set $\{\mu\colon \mu < \theta\}$ endowed with the order topology. The space $[0, \theta + 1)$ will be denoted by $[0, \theta]$. For a cardinal α, we say that a subset B of a space X is *C_α-compact* in X if for every continuous function $f\colon X \to \mathbb{R}^\alpha$, $f[B]$ is a compact subset of $\mathbb{R}^\alpha$. If $\alpha = \omega$, then we simply say

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C-compact instead of C_ω-compact. The C-compact subsets were studied, under a different name, by Isiwata [Is] and for arbitrary cardinals in [GST]. C_α-compactness is a natural generalization of the notion of α-pseudocompactness which was introduced by J. F. Kennison [Ke]: we say that a space X is *α-pseudocompact* if X is C_α-compact into itself. Notice that ω-pseudocompactness agrees with pseudocompactness and if $\gamma < \alpha$, then every C_α-compact subset is C_γ-compact. We remark that C-compact subsets are bounded; that is, every real-valued continuous function on the main space is bounded on every C-compact subset. There are examples of bounded subsets that are not C-compact, even for closed subsets (see [Is, p. 460]). A useful characterization of C_α-compactness is that B is C_α-compact in X iff B is G_α-dense in $\mathrm{cl}_{\beta(X)} B$ (see [GST, Th. 1.2]). The degree of C_α-compactness of a C-compact subset B of a space X was introduced in [GST] and is defined as follows: if B is a C-compact subset of X, then we define

$$\varrho(B, X) = \infty$$

if B is compact, and

$$\varrho(B, X) = \sup\{\alpha\colon\ B \text{ is } C_\alpha\text{-compact in } X\}$$

if B is not compact. The authors of [GST, Th. 4.2] established the inequality

$$\varrho\Bigl(\prod_{i\in I} A_i, \prod_{i\in I} X_i\Bigr) \leqslant \min\{\varrho(A_i, X_i)\colon\ i \in I\},$$

whenever $\prod_{i\in I} A_i$ is C-compact in $\prod_{i\in I} X_i$. Unfortunately, we do not know whether the equality must hold.

In this article, we shall answer a question posed by T. Retta [Re] and study several interesting products of two C-compact subsets in which the above equality holds. The example of an α-pseudocompact space, for each cardinal α, whose square is not pseudocompact is presented in the first Section. In the second Section, we shall study the boundedness of the product of two bounded subsets with some additional assumptions. We start the second Section with a version of the classical Gilcksberg's Theorem on pseudocompactness of the product of two spaces in the real of boundedness.

It is wellknown that there is a pseudocompact space X such that $X \times X$ is not pseudocompact (see [GJ, 3.18]). This example lead T. Retta ([Re]) to ask whether, for every α, there are α-pseudocompact spaces whose product is not pseudocompact. For every cardinal number α, we will give an example of an α-pseudocompact space whose square is not α-pseudocompact. In this example, Noble's ([No1, 2.3]) construction plays a very important rule. His construction is the following:

Let X be a space and let α be a cardinal number such that $\alpha > |\beta(X)|$ and $cf(\alpha) > \omega$. Then we have that $Y = (\beta(X) \times [0, \alpha)) \cup (X \times \{\alpha\})$ is a pseudocompact space that contains X as a closed subspace and $\beta(Y) = \beta(X) \times [0, \alpha]$. It is not hard to see that Y is γ-pseudocompact for every $\gamma < cf(\alpha)$. In particular, we have that, for every cardinal number α, every space X can be embedded as a closed subspace in an α-pseudocompact space.

Next, we shall slightly modify Noble's construction:

Lemma 1.1. *Let X be a pseudocompact subspace of $\beta(\omega)$ with $\omega \subseteq X$. Then X can be embedded as a closed subspace in an α-pseudocompact space Y such that ω is open in Y for every α.*

Proof. First, we observe that a subspace X of $\beta(\omega)$ that contains ω is pseudocompact iff $X \cap \omega^*$ is dense in ω^*. Fix a pseudocompact subspace X of $\beta(\omega)$ with $\omega \subseteq X$ and a cardinal α. Choose a cardinal κ with $cf(\kappa) > \max\{\alpha, 2^c\}$. Put $Y = (X \times \{\kappa\}) \cup ((\beta(\omega) - \omega) \times [0, \kappa))$. Notice that $(\beta(\omega) \times \{\kappa\}) \cup (\omega^* \times [0, \kappa])$ is compact since it is the complement of $\omega \times [0, \kappa)$ in $\beta(\omega) \times [0, \kappa]$. By using the density of $X \cap \omega^*$ in ω^* and by standard argument, we may show that Y is C^*-embedded in $(\beta(\omega) \times \{\kappa\}) \cup (\omega^* \times [0, \kappa])$; hence $\beta(Y) = (\beta(\omega) \times \{\kappa\}) \cup (\omega^* \times [0, \kappa])$. Thus Y is κ-pseudocompact, because $cf(\kappa) > \alpha$. Therefore, Y is α-pseudocompact and contains X as a closed subspace. It is clear that ω is open in Y. □

If X is a pseudocompact subspace of $\beta(\omega)$ with $\omega \subseteq X$ and α and κ are cardinal numbers such that $cf(\kappa) > \max\{\alpha, 2^c\}$, then the space constructed as in Lemma 1.1 will be denoted by $N(X, \alpha, \kappa)$.

Example 1.2. For every cardinal α there is an α-pseudocompact, countably compact space X such that $X \times X$ is not α-pseudocompact.

Proof. Fix a cardinal α and choose a cardinal κ with $cf(\kappa) > \max\{\alpha, 2^c\}$. Consider the countably compact subspace X of $\beta(\omega)$ constructed in [GJ, 9.15] which is defined by a permutation σ of ω (this permutation satisfies that σ^2 is the identity and it does not have fixed points) and has cardinality equal to c. By the construction of this space, we have that $D = \{(n, \sigma(n)) \colon n < \omega\}$ is a clopen discrete subspace of $X \times X$; this fact witnesses that $X \times X$ is not pseudocompact. By Lemma 1.1,

$N(X,\alpha,\kappa)$ is an α-pseudocompact space that contains X as a closed subspace. Hence, we have that D is a closed discrete subspace of $N(X,\alpha,\kappa) \times N(X,\alpha,\kappa)$. It follows from Lemma 1.1 that D is open in $N(X,\alpha,\kappa) \times N(X,\alpha,\kappa)$. Therefore, $N(X,\alpha,\kappa) \times N(X,\alpha,\kappa)$ cannot be pseudocompact and hence $N(X,\alpha,\kappa) \times N(X,\alpha,\kappa)$ is not α-pseudocompact. □

It should be remarked that the condition X is pseudocompact in Lemma 1.1 is essential. In fact, we have the following two corollaries.

Corollary 1.3. *Let X be a subspace of $\beta(\omega)$ with $\omega \subseteq X$ and let α be a cardinal with $cf(\alpha) > 2^c$. Then X is pseudocompact if and only if $N(X,c,\alpha)$ is α-pseudocompact.*

Proof. The necessity is Lemma 1.1. To prove the sufficiency suppose that $N(X,c,\alpha)$ is α-pseudocompact. In particular, $N(X,c,\alpha)$ is pseudocompact. Since X is identified with $X \times \{\alpha\}$, X is a regular-closed subset of $N(X,c,\alpha)$. This implies that X is pseudocompact. □

Corollary 1.4. *If α is a cardinal with $cf(\alpha) > 2^c$, then α-pseudocompactness is not regular-closed hereditary.*

Proof. Let α be a cardinal with $cf(\alpha) > 2^c$. Now, let X be a pseudocompact subspace of $\beta(\omega)$ with $\omega \subseteq X$ and $|X| = c$. It is clear that X cannot be c-pseudocompact and so X is not α-pseudocompact. Then $N(X,c,\alpha)$ is an α-pseudocompact space that contains a regular-closed subset, say X, that is not α-pseudocompact. □

The previous corollary suggests the question: *If α is an uncountable cardinal with $cf(\alpha) \leqslant 2^c$, must α-pseudocompactness be regular-closed hereditary?*

The following example due to H. Ohta answers this question in the negative (see also [GO]).

Example 1.5. Given a space K, we shall denote by $A(K)$ the Alexandroff duplicate of K. Let $T = [0,\omega_2] \times [0,\omega_1] \setminus \{p\}$, where p is the corner point (ω_2,ω_1) and let $X = A(T)$. We shall show that X is an ω_1-pseudocompact space including a regular-closed subset Y which is not ω_1-pseudocompact.

First, we consider the subspace $X \cup \{(p,0)\}$ of $A([0,\omega_2] \times [0,\omega_1])$. The space X has the following properties:

Claim 1: $\beta(X) = X \cup \{(p,0)\}$.

Since the Alexandroff duplicate of a compact space is compact, $A([0,\omega_2] \times [0,\omega_1])$ is compact. Moreover, since $(p,1)$ is an isolated point and

$$X \cup \{(p,0)\} = A([0,\omega_2] \times [0,\omega_1]) \setminus \{(p,1)\},$$

$X \cup \{(p,0)\}$ is compact. So, we only need to prove that X is C^*-embedded in $X \cup \{(p,0)\}$ [GJ, Th. 6.5]. To see this, let $f \in C^*(X)$. Then there are $\alpha < \omega_2$ and $\beta < \omega_1$ such that f takes constant value r on $T_0 \times \{0\}$, where $T_0 = [\alpha, \omega_2] \times [\beta, \omega_1] \backslash \{p\}$.

Now, suppose that there is an uncountable set $\{m_\lambda\}_{\lambda<\omega_1} \subseteq T_0$ such that

$$f(m_\lambda, 1) \neq r \quad \text{for each } \lambda < \omega_1.$$

Then there exist a countably infinite subset $I \subseteq \omega_1$ and $\delta > 0$ such that

$$|f(m_\lambda, 1) - r| > \delta \quad \text{for each } \lambda \in I. \tag{1}$$

Since T_0 is countably compact, $\{m_\lambda\}_{\lambda \in I}$ has an accumulation point $m \in T_0$. Then,

$$(g, 0) \in \mathrm{cl}_X \{(m_\lambda, 0) : \lambda \in I\} \cap \mathrm{cl}_X \{(m_\lambda, 1) : \lambda \in I\}. \tag{2}$$

Since $f(m_\lambda, 0) = r$ for each $\lambda \in I$, (1) and (2) contradict the continuity of f. Thus,

$$|\{t \in T_0 : f(t,1) \neq r\}| \leqslant \omega.$$

Hence there exist α_1 with $\alpha < \alpha_1 < \omega_2$ and β_1 with $\beta < \beta_1 < \omega_1$ such that $f(t,1) = r$ for each $t \in T_1$, where T_1 means for $[\alpha_1, \omega_2] \times [\beta_1, \omega_1] \setminus \{p\}$. Since $f[A(T_1)] = \{r\}$, f extends continuously over $X \cup \{(p,0)\}$. Hence, $\beta(X) = X \cup \{(p,0)\}$. This completes the proof of the Claim 1.

Notice that X is pseudocompact, because the cardinality of the remainder of $\beta(X)$ is one.

Claim 2: X is ω_1-pseudocompact.

Let f be a continuous function from X into $\mathbb{R}^{\omega_1}$. For each $\alpha < \omega_1$, let $\pi_\alpha : \mathbb{R}^{\omega_1} \longrightarrow \mathbb{R}$ be the α-th projection. For each $\alpha < \omega_1$, since X is pseudocompact, $\pi_\alpha \circ f : X \longrightarrow \mathbb{R}$ extends to a continuous function $\overline{f}_\alpha$ from $\beta(X)$ into $\mathbb{R}$. Then the diagonal mapping $\overline{f} = \Delta_{\alpha<\omega_1} \overline{f}_\alpha$ is a continuous extension of f to $\beta(X)$. By Claim 1, the remainder of $\beta(X)$ is the point $(p,0)$. We shall prove that $x \in f(X)$ where $x = \overline{f}(p,0)$. Since the weight of $\mathbb{R}^{\omega_1}$ is ω_1, the point x has a neighbourhood base $\{U_\gamma\}_{\gamma<\omega_1}$. For each $\alpha < \omega_2$, let $F_\alpha = A([0,\alpha] \times [0, \omega_1])$. Since $(p,0) \in \mathrm{cl}_{\beta(X)} \bigcup_{\alpha<\omega_2} F_\alpha$,

$$x \in \overline{f}\left(\mathrm{cl}_{\beta(X)} \bigcup_{\alpha<\omega_2} F_\alpha\right) \subseteq \mathrm{cl}_{\mathbb{R}^{\omega_1}} \left(\bigcup_{\alpha<\omega_2} f(F_\alpha)\right). \tag{3}$$

Now, suppose that $x \notin f(X)$. For each $\alpha < \omega_2$, since $x \notin f(F_\alpha)$, there is $\varphi(\alpha) < \omega_1$ such that $U_{\varphi(\alpha)} \cap f(F_\alpha) = \emptyset$. Hence there exist a cofinal subset $J \subseteq \omega_2$ and $\gamma < \omega_1$ such that $\varphi(\alpha) = \gamma$ for each $\alpha \in J$. This means that $U_\gamma \cap f(F_\alpha) = \emptyset$ for each $\alpha \in J$.

Since $\{f(F_\alpha): \alpha < \omega_2\}$ is increasing, $U_\gamma \cap \Big(\bigcup_{\alpha<\omega_2} f(F_\alpha)\Big) = \emptyset$ which contradicts (3). Hence, $x \in f(X)$ and consequently, $f(X)$ is compact. Thus, X is ω_1-pseudocompact. This proves Claim 2.

Let $S = \{(\omega_2, \beta): \beta < \omega_1\} \subseteq T$, and let $Y = A(S)$. Then Y is regular closed in X, but we have:

Claim 3: Y is not ω_1-pseudocompact.

Since S is homeomorphic to $[0, \omega_1)$, it suffices to show that $A([0, \omega_1))$ is not ω_1-pseudocompact. For each $\alpha < \omega_1$, let $U_\alpha = A([0, \alpha])$. Then $\{U_\alpha\}_{\alpha<\omega_1}$ is a cozero-set cover of $A([0, \omega_1])$ which has no finite subcover. This means that $A([0, \omega_1))$ is not ω_1-pseudocompact.

2. Products of bounded subsets

We start this section with some basic notation and some preliminary results.

If $f \in C(X \times Y)$, then $f(x, \_)\colon Y \to \mathbb{R}$ is the function defined on Y by $f(x, \_)(y) = f(x, y)$ for each $y \in Y$, the definition of $f(\_, y)\colon X \to \mathbb{R}$ should be clear, for each $y \in Y$. For each $f \in C(X \times Y)$ and for each $(x, y) \in X \times Y$, the domain of $f(x, \_)$ and $f(\_, y)$ will be sometimes identified with $\{x\} \times Y$ and $X \times \{y\}$, respectively.

We know that $A \subseteq X$ is bounded in X iff $\mathrm{cl}_{\beta(X)} A \subseteq \upsilon(X)$ (see [GG, Lemma 2.2]). Let B be bounded in Y and let $f \in C(X \times Y)$. Then, for each $a \in A$, we have that $f(a, \_)\colon \{a\} \times Y \to \mathbb{R}$ can be extended continuously to $\{a\} \times \upsilon(Y)$. Henceforth, the restriction of this extension to $\{a\} \times \mathrm{cl}_{\beta(Y)} B$ will be denoted by $\hat{f}_a$. In a similar way, if A is a bounded subset of X, then we let $\hat{f}^b\colon (\mathrm{cl}_{\beta(X)} A) \times \{b\} \to \mathbb{R}$ be the restriction of the continuous extension of $f(\_, b)$ to $\upsilon(X)$, for each $b \in B$. Now, if A and B are bounded subsets of X and Y, respectively, and $f \in C(X \times Y)$, then we define $\hat{f}_A\colon A \times \mathrm{cl}_{\beta(Y)} B \to \mathbb{R}$ and $\hat{f}_B\colon (\mathrm{cl}_{\beta(X)} A) \times B \to \mathbb{R}$ by $\hat{f}_A(a, b) = \hat{f}_a(b)$ for every $(a, b) \in A \times \mathrm{cl}_{\beta(Y)} B$ and $\hat{f}_B(a, b) = \hat{f}^b(a)$ for every $(a, b) \in (\mathrm{cl}_{\beta(X)} A) \times B$. Notice that $\hat{f}_A|_{A\times B} = \hat{f}_B|_{A\times B} = f|_{A\times B}$. If $f \in C(X \times Y)$ and B is bounded in Y (A is bounded in X), then the set $\{\hat{f}_A(\_, b)\colon b \in \mathrm{cl}_{\beta(Y)} B\}$ ($\{\hat{f}_B(a, \_)\colon a \in \mathrm{cl}_{\beta(X)} A\}$) is called the *natural extension* of $\{f(\_, b)\colon b \in B\}$ ($\{f(a, \_)\colon a \in A\}$). We shall show that $\hat{f}_A$ and $\hat{f}_B$ are sometimes continuous. This result was implicitly used in [Bu, Cor. 5.2] and it is surely known to various people. We begin with a lemma which easily follows from Proposition 6 of [Bo, Chapter X].

Lemma 2.1. *Let A and B be subsets of X and Y, respectively. For $f \in C(X \times Y)$, the following conditions are equivalent.*

1. *$\{f(\_, b)\colon b \in B\}$ ($\{f(a, \_)\colon a \in A\}$) is equicontinuous on A (B);*
2. *the natural extension $\{\hat{f}_A(\_, b)\colon b \in \mathrm{cl}_{\beta(Y)} B\}$ ($\{\hat{f}_B(a, \_)\colon a \in \mathrm{cl}_{\beta(X)} A\}$) of $\{f(\_, b)\colon b \in B\}$ ($\{f(a, \_)\colon a \in A\}$) is equicontinuous on A (B).*

Now, we state a theorem due to W. W. Comfort and T. Hager [CH] (a proof is available in [Wa, Th. 8.6]). We recall that a *z-closed* mapping is the one in which the image of every zero set is closed. We consider the ring $C^*(X)$ endowed with the topology induced by the norm $\|f\| = \sup\{f(x)\colon\ x \in X\}$.

Lemma 2.2. *The following conditions in the product space $X \times Y$ are equivalent:*

(1) *the projection map $\pi_X\colon\ X \times Y \to X$ is z-closed;*
(2) *for each $f \in C^*(X \times Y)$ the function $\Phi\colon\ X \to C^*(Y)$ defined by $\Phi(x) = f(x, \_)$ is continuous;*
(3) *for each $f \in C^*(X \times Y)$, $\{f(\_, y)\colon\ y \in Y\}$ is equicontinuous on X;*
(4) *every bounded real-valued continuous function on $X \times Y$ admits a continuous extension to $X \times \beta(Y)$.*

Lemma 2.3. *Let A and B be subsets of X and Y, respectively, such that B is bounded in Y. For $f \in C(X \times Y)$, the following conditions are equivalent:*

(1) *$\{f(\_, b)\colon\ b \in B\}$ is equicontinuous on A;*
(2) *the natural extension $\{\hat{f}_A(\_, b)\colon\ b \in \mathrm{cl}_{\beta(Y)} B\}$ of $\{f(\_, b)\colon\ b \in B\}$ is equicontinuous on A;*
(3) *$\hat{f}_A$ is continuous.*

Proof. (1) $\Leftrightarrow$ (2). This is Lemma 2.1.

(2) $\Rightarrow$ (3). This follows from Proposition 6 of [Bo, Chapter X, 2.3].

(3) $\Rightarrow$ (2). By Kuratowski's Theorem the projection $\pi\colon\ A \times \mathrm{cl}_{\beta(Y)} B \to A$ is closed. According to Lemma 2.2, we have that $\{f(\_, b)\colon\ b \in \mathrm{cl}_{\beta(Y)} B\}$ is equicontinuous on A. □

The following lemma is a particular case of a result proved by R. Pupier [Pu] and it is a generalization of Lemma 1.3 of [Fro].

Lemma 2.4. *Let A and B be bounded subsets of X and Y, respectively. If $A \times B$ is bounded in $X \times Y$, then $\{f(\_, b)\colon\ b \in B\}$ is equicontinuous on A and $\{f(a, \_)\colon\ a \in A\}$ is equicontinuous on B for every $f \in C(X \times Y)$. In particular, $\hat{f}_A\colon\ A \times \mathrm{cl}_{\beta(Y)} B \to \mathbb{R}$ and $\hat{f}_B\colon\ (\mathrm{cl}_{\beta(X)} A) \times B \to \mathbb{R}$ are continuous for every $f \in C(X \times Y)$.*

Let X be a space. A compact space K is called a compactification of X if X is a dense subspace of K. We recall that two compactifications K_1 and K_2 of X are called *equivalent* if there exists a homeomorphism Φ from K_1 onto K_2 such that $\Phi|_X$ is the identity map. In this case we will write $K_1 \cong K_2$.

Lemma 2.5. *Let A and B be bounded subsets of X and Y, respectively. If $(\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B) \cong \mathrm{cl}_{\beta(X \times Y)}(A \times B)$, then $A \times B$ is bounded in $X \times Y$.*

P r o o f. Suppose that there is a continuous function $f\colon X\times Y\to\mathbb{R}$ such that $f(x,y)>0$ for every $(x,y)\in X\times Y$ and $f|_{A\times B}$ is unbounded. Set $g=\frac{1}{f}$. Then $g\colon X\times Y\to\mathbb{R}$ is a continuous bounded function. By hypothesis, g can be extended to a continuous function $g^*\colon \mathrm{cl}_{\beta(X)} A\times \mathrm{cl}_{\beta(Y)} B\to\mathbb{R}$. For each $n<\omega$, choose $(a_n,b_n)\in A\times B$ such that $g^*((a_n,b_n))\to 0$ and let $(a,b)\in \mathrm{cl}_{\beta(X)} A\times \mathrm{cl}_{\beta(Y)} B$ be an accumulation point of $\{(a_n,b_n)\colon n<\omega\}$. For each $k<\omega$, let $V_k\times U_k$ be a basic neighbourhood of (a,b) such that $0\leqslant g^*((x,y))<\frac{1}{k+1}$ for each $(x,y)\in V_k\times U_k$. For each $k<\omega$, pick $(a_{n_k},b_{n_k})\in V_k\times U_k$. We have that either $a\notin A$ or $b\notin B$, since $g^*((a,b))=0$. Without loss of generality, we may assume that $a\notin A$. Consider $g^*|_{A\times\{b\}}$. It then follows that $f|_{A\times\{b\}}$ is unbounded, which contradicts the boundedness of A in X. $\square$

Glicksberg's Theorem on pseudocompactness says that if X and Y are pseudocompact spaces, then $X\times Y$ is pseudocompact if and only if $\beta(X\times Y)=\beta(X)\times\beta(Y)$ if and only if every real-valued continuous function on $X\times Y$ has a continuous extension to $\beta(X)\times\beta(Y)$. The following theorem is a slight generalization of Glicksberg's Theorem.

Theorem 2.6. *Let A and B be bounded subsets of X and Y, respectively. Then, the following assertions are equivalent:*

(1) *if $f\in C(X\times Y)$, then $\{\hat f_A(a,\_)\colon a\in A\}$ is equicontinuous on $\mathrm{cl}_{\beta(Y)} B$;*
(2) *if $f\in C(X\times Y)$, then $\{\hat f_A(a,\_)\colon a\in A\}$ and $\{\hat f_B(\_,b)\colon b\in B\}$ are equicontinuous on $\mathrm{cl}_{\beta(Y)} B$ and $\mathrm{cl}_{\beta(X)} A$, respectively;*
(3) *if $f\in C(X\times Y)$, then $f|_{A\times B}$ has a continuous extension to $\mathrm{cl}_{\beta(X)} A\times \mathrm{cl}_{\beta(Y)} B$;*
(4) $\mathrm{cl}_{\beta(X)} A\times \mathrm{cl}_{\beta(Y)} B\cong \mathrm{cl}_{\beta(X\times Y)}(A\times B)$.

P r o o f. (1) $\Longrightarrow$ (2). Let $y_0\in \mathrm{cl}_{\beta(Y)} B-B$. First, we shall prove that $\hat f_A(\_,y_0)\colon A\times\{y\}\to\mathbb{R}$ has a continuous extension to $\mathrm{cl}_{\beta(X)} A\times\{y\}$ for every $y\in \mathrm{cl}_{\beta(Y)} B$. If $y\in B$, then $\hat f_B(\_,y)$ is such an extension. Suppose that $y\in \mathrm{cl}_{\beta(Y)} B-B$ and take a net $\{y_\delta\}_{\delta\in D}$ in B that converges to y. From the hypothesis it follows that $\hat f_A(\_,y)$ is the uniform limit in $C(A)$ of the net $\{\hat f_A(\_,y_\delta)\}_{\delta\in D}$ and hence $\hat f_A(\_,y)$ admits a continuous extension to $\mathrm{cl}_{\beta(X)} A$ since each $\hat f_A(\_,y_\delta)$ has a continuous extension to $\mathrm{cl}_{\beta(X)} A$ for each $\delta\in D$. This extension of $\hat f_A(\_,y)$ to $\mathrm{cl}_{\beta(X)} A$ will be denoted by $\hat f_B(\_,y)$ for every $y\in \mathrm{cl}_{\beta(Y)} B$. Next we shall show that $\{\hat f_B(\_,b)\colon b\in B\}$ is equicontinuous on $\mathrm{cl}_{\beta(X)} A$. Fix $\varepsilon>0$ and $x\in \mathrm{cl}_{\beta(X)} A$. Since $\{\hat f_A(a,\_)\colon a\in A\}$ is equicontinuous on $\mathrm{cl}_{\beta(Y)} B$, for each $y\in \mathrm{cl}_{\beta(Y)} B$ we may find an open neighbourhood V_y of y such that

$$|\hat f_A(a,y')-\hat f_A(a,y)|<\varepsilon \tag{1}$$

for every $a\in A$ and for every $y'\in V_y$. Since $\mathrm{cl}_{\beta(Y)} B$ is compact, there is a finite set $\{y_0,y_1,\dots,y_n\}$ such that $\bigcup_{i=0}^{n} V_{y_i}=\mathrm{cl}_{\beta(Y)} B$. Now choose and consider a net $\{x_\delta\}_{\delta\in D}$

in A which converges to x. Since $\hat{f}_B(\_, y_i)$ is continuous for each $i = 0, 1, \dots, n$, there exists $\delta_0 \in D$ such that

$$(2) \qquad |\hat{f}_B(x, y_i) - \hat{f}_B(x_\delta, y_i)| < \varepsilon$$

for every $i = 0, 1, \dots, n$ and for every $\delta \geqslant \delta_0$. Fix $y \in B$. Then there exists i such that $y \in V_{y_i}$. By (1), we have that

$$(3) \qquad |\hat{f}_B(x, y) - \hat{f}_B(x, y_i)| \leqslant \varepsilon \quad \text{and} \quad |\hat{f}_B(x_\delta, y_i) - \hat{f}_B(x_\delta, y)| < \varepsilon$$

for every $\delta \geqslant \delta_0$. Applying (2) and (3) we obtain that

$$\begin{aligned} |\hat{f}_B(x, y) - \hat{f}_B(x_\delta, y)| &\leqslant |\hat{f}_B(x, y) - \hat{f}_B(x, y_i)| + |\hat{f}_B(x, y_i) - \hat{f}_B(x_\delta, y_i)| \\ &\quad + |\hat{f}_B(x_\delta, y_i) - \hat{f}_B(x_\delta, y)| \leqslant 3\varepsilon \end{aligned}$$

for every $\delta \geqslant \delta_0$. Notice that δ_0 does not depend on the choice of y. Therefore, $\{\hat{f}_B(\_, b) \colon b \in B\}$ is equicontinuous on $\mathrm{cl}_{\beta(X)} A$.

(2) $\Longrightarrow$ (3). We shall first show that $\hat{f}_B$ has a separately continuous extension f^* to $\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B$ (i. e., $f^*|_{\{x\} \times \mathrm{cl}_{\beta(Y)} B}$ is continuous for every $x \in \mathrm{cl}_{\beta(X)} A$). We shall verify that $\hat{f}_B|_{\{x\} \times B}$ has a continuous extension to $\{x\} \times \mathrm{cl}_{\beta(Y)} B$ for every $x \in \mathrm{cl}_{\beta(X)} A$. Fix $x_0 \in \mathrm{cl}_{\beta(X)} A$. According to Problem 6H of [GJ], it suffices to prove that $\hat{f}_B$ admits a continuous extension to $\{x_0\} \times (B \cup \{y\})$ for every $y \in \mathrm{cl}_{\beta(Y)} B$. Fix $y_0 \in \mathrm{cl}_{\beta(Y)} B - B$. Let $\varepsilon > 0$ and let $\{y_\delta\}_{\delta \in D}$ be a net in B converging to y_0. Since $\{\hat{f}_A(\_, b) \colon b \in B\}$ is equicontinuous on $\mathrm{cl}_{\beta(X)} A$, there exists an open neighbourhood V of x_0 such that

$$|\hat{f}_A(x_0, y) - \hat{f}_A(x, y)| < \varepsilon$$

for every $x \in V$ and for every $y \in B$. Since $\{\hat{f}_B(a, \_) \colon a \in A\}$ is equicontinuous on the compact set $\mathrm{cl}_{\beta(Y)} B$, it is uniformly equicontinuous on $\mathrm{cl}_{\beta(Y)} B$ ([Bo, Chap. X, Sec. 2, Cor. 2]). Since the net $\{y_\delta\}_{\delta \in D}$ converges to y_0, we can find $\delta_0 \in D$ such that

$$|f(x, y_\delta) - f(x, y_{\delta'})| < \varepsilon$$

for every $\delta, \delta' \geqslant \delta_0$ and for every $x \in A$. Fix $x \in V$. Applying the previous two inequalities, we have

$$\begin{aligned} |\hat{f}_B(x_0, y_\delta) - \hat{f}_B(x_0, y_{\delta'})| &\leqslant |\hat{f}_B(x_0, y_\delta) - \hat{f}_B(x, y_\delta)| + |\hat{f}_B(x, y_\delta) - \hat{f}_B(x, y_{\delta'})| \\ &\quad + |\hat{f}_B(x, y'_\delta) - \hat{f}_B(x_0, y_{\delta'})| \leqslant 3\varepsilon \end{aligned}$$

for every $\delta, \delta' \geqslant \delta_0$. Thus, $\{\hat{f}_B(x_0, y_\delta)\}_{\delta \in D}$ is a Cauchy sequence in $\mathbb{R}$. Let us denote the limit of $\{\hat{f}_B(x_0, y_\delta)\}_{\delta \in D}$ by $\hat{f}(x_0, y_0)$. It is not hard to show that $\hat{f}(x_0, y_0)$ does not depend on the choice of the net $\{y_\delta\}_{\delta\in D}$. Thus, $\hat{f}_B|_{\{x_o\}\times B} \cup ((x_0, y_0), \hat{f}(x_0, y_0))$ is a continuous extension of $\hat{f}_B|_{\{x_o\}\times B}$ as required. So, $\hat{f}_B$ admits a separately continuous extension, say f^*, to $\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B$. From Lemma 2.3 it follows that $\{\hat{f}_A(\_, b)\colon b \in \mathrm{cl}_{\beta(Y)} B\}$ is equicontinuous on $\mathrm{cl}_{\beta(X)} A$. Using this result and reasoning as in the proof of (2) $\Longrightarrow$ (3) in Lemma 2.3, we may show that f^* is a continuous extension of $\hat{f}_B$ to $\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B$.

(3) $\Longrightarrow$ (4). We know that if $\mathscr{F} \subseteq C^*(X)$ separates points from closed subsets, then $\alpha_{\mathscr{F}}(X)$ is a compactification of X, where $\alpha_{\mathscr{F}}\colon X \to [0,1]^{\mathscr{F}}$ is the closure of the image of the evaluation map induced by $\mathscr{F}$. Now, let $\mathscr{A} = \{f|_{A\times B}\colon f \in C(\mathrm{cl}_{\beta(X\times Y)}(A \times B))\}$ and $\mathscr{B} = \{f|_{A\times B}\colon f \in C((\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B))\}$. By Theorem 2.5 of [Ch], we obtain that $\alpha_{\mathscr{A}}(A \times B) \cong \mathrm{cl}_{\beta(X\times Y)}(A \times B)$ and $\alpha_{\mathscr{B}}(A \times B) \cong (\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B))$. By assumption, we have that $\mathscr{A} = \mathscr{B}$ and so $\alpha_{\mathscr{A}}(A \times B) \cong \alpha_{\mathscr{B}}(A \times B)$. That is, $(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B) \cong \mathrm{cl}_{\beta(X\times Y)}(A \times B)$.

(4) $\Longrightarrow$ (1). By Lemma 2.5, we have that $A \times B$ is bounded in $X \times Y$. Let $f \in C(X \times Y)$. Since $A \times B$ is bounded in $X \times Y$ we may find a bounded function $g \in C(X \times Y)$ such that $g|_{A\times B} = f|_{A\times B}$. Let g^* be the Stone-Čech continuous extension of g to $\beta(X \times Y)$ and set $h = g^*|_{\mathrm{cl}_{\beta(X\times Y)}(A\times B)}$. By hypothesis, we may assume that the domain of h is $\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B$. Since $\mathrm{cl}_{\beta(X)} A$ is compact, by Kuratowski's Theorem, the projection map from $\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B$ onto $\mathrm{cl}_{\beta(Y)} B$ is closed. According to Lemma 2.2, the family $\{h(x, \_)\colon x \in \mathrm{cl}_{\beta(X)} A\}$ is equicontinuous on $\mathrm{cl}_{\beta(Y)} B$. Since $h|_{A\times B} = f|_{A\times B}$, we obtain that $\{\hat{f}_A(a, \_)\colon a \in A\}$ is equicontinuous on $\mathrm{cl}_{\beta(Y)} B$. □

Question 2.7. *Are the conditions of Theorem* 2.6 *equivalent to the condition $A \times B$ is bounded in $X \times Y$?*

We do not know whether the converse of Lemma 2.5 holds as well. An interesting case where the previous question has the affirmative answer is when the set A is pseudocompact.

Theorem 2.8. *Let A and B be subsets of X and Y, respectively. If A is pseudocompact and B is bounded in Y, then the following assertions are equivalent:*

(1) *$A \times B$ is bounded in $X \times Y$;*
(2) *if $f \in C(X \times Y)$, then $\hat{f}_A$ is continuous;*
(3) *if $f \in C(X \times Y)$, then $f|_{A\times B}$ has a continuous extension to $(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B)$;*
(4) *$(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B) \cong \mathrm{cl}_{\beta(X\times Y)}(A \times B)$.*

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) follow from Lemma 2.4 and from Theorem 2.6, respectively. The implication (4) $\Longrightarrow$ (1) is a direct consequence of Lemma 2.5. It remains to prove (2) $\Longrightarrow$ (3). Let $f \in C(X \times Y)$ and suppose that $\hat{f}_A$ is continuous. Since $\operatorname{cl}_{\beta(Y)} B$ is compact, $A \times \operatorname{cl}_{\beta(Y)} B$ is pseudocompact (see [Fro, Th. 3.4]). So, by Tamano's theorem [Ta] the projection map π from $A \times \operatorname{cl}_{\beta(Y)} B$ onto $\operatorname{cl}_{\beta(Y)} B$ is z-closed, that is, $\pi(Z)$ is closed in $\operatorname{cl}_{\beta(Y)} B$ for each zero-set Z in $A \times \operatorname{cl}_{\beta(Y)} B$. According to Lemma 2.2, the family $\{\hat{f}_A(a, \_)\colon\ a \in A\}$ is equicontinuous on $\operatorname{cl}_{\beta(Y)} B$. The conclusion follows from Theorem 2.6. □

The following examples point out that we can not replace $\hat{f}_A$ by $\hat{f}_B$ in condition (2) of the previous theorem and that $\hat{f}_A$ and $\hat{f}_B$ continuous does not imply that $A \times B$ is bounded.

Example 2.9. Let P be a pseudocompact subspace of $\beta(\omega)$ such that $\omega \subseteq P$ and $P \times P$ is not pseudocompact (for a construction of this kind of spaces see [GJ, 9.15]). Let $Y = P \times K$ where K is an infinite compact space. Since K is compact, $P \times K$ is pseudocompact [Fro, Th. 3.4]. So, $\omega \times K$ is bounded in Y. Since $P \times (\omega \times K)$ is dense in $P \times Y$ and $P \times P$ is not pseudocompact, we have that $P \times (\omega \times K)$ is not bounded in $P \times Y$. However, for every $f \in C(P \times Y)$, $f|_{P \times (\omega \times K)}$ has a continuous extension to $\beta(\omega) \times (\omega \times K)$. To see this, notice that it suffices to extend $f|_{P \times (\{n\} \times K)}$ to $\beta(\omega) \times (\{n\} \times K)$ for all $n < \omega$. But this immediately follows from the fact that $P \times (\{n\} \times K)$ is pseudocompact and from the classical Glicksberg's Theorem on pseudocompactness [Gl].

Remark 2.10. We have just proved that every real-valued continuous function on $P \times (\omega \times K)$ admits a continuous extension to $\beta(\omega) \times (\omega \times K)$. By Lemma 2.2 the projection map from $P \times (\omega \times K)$ onto $\omega \times K$ is z-closed. So, Example 2.9 points out that *sufficient* in Theorem 2.5 of [No2] is not correct.

Example 2.11. Let P be a pseudocompact space as in Example 2.9. Consider $X = P \times K$ and $Y = K \times P$ where K is an infinite compact space. Let $A = \omega \times K$ and let $B = K \times \omega$. A similar argument to the one used in Example 2.9 shows that A and B are bounded subsets of X and Y, respectively, and that the restriction of every continuous function on $X \times Y$ to $A \times B$ has continuous extensions $\hat{f}_A$ and $\hat{f}_B$ to $A \times \operatorname{cl}_{\beta(Y)} B$ and $\operatorname{cl}_{\beta(X)} A \times B$, respectively. Since $A \times B$ is dense in $X \times Y$ and $P \times P$ is not pseudocompact, $A \times B$ is not bounded in $X \times Y$.

We can apply Theorem 2.8 in order to obtain the following result.

Theorem 2.12. *Let A and B be subsets of X and Y, respectively, such that A is pseudocompact and B is C-compact in Y. Then the following assertions are equivalent:*

(1) *$A \times B$ is C-compact in $X \times Y$;*
(2) *$A \times B$ is bounded in $X \times Y$;*
(3) *if $f \in C(X \times Y)$, then $\hat{f}_A$ and $\hat{f}_B$ are continuous;*
(4) *$\hat{f}_A$ is continuous for every $f \in C(X \times Y)$;*

(5) *if* $f \in C(X \times Y)$, *then* $f|_{A\times B}$ *has a continuous extension to* $(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B)$*;*
(6) $(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B) \cong \mathrm{cl}_{\beta(X\times Y)}(A \times B)$*;*
(7) $A \times B$ *is* C*-compact in* $X \times Y$ *and*

$$\varrho(A \times B, X \times Y) = \min\{\varrho(A, X), \varrho(B, Y)\}.$$

Proof. (1) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (4), (7) $\Longrightarrow$ (1) are evident. (2) $\Longrightarrow$ (3) is a direct application of Lemma 2.4. (4) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (6) follow from Theorem 2.8.

(6) $\Longrightarrow$ (7). We have that A and B are G_δ-dense in $\mathrm{cl}_{\beta(X)} A$ and $\mathrm{cl}_{\beta(Y)} B$, respectively. Hence, $A \times B$ is G_δ-dense in $(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B)$ and so $A \times B$ is G_δ-dense in $\mathrm{cl}_{\beta(X\times Y)}(A \times B)$. Thus, $A \times B$ is C-compact in $X \times Y$. By assumption and by the α-density characterization of C_α-compactness (see [GST]), we have

$$\varrho(A \times B, X \times Y) = \varrho(A \times B, (\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B)) =$$

$$\varrho(A \times B, \mathrm{cl}_{\beta(X\times Y)}(A \times B)).$$

According to Theorem 4.3 of [GST], we obtain

$$\varrho(A \times B, (\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B)) = \min\{\varrho(A, \mathrm{cl}_{\beta(X)} A), \varrho(B, \mathrm{cl}_{\beta(Y)} B)\} =$$

$$\min\{\varrho(A, X), \varrho(B, Y)\}.$$

□

Question 2.13. *Is Theorem* 2.12 *true when* A *is* C*-compact in* X*?*

We will denote by $\mathscr{P}$ the Frolík class consisting of all spaces X such that $X \times Y$ is pseudocompact for each pseudocompact space Y (for details see [Fro]). Applying Theorem 3.6 of [Fro] and Corollary 6 of [BS], we obtain the following characterization of $\mathscr{P}$.

Corollary 2.14. *A pseudocompact space* X *belongs to* $\mathscr{P}$ *if and only if for each space* Y *and each* C*-compact subset* B *of* Y, $X \times B$ *satisfies (in* $X \times Y$*) any one of the conditions of Theorem* 2.12.

Next we shall study the product of two C-compact subsets $A \subseteq X$ and $B \subseteq Y$ for a wide class of spaces X. Let us first recall some definitions.

Let α be a cover of X directed by inclusion. A space X is said to be an α_r*-space* if a real-valued function f on X is continuous whenever its restriction to every subset $B \in \alpha$ is continuous. We will say that X is a b_r*-space* (a c_r*-space*) if it is an α_r-space for the cover α of all bounded (C-compact) subsets of X. The k_r*-spaces*, where k

is the set of all compact subsets of a space, are examples of c_r-spaces. Then, first countable and locally pseudocompact spaces are c_r-spaces as well. The b_r-spaces arose in the study of z-closed projections in [No3], and also in the problem of the distribution of the functor of the topological completion (see [Bu], [Pu], [Sa1]). This class of spaces also appears in the study of compactness of function spaces (see [Ar], [Sa2]).

Lemma 2.15. *Let X and Y be two topological spaces. If X is an α_r-space and B a bounded subset of Y such that $\hat{f}_A$ is continuous for each $A \in \alpha$, then $f|_{X\times B}$ has a continuous extension $\hat{f}\colon X \times \operatorname{cl}_{\beta(Y)} B \to \mathbb{R}$ for every $f \in C(X \times Y)$.*

P r o o f. Fix $f \in C(X\times Y)$ and let $\hat{f}$ be the real-valued function on $X\times \operatorname{cl}_{\beta(Y)} B$ defined by $\hat{f}(x,y) = \hat{f}_A(x,y)$ for $x \in A$ and for $A \in \alpha$. It is easy to verify that $\hat{f}$ is well-defined. We claim that $\hat{f}$ is continuous. In fact, let $C_u(\operatorname{cl}_{\beta(Y)} B)$ be the space $C(\operatorname{cl}_{\beta(Y)} B)$ endowed with the topology of uniform convergence and let $A \in \alpha$. Since $\operatorname{cl}_{\beta(Y)} B$ is compact, by Kuratowski's Theorem, the projection map from $A\times \operatorname{cl}_{\beta(Y)} B$ onto A is closed. According to Lemma 2.2, we have that the function $\varphi_A\colon A \to C_u(\operatorname{cl}_{\beta(Y)} B)$ defined by $\varphi_A(x) = \hat{f}_A(x,y)$ for every $x \in A$ and for every $y \in \operatorname{cl}_{\beta(Y)} B$ is continuous. Consider the function $\varphi\colon X \to C_u(\operatorname{cl}_{\beta(Y)} B)$ defined by $\varphi(x) = \varphi_A(x)$ for every $x \in A$ and for every $A \in \alpha$. Then φ is well-defined and, since X is an α_r-space, φ is continuous. It is now a routine to prove that $\hat{f}$ is continuous. □

Theorem 2.16. *Let X and Y be two topological spaces. If X is a c_r-space and B is C-compact in Y, then the following assertions are equivalent:*

(1) *$A \times B$ is C-compact in $X \times Y$ for each C-compact subset A of X;*
(2) *$A \times B$ is bounded in $X \times Y$ for each C-compact subset A of X;*
(3) *if $f \in C(X \times Y)$, then $\hat{f}_A$ is continuous for each C-compact subset A of X;*
(4) *if $f \in C(X\times Y)$, then $f|_{A\times B}$ has a continuous extension to $(\operatorname{cl}_{\beta(X)} A)\times(\operatorname{cl}_{\beta(Y)} B)$ for each C-compact subset A of X;*
(5) *$(\operatorname{cl}_{\beta(X)} A) \times (\operatorname{cl}_{\beta(Y)} B) \cong \operatorname{cl}_{\beta(X\times Y)}(A \times B)$ for each C-compact subset A of X;*
(6) *$A \times B$ is C-compact in $X \times Y$ and*

$$\varrho(A \times B, X \times Y) = \min\{\varrho(A, X), \varrho(B, Y)\}$$

for each C-compact subset A of X.

P r o o f. The implications (1) $\Longrightarrow$ (2) and (6) $\Longrightarrow$ (1) are obvious. On the other hand, (2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5) follow from Lemma 2.4 and Theorem 2.6, respectively, and the proof of (5) $\Longrightarrow$ (6) is similar to that of (6) $\Longrightarrow$ (7) in Theorem 2.12. We only need to prove (3) $\Longrightarrow$ (4). Let $f \in C(X \times Y)$ and let A be a C-compact subset of X. Applying Lemma 2.15, we conclude that $f|_{A\times B}$ has a continuous extension $\hat{f}\colon X \times \operatorname{cl}_{\beta(Y)} B \to \mathbb{R}$. Since $\operatorname{cl}_{\beta(Y)} B$ is compact, $A\times \operatorname{cl}_{\beta(Y)} B$ is bounded in

$X \times \mathrm{cl}_{\beta(Y)} B$ (by Proposition 1.4 of [Bla]). By Lemma 2.6, we have that $\hat{f}|_{A\times \mathrm{cl}_{\beta(Y)} B}$ admits a continuous extension to $(\mathrm{cl}_{\beta(X)} A) \times (\mathrm{cl}_{\beta(Y)} B)$. □

To state the proof of the next theorem we need the following notions:

A subset A of a space X is *strongly bounded* in X if for every space Y and for each bounded subset B of Y, $A \times B$ is bounded in $X \times Y$ (this concept was introduced, in an equivalent form, by Tkachenko in [Tk]). Following [BS], we say that a space X has property (b) if for each space Y, the product $A \times B$ of each pair of bounded subsets $A \subseteq X$ and $B \subseteq Y$ is bounded in $X \times Y$. By Corollary 2 of [BS], X has property (b) if and only if every bounded subset of X is strongly bounded. According to Corollaries 4 and 6 of [BS], the k_r-spaces and spaces locally in the Frolík class $\mathscr{P}$ are examples of c_r-spaces which have property (b). Locally pseudocompact groups are also c_r-spaces enjoying property (b) [Sa3].

Theorem 2.17. *For a c_r-space X, the following assertions are equivalent:*

(1) *for each pseudocompact space P, $A \times P$ is C-compact in $X \times P$ for each C-compact subset A of X;*

(2) *for each pseudocompact space P, $A \times P$ is bounded in $X \times P$ for each C-compact subset A of X;*

(3) *every C-compact subset of X is strongly bounded in X;*

(4) *for each space Y, $A \times B$ is C-compact in $X \times Y$ for each C-compact subset A of X and each C-compact subset B of Y;*

(5) *for each space Y, $A \times B$ is bounded in $X \times Y$ for each C-compact subset A of X and each C-compact subset B of Y;*

(6) *for each pseudocompact space P, $A \times P$ is C-compact in $X \times P$ and*

$$\varrho(A \times P, X \times P) = \min\{\varrho(A, X), \varrho(P, P)\}$$

for each C-compact subset A of X;

(7) *for each space Y, $A \times B$ is C-compact in $X \times Y$ and*

$$\varrho(A \times B, X \times Y) = \min\{\varrho(A, X), \varrho(B, Y)\}$$

for each C-compact subset A of X and each C-compact subset B of Y.

Proof. The implications (1) $\Longrightarrow$ (2), (4) $\Longrightarrow$ (5) and (7) $\Longrightarrow$ (1) are clear. The implication (5) $\Longrightarrow$ (6) follows from Theorem 2.16 and (2) $\Longleftrightarrow$ (3) is a direct consequence of Proposition 1 from [BS].

(3) $\Longrightarrow$ (4). Since every C-compact subset of X is strongly bounded, for each space Y, $A\times B$ is bounded in $X\times Y$ for each C-compact subset A of X and each C-compact subset B of Y. So, the conclusion follows from (2) $\Longrightarrow$ (1) of Theorem 2.16.

(6) $\Longrightarrow$ (7). By Proposition 1 of [BS], every C-compact subset of X is strongly bounded. So, the result is a consequence of the equivalence (2) $\Longleftrightarrow$ (6) in Theorem 2.16. □

Corollary 2.18. *Let X be a c_r-space which has property (b). Then, for each space Y and each C-compact subset B of Y, $A \times B$ is C-compact in $X \times Y$ for each C-compact subset A of X.*

The following questions seem to be worth of study.

Question 2.19. *Is there a space X which has property (b) and does not satisfy the conclusions of Theorem* 2.17*?*

Question 2.20. *Is there an example of a C-compact subset $A \subseteq X$ such that $A \times P$ is C-compact in $X \times P$ for each pseudocompact space P but there exist a space Y and a C-compact subset B of Y such that $A \times B$ is not C-compact in $X \times Y$?*

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