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REMARK TO TRANSFORMATIONS OF LINEAR DIFFERENTIAL
AND FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. For linear differential and functional-differential equations of the n -th order criteria of equivalence with respect to the pointwise transformation are derived.

Keywords: ordinary differential equations, functional-differential equations, transformations

MSC 2000: 34A30, 34A34, 34K05, 34K15

1. INTRODUCTION

The theory of global pointwise transformations of homogeneous linear differential equations was developed in the monograph of F. Neuman [6] (see historical remarks, definitions, results and applications). The criterion of global equivalence of the second order equations was published by O. Borůvka [1], of the third and higher order equations by F. Neuman [6]. Transformations of functional-differential equations were studied in [2, 3, 4, 5, 7, 8]. In this paper we derive criteria of equivalence for ordinary differential equations and functional-differential equations of the first and higher orders, exploiting some results from [6, 9].

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2. NOTATION

Let \mathbf{V}_{n+1} denote an $(n+1)$ -dimensional vector space, $\vec{c} = [c_0, c_1, \dots, c_n]^T = [c_i]_{i=0}^n \in \mathbf{V}_{n+1}$ being a point, a vector of the space written in the column form; T means the transposition. Let \mathbf{V}_{n+1} be equipped with the scalar product $(\vec{p}, \vec{q}) = \sum_{i=0}^n p_i q_i$ for any pair \vec{p}, \vec{q} of its vectors. Let $\vec{p}_0, \vec{p}_1, \dots, \vec{p}_m$ be $m+1$ vectors from \mathbf{V}_{n+1} . Notation $P = [\vec{p}_0, \vec{p}_1, \dots, \vec{p}_m] = [p_{ij}]_{j=0, \dots, m}^{i=0, \dots, n}$ denotes a matrix and $(P, Q) = \sum_j^i p_{ij} q_{ij}$ the scalar product of two matrices of the same type. Similarly $P_{(j, \dots, k)} = [\vec{p}_j, \dots, \vec{p}_k]$ means a submatrix, $PQ = P_{(0, \dots, n)} Q_{(0, \dots, n)}$ is the matrix multiplication. Consider real functions $y \in C^{n+1}(I)$, $I \subseteq \mathbb{R}$ being an interval, $\xi_1, \xi_2, \dots, \xi_m \in C^n(I)$, $\xi_j: I \rightarrow I$, $\xi_0 = \text{id}_I$, $\xi_j \neq \xi_k$ for $j \neq k$; $j, k \in \{0, \dots, m\}$; $m, n \in \mathbb{N} = \{1, 2, \dots\}$. We denote $(y(\xi_j(x)))^{(i)} = d^i y(\xi_j(x))/dx^i$, $y^{(i)}(\xi_j(x)) = d^i y(\xi_j(x))/d\xi_j(x)^i$, $x \in I$ and $y_i(x) = y^{(i)}(x)$, $y_{ij}(x) = y^{(i)}(\xi_j(x))$. Then $\vec{y}(x) = [y_0(x), y_1(x), \dots, y_n(x)]^T = [y(x), y'(x), \dots, y^{(n)}(x)]^T \in \mathbf{V}_{n+1}$ for each $x \in I$ and $Y_{(j, \dots, k)}(x) = [\vec{y}(\xi_j(x)), \dots, \vec{y}(\xi_k(x))]$, $x \in I$.

3. PRELIMINARY RESULTS

Lemma 1 (A modification of Lemma 8.5.1, [6]). *Let $n \in \mathbb{N}$ and let the relation*

$$z(t) = L(t)y(\varphi(t))$$

be satisfied where the real functions $y: I \rightarrow \mathbb{R}$, $z: J \rightarrow \mathbb{R}$ belong to the classes $C^{n+1}(I)$, $C^{n+1}(J)$ respectively, and $L: J \rightarrow \mathbb{R}$, $L \in C^r(J)$, $L(t) \neq 0$ on J , and φ is a C^r diffeomorphism of J onto I for some integer $r \geq n+1$. Then

$$\begin{aligned} z^{(i)}(t) &= \sum_{j=0}^i a_{ij}(t)y^{(j)}(\varphi(t)) \\ &= a_{i0}(t)y(\varphi(t)) + a_{i1}(t)y'(\varphi(t)) + \dots + a_{ii}(t)y^{(i)}(\varphi(t)), \\ & \quad i \in \{0, 1, \dots, n+1\}, \end{aligned}$$

where

$$\begin{aligned} a_{00}(t) &= L(t), \dots, a_{i0}(t) = a'_{i-10}(t), \quad i \geq 1; \\ a_{ij}(t) &= a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t), \quad i > j > 1; \\ a_{ii}(t) &= a_{i-1i-1}(t)\varphi'(t), \quad i \in \{0, 1, \dots, n+1\} \end{aligned}$$

are real functions, $a_{ij}(t) \in C^{r-(i-j)-1}(J)$ for $j > 0$, and $a_{i0}(t) \in C^{r-i}(J)$. Moreover

$$\begin{aligned}
 a_{i0}(t) &= L^{(i)}(t), \quad i \geq 0; \\
 a_{i1}(t) &= (L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t) = \sum_{j=0}^{i-1} \binom{i}{j} L^{(j)}(t)\varphi^{(i-j)}(t), \quad i \geq 1; \\
 a_{ij}(t) &= \binom{i}{j} L^{(i-j)}(t)\varphi'(t)^j + \binom{i}{j-1} L(t)\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t) \\
 &\quad + r_{ij}(L, \dots, L^{(i-j-1)}, \varphi', \dots, \varphi^{(i-j)})(t), \quad i > j > 1; \\
 a_{ii-2}(t) &= \binom{i}{2} L''(t)\varphi'(t)^{i-2} + \binom{i}{3} (L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-3} \\
 &\quad + 3\binom{i}{4} L(t)\varphi'(t)^{i-4}\varphi''(t)^2, \quad i \geq 2; \\
 a_{ii-1}(t) &= \binom{i}{1} L'(t)\varphi'(t)^{i-1} + \binom{i}{2} L(t)\varphi'(t)^{i-2}\varphi''(t), \quad i \geq 2; \\
 a_{ii}(t) &= L(t)\varphi'(t)^i, \quad i \geq 0
 \end{aligned}$$

and

$$\begin{aligned}
 a_{i0}(t) &= a_{i0}(L^{(i)})(t), \quad i \geq 0; \\
 a_{ij}(t) &= a_{ij}(L, \dots, L^{(i-j)}, \varphi', \dots, \varphi^{(i-j+1)})(t), \quad i \geq j > 0; \quad i \in \{0, 1, \dots, n+1\}.
 \end{aligned}$$

P r o o f. From the relation

$$z(t) = a_{00}(t)y(\varphi(t)), \quad a_{00}(t) = L(t),$$

we have

$$z'(t) = a'_{00}(t)y(\varphi(t)) + a_{00}(t)\varphi'(t)y'(\varphi(t)) = a_{10}(t)y(\varphi(t)) + a_{11}(t)y'(\varphi(t)).$$

Suppose by induction that

$$\begin{aligned}
 z^{(i-1)}(t) &= a_{i-10}(t)y(\varphi(t)) + \dots + a_{i-1j-1}(t)y^{(j-1)}(\varphi(t)) \\
 &\quad + a_{i-1j}(t)y^{(j)}(\varphi(t)) + \dots + a_{i-1i-1}(t)y^{(i-1)}(\varphi(t)).
 \end{aligned}$$

Then

$$\begin{aligned}
 z^{(i)}(t) &= a'_{i-10}(t)y(\varphi(t)) + \dots + (a_{i-1j-1}(t)\varphi'(t) \\
 &\quad + a'_{i-1j}(t))y^{(j)}(\varphi(t)) + \dots + a_{i-1i-1}(t)\varphi'(t)y^{(i)}(\varphi(t)) \\
 &= a_{i0}(t)y(\varphi(t)) + \dots + a_{ij}(t)y^{(j)}(\varphi(t)) + \dots + a_{ii}(t)y^{(i)}(\varphi(t)).
 \end{aligned}$$

Hence

$$z^{(i)}(t) = \sum_{j=0}^i a_{ij}(t)y^{(j)}(\varphi(t))$$

where

$$\begin{aligned} a_{00}(t) &= L(t), \dots, a_{i0}(t) = a'_{i-10}(t), & i \geq 1; \\ a_{ij}(t) &= a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t), & i > j > 1; \\ a_{ii}(t) &= a_{i-1i-1}(t)\varphi'(t); & i \in \{0, 1, \dots, r\}. \end{aligned}$$

By induction we get

$$a_{00}(t) = L(t), \quad a_{10}(t) = a'_{00}(t) = L'(t), \quad \dots$$

and

$$a_{i0}(t) = a'_{i-10}(t) = L^{(i)}(t), \quad i \geq 0.$$

Similarly

$$a_{00}(t) = L(t), \quad a_{11}(t) = a_{00}(t)\varphi'(t) = L(t)\varphi'(t), \quad \dots$$

and

$$a_{ii}(t) = a_{i-1i-1}(t)\varphi'(t) = L(t)\varphi'(t)^i, \quad i \geq 0.$$

We have

$$a_{21}(t) = a'_{11}(t) + a_{10}(t)\varphi'(t) = 2L'(t)\varphi'(t) + L(t)\varphi''(t) = (L(t)\varphi(t))'' - L''(t)\varphi(t)$$

and by induction we get

$$a_{i1}(t) = (L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t); \quad i \geq 2$$

because

$$\begin{aligned} a_{i+11}(t) &= a'_{i1}(t) + a_{i0}(t)\varphi'(t) = ((L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t))' + L^{(i)}(t)\varphi'(t) \\ &= (L(t)\varphi(t))^{(i+1)} - L^{(i+1)}(t)\varphi(t). \end{aligned}$$

Then

$$\begin{aligned} a_{21}(t) &= 2L'(t)\varphi'(t) + L(t)\varphi''(t), \\ a_{32}(t) &= a'_{22}(t) + a_{21}(t)\varphi'(t) \\ &= \binom{3}{1}L'(t)\varphi'(t)^{3-1} + \binom{3}{2}L(t)\varphi'(t)^{3-2}\varphi''(t) \end{aligned}$$

and

$$a_{ii-1}(t) = \binom{i}{1}L'(t)\varphi'(t)^{i-1} + \binom{i}{2}L(t)\varphi'(t)^{i-2}\varphi''(t), \quad i \geq 2$$

since

$$\begin{aligned}
 a_{i+1i}(t) &= a'_{ii}(t) + a_{ii-1}(t)\varphi'(t) \\
 &= (L(t)\varphi'(t)^i)' + \binom{i}{1}L'(t)\varphi'(t)^i + \binom{i}{2}L(t)\varphi'(t)^{i-1}\varphi''(t) \\
 &= \binom{i+1}{1}L'(t)\varphi'(t)^i + \binom{i+1}{2}L(t)\varphi'(t)^{i-1}\varphi''(t).
 \end{aligned}$$

In the same way,

$$\begin{aligned}
 a_{20}(t) &= L''(t), \\
 a_{31}(t) &= a'_{21}(t) + a_{20}(t)\varphi'(t) = L'''(t)\varphi(t) + 3L'(t)\varphi''(t) + 3L''(t)\varphi'(t) \\
 &= \binom{3}{2}L''(t)\varphi'(t)^{3-2} + \binom{3}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{3-3}
 \end{aligned}$$

and

$$\begin{aligned}
 a_{ii-2}(t) &= \binom{i}{2}L''(t)\varphi'(t)^{i-2} + \binom{i}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-3} \\
 &\quad + 3\binom{i}{4}L(t)\varphi'(t)^{i-4}\varphi''(t)^2; \quad i \geq 2
 \end{aligned}$$

because then

$$\begin{aligned}
 a_{i+1i-1}(t) &= a'_{ii-1}(t) + a_{ii-2}(t)\varphi'(t) \\
 &= \left(i + \binom{i}{2}\right)L''(t)\varphi'(t)^{i-1} + \left(\binom{i}{2} + \binom{i}{3}\right)L(t)\varphi'(t)^{i-3}\varphi'''(t) \\
 &\quad + \left(i(i-1) + \binom{i}{2} + 3\binom{i}{3}\right)L'(t)\varphi'(t)^{i-2}\varphi''(t) \\
 &\quad + \left((i-2)\binom{i}{2} + 3\binom{i}{4}\right)L(t)\varphi'(t)^{i-3}\varphi''(t)^2 \\
 &= \binom{i+1}{2}L''(t)\varphi'(t)^{i-1} + \binom{i+1}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-2} \\
 &\quad + 3\binom{i+1}{4}L(t)\varphi'(t)^{i-3}\varphi''(t)^2.
 \end{aligned}$$

Using the above results we can suppose that

$$\begin{aligned}
 a_{ij}(t) &= \binom{i}{j}L^{(i-j)}(t)\varphi'(t)^j + \binom{i}{j-1}L(t)\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t) \\
 &\quad + r_{ij}(L, \dots, L^{(i-j-1)}, \varphi', \dots, \varphi^{(i-j)})(t).
 \end{aligned}$$

Then

$$\begin{aligned}
a_{i+1k}(t) &= a'_{ik}(t) + a_{ik-1}(t)\varphi'(t) \\
&= \left(\binom{i}{k} L^{(i-k)}(t)\varphi'(t)^k + \binom{i}{k-1} L(t)\varphi'(t)^{k-1}\varphi^{(i-k+1)}(t) \right. \\
&\quad \left. + r_{ik}(L, \dots, L^{(i-k-1)}, \varphi', \dots, \varphi^{(i-k)})(t) \right)' \\
&\quad + \left(\binom{i}{k-1} L^{(i-k+1)}(t)\varphi'(t)^{k-1} + \binom{i}{k-2} L(t)\varphi'(t)^{k-2}\varphi^{(i-k+2)}(t) \right. \\
&\quad \left. + r_{ik-1}(L, \dots, L^{(i-k)}, \varphi', \dots, \varphi^{(i-k+1)})(t) \right)\varphi'(t) \\
&= \left(\binom{i}{k} + \binom{i}{k-1} \right) L^{(i-k+1)}(t)\varphi'(t)^k \\
&\quad + \left(\binom{i}{k-1} + \binom{i}{k-2} \right) L(t)\varphi'(t)^{k-1}\varphi^{(i-k+2)}(t) \\
&\quad + [r'_{ik}(L, \dots, L^{(i-k-1)}, \varphi', \dots, \varphi^{(i-k)})(t) \\
&\quad + r_{ik-1}(L, \dots, L^{(i-k)}, \varphi', \dots, \varphi^{(i-k+1)})(t)\varphi'(t) \\
&\quad + k \binom{i}{k} L^{(i-k)}(t)\varphi'(t)^{k-1}\varphi''(t) + \binom{i}{k-1} L'(t)\varphi'(t)^{k-1}\varphi^{(i-k+1)}(t) \\
&\quad + (k-1) \binom{i}{k-1} L(t)\varphi'(t)^{k-2}\varphi''(t)\varphi^{(i-k+1)}(t)] \\
&= \binom{i+1}{k} L^{((i+1)-k)}(t)\varphi'(t)^k + \binom{i+1}{k-1} L(t)\varphi'(t)^{k-1}\varphi^{((i+1)-k+1)}(t) \\
&\quad + r_{i+1k}(L, \dots, L^{((i+1)-k+1)}, \varphi', \dots, \varphi^{((i+1)-k)})(t)
\end{aligned}$$

for every k , $0 < k \leq i+1$. Moreover,

$$a_{i0}(t) = a_{i0}(L^{(i)}(t)) = L^{(i)}(t).$$

We can suppose by induction that

$$a_{ij}(t) = a_{ij}(L(t), \dots, L^{(i-j)}(t), \varphi'(t), \dots, \varphi^{(i-j+1)}(t))$$

for $i \geq j > 0$ because $a_{i1}(t) = L(t)\varphi'(t) = a_{i1}(L(t), \varphi'(t))$ and

$$\begin{aligned}
a_{i+1k}(t) &= a'_{ik}(L(t), \dots, L^{(i-k)}(t), \varphi'(t), \dots, \varphi^{(i-k+1)}(t)) \\
&\quad + a_{ik-1}(L(t), \dots, L^{(i-k+1)}(t), \varphi'(t), \dots, \varphi^{(i-k+2)}(t))\varphi'(t) \\
&= a_{i+1k}(L(t), \dots, L^{((i+1)-k)}(t), \varphi'(t), \dots, \varphi^{((i+1)-k+1)}(t))
\end{aligned}$$

for every k , $0 < k \leq i+1$.

Finally,

$$a_{i0}(t) = L^i(t) \in C^{r-i}(J),$$

$$a_{ij}(t) = a_{ij}(L(t), \dots, L^{(i-j)}(t), \varphi'(t), \dots, \varphi^{(i-j+1)}(t)) \in C^{r-(i-j)-1}(J)$$

for $j > 0$, with respect to the assumptions $L, \varphi \in C^r(J)$. □

4. LINEAR DIFFERENTIAL EQUATIONS

Consider two ordinary homogeneous linear differential equations

$$(1) \quad y_{n+1}(x) = (\vec{p}(x), \vec{y}(x)) = p_0(x)y_0(x) + p_1(x)y_1(x) + \dots + p_n(x)y_n(x),$$

$$y_i(x) = y^{(i)}(x), \quad x \in I;$$

$$(2) \quad z_{n+1}(t) = (\vec{q}(t), \vec{z}(t)) = q_0(t)z_0(t) + q_1(t)z_1(t) + \dots + q_n(t)z_n(t),$$

$$z_i(t) = z^{(i)}(t), \quad t \in J$$

with real coefficients, defined on an interval $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, respectively.

Definition 1 ([6], p. 15). We say that (1) is globally transformable into (2) if there exist two functions L, φ such that

- the function L is of the class $C^{n+1}(J)$ and is nonvanishing on J ,
- the function φ is a C^{n+1} diffeomorphism of the interval J onto I ,

and the function

$$(3) \quad z(t) = L(t)y(\varphi(t))$$

is a solution of (2) whenever y is a solution of (1).

If (1) is globally transformable into (2) then we say that (1), (2) are *equivalent equations*. We say that (3) is a *stationary transformation* if it globally transforms an equation (1) into itself on I , i.e. if $L \in C^{n+1}(I)$, $L(x) \neq 0$ on I , φ is a C^{n+1} diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = L(x)y(\varphi(x))$ is a solution of $z_{n+1}(x) = (\vec{p}(x), \vec{z}(x))$ whenever y is a solution of $y_{n+1}(x) = (\vec{p}(x), \vec{y}(x))$, $x \in I$.

Theorem 1. Let $n, r \in \mathbb{N}$ and $r \geq n + 1$. Let L, φ satisfy the assumptions of Lemma 1. Then (1) is globally transformable into (2) by means of a transformation $z(t) = L(t)y(\varphi(t))$ if and only if

$$(4) \quad \vec{a}_{n+1}(t) = A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)), \quad t \in J$$

is satisfied for the vectors of coefficients of the equations (1), (2) and

$$A(t) = [a_{ij}(t)]_{j=0, \dots, n}^{i=0, \dots, n}, \vec{a}_{n+1}(t) = [a_{n+10}(t), a_{n+11}(t), \dots, a_{n+1n}(t)]^T$$

where the functions $a_{ij}(t)$ are defined by Lemma 1.

P r o o f. Using Lemma 1, we obtain

$$z_i(t) = z^{(i)}(t) = \sum_{j=0}^i a_{ij}(t)y^{(j)}(\varphi(t)) = \sum_{j=0}^i a_{ij}(t)y_j(\varphi(t)), i \in \{0, 1, \dots, n+1\}, t \in J.$$

Thus

$$\begin{aligned} \vec{z}(t) &= [z_i(t)]_{i=0}^n = A(t)\vec{y}(\varphi(t)), \\ z_{n+1}(t) &= a_{n+10}(t)y_0(\varphi(t)) + \dots + a_{n+1n}(t)y_n(\varphi(t)) + a_{n+1n+1}(t)y_{n+1}(\varphi(t)) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t)(\vec{p}(\varphi(t)), \vec{y}(\varphi(t))) \\ &= (\vec{q}(t), \vec{z}(t)) = (\vec{q}(t), A(t)\vec{y}(\varphi(t))), \quad t \in J. \end{aligned}$$

Hence

$$(\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) = (A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)), \vec{y}(\varphi(t)))$$

and

$$(5) \quad \vec{a}_{n+1}(t) = A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)),$$

i.e.

$$\begin{aligned} a_{n+10}(t) &= a_{00}(t)q_0(t) + a_{10}(t)q_1(t) + \dots + a_{n0}(t)q_n(t) - a_{n+1n+1}(t)p_0(\varphi(t)), \\ a_{n+11}(t) &= a_{11}(t)q_1(t) + \dots + a_{n1}(t)q_n(t) - a_{n+1n+1}(t)p_1(\varphi(t)), \\ &\dots \\ a_{n+1n}(t) &= a_{nn}(t)q_n(t) - a_{n+1n+1}(t)p_n(\varphi(t)), \\ &t \in J. \end{aligned}$$

□

Corollary 1. If (1) is globally transformable into (2) by means of the transformation (3), then the $(n+1)$ -st derivatives ($n \geq 1$) of the functions L and φ satisfy

differential equations

$$\begin{aligned}
 L^{(n+1)}(t) &= h(t, \varphi(t), \varphi'(t), L(t), \dots, L^{(n)}(t)) \\
 &= q_0(t)L(t) + \dots + q_n(t)L^{(n)}(t) - L(t)\varphi'(t)^{n+1}p_0(\varphi(t)); \\
 \varphi^{(n+1)}(t) &= g(t, \varphi(t), \dots, \varphi^{(n)}(t), L(t), \dots, L^{(n)}(t)) \\
 &= \frac{1}{L(t)} \sum_{k=1}^n (a_{k1}(t)q_k(t) - \binom{n+1}{k} L^{(k)}(t)\varphi^{(n+1-k)}(t)) - p_1(\varphi(t))\varphi'(t)^{n+1}, \\
 a_{k1}(t) &= a_{k1}(L, \dots, L^{(k-1)}, \varphi', \dots, \varphi^{(k)})(t); \quad t \in J.
 \end{aligned}$$

We obtain the assertion using the first two equations in (5) together with the definitions of the functions $a_{ij}(t)$ in Lemma 1.

Corollary 2. The transformation (3) is a stationary transformation of (1) if and only if $\varphi(I) = I$ and the functions $L, \varphi \in C^{n+1}(I)$ satisfy the conditions

$$\bar{a}_{n+1}(x) = A^T(x)\bar{p}(x) - a_{n+1n+1}(x)\bar{p}(\varphi(x)), \quad x \in I,$$

where the functions a_{ij} are defined by Lemma 1; $i, j \in \{0, 1, \dots, n+1\}$.

Remark 1. Using Theorem 1, if we suppose that $q_n(t) \equiv 0$ on J , $p_n(x) \equiv 0$ on I , then $a_{n+1n}(t) = (n+1)\varphi'(t)^{n-1}(L''(t)\varphi(t) + \frac{n}{2}L(t)\varphi(t)) \equiv 0$ on J and $a_{n+1n}(t) = -\{\varphi, t\} \binom{n+2}{3} L(t)\varphi'(t)^{n-1}$ for $\varphi \in C^r(J)$, $r > n+1$ ($r \geq n+1$ if $n > 1$), $\varphi'(t) \neq 0$, where the symbol $\{\varphi, t\} := \frac{1}{2} \frac{\varphi'''(t)}{\varphi'(t)} - \frac{3}{4} \left(\frac{\varphi''(t)}{\varphi'(t)} \right)^2$ is used for the Schwarzian derivative of φ ; $t \in J$ (See [6], p. 7). In the case $n = 1$ we obtain for $\varphi \in C^3(J)$ the Kummer equation $q_0(t) = p_0(\varphi(t))\varphi'(t)^2 - \{\varphi, t\}$, $t \in J$ (see [6], p. 89). We can obtain more useful formulas (as in [6], pp. 49–52) using (5).

5. LINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

Let $y \in C^{n+1}(I)$; $\xi_1, \xi_2, \dots, \xi_m \in C^n(I)$, $I \subseteq \mathbb{R}$ being an interval, $\xi_j: I \rightarrow I$, $\xi_0 = \text{id}_I$, $\xi_j \neq \xi_k$ for $j \neq k$; $j, k \in \{0, \dots, m\}$; $m, n \in \mathbb{N}$. We consider a homogeneous linear functional-differential equation with real coefficients p_i , p_{ij} and m delays ξ_1, \dots, ξ_m of the form

$$\begin{aligned}
 y_{n+1}(x) &= (\bar{p}(x), \bar{y}(x)) + (P(x), Y_{(1, \dots, m)}(x)) \\
 &= ([p_i(x)]_{i=0}^n, [y_i(x)]_{i=0}^n) + ([p_{ij}(x)]_{j=1, \dots, m}^{i=0, \dots, n}, [y_i(\xi_j(x))]_{j=1, \dots, m}^{i=0, \dots, n}) \\
 &= \sum_{i=0}^n (p_i(x)y_i(x) + \sum_{j=1}^m p_{ij}(x)y_i(\xi_j(x))), \quad y_i(x) = y^{(i)}(x), \quad x \in I.
 \end{aligned}$$

Consider two such equations

$$(6) \quad y_{n+1}(x) = (\vec{p}(x), \vec{y}(x)) + (P(x), Y(x)), \quad Y(x) = [y_i(\xi_j(x))]_j^i, \quad x \in I,$$

$$(7) \quad z_{n+1}(t) = (\vec{q}(t), \vec{z}(t)) + (Q(t), Z(t)), \quad Z(t) = [z_i(\eta_j(t))]_j^i, \quad t \in J.$$

In accordance with Definition 1 (see [2], [5], [7], [8]), an equation (6) is globally transformable into an equation (7) with respect to the transformation $z(t) = L(t)y(\varphi(t))$, if $L \in C^{n+1}(J)$, $L(t) \neq 0$ on J , φ is a C^{n+1} diffeomorphism of J onto I and the function z is a solution of (7) whenever y is a solution of (6). Moreover,

$$(8) \quad \xi_j(\varphi(t)) = \varphi(\eta_j(t)), \quad j = 1, 2, \dots, m; \quad t \in J$$

is true for the deviations $\xi_j(x)$, $\eta_j(t)$ and the function φ .

Theorem 2. *Let $m, n, r \in \mathbb{N}$ and $r \geq n + 1$. Let L, φ satisfy the assumptions of Lemma 1. Then (6) is globally transformable into (7) by means of a transformation $z(t) = L(t)y(\varphi(t))$ if and only if the function φ is a solution of*

$$\xi_j(\varphi(t)) = \varphi(\eta_j(t)), \quad j = 1, 2, \dots, m; \quad t \in J, \quad \varphi(J) = I,$$

and

$$(9) \quad \vec{a}_{n+1}(t) = A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t));$$

$$(10) \quad A^T(\eta_j(t))\vec{q}_j(t) = a_{n+1n+1}(t)\vec{p}_j(\varphi(t)), \quad j = 1, 2, \dots, m$$

are satisfied on J , where $a_{ij}(t)$ are the functions defined by Lemma 1, $\vec{a}_{n+1}(t) = [a_{n+10}(t), a_{n+11}(t), \dots, a_{n+1n}(t)]^T$, $\vec{p}_j(\varphi(t))$ and $\vec{q}_j(t)$ denotes the j -th column of the matrix $P(\varphi(t))$ and $Q(t)$ respectively.

P r o o f. By using Lemma 1 we have

$$z_i(t) = z^{(i)}(t) = \sum_{j=0}^i a_{ij}(t)y^{(j)}(\varphi(t)) = \sum_{j=0}^i a_{ij}(t)y_j(\varphi(t)), \quad i \in \{0, 1, \dots, n+1\}, \quad t \in J.$$

Thus

$$\vec{z}(t) = [z_i(t)]_{i=0}^n = A(t)\vec{y}(\varphi(t))$$

and for $\xi_j(\varphi(t)) = \varphi(\eta_j(t))$ we get

$$\vec{z}(\eta_j(t)) = A(\eta_j(t))\vec{y}(\varphi(\eta_j(t))) = A(\eta_j(t))\vec{y}(\xi_j(\varphi(t))); \quad j = 1, 2, \dots, m; \quad t \in J.$$

Then

$$\begin{aligned}
z_{n+1}(t) &= (\bar{a}_{n+1}(t), \bar{y}(\varphi(t))) + a_{n+1n+1}(t)y_{n+1}(\varphi(t)) \\
&= (\bar{a}_{n+1}(t), \bar{y}(\varphi(t))) + a_{n+1n+1}(t)(\bar{p}(\varphi(t)), \bar{y}(\varphi(t))) \\
&\quad + a_{n+1n+1}(t)(P(\varphi(t)), Y(\varphi(t))) \\
&= (\bar{a}_{n+1}(t) + a_{n+1n+1}(t)\bar{p}(\varphi(t)), \bar{y}(\varphi(t))) \\
&\quad + \sum_{j=1}^n (a_{n+1n+1}(t)\bar{p}_j(\varphi(t)), \bar{y}(\xi_j(\varphi(t))))), \quad t \in J
\end{aligned}$$

where \bar{p}_j denotes the j -th column of the matrix P . Hence

$$\begin{aligned}
z_{n+1}(t) &= (\bar{q}(t), \bar{z}(t)) + (Q(t), Z(t)) = (\bar{q}(t), \bar{z}(t)) + \sum_{j=1}^m (\bar{q}_j(t), \bar{z}(\eta_j(t))) \\
&= (\bar{q}(t), A(t)\bar{y}(\varphi(t))) + \sum_{j=1}^m (\bar{q}_j(t), A(\eta_j(t))\bar{y}(\xi_j(\varphi(t)))) \\
&= (A^T(t)\bar{q}(t), \bar{y}(\varphi(t))) + \sum_{j=1}^m (A^T(\eta_j(t))\bar{q}_j(t), \bar{y}(\xi_j(\varphi(t)))).
\end{aligned}$$

Comparison of the last two expressions yields

$$\begin{aligned}
&(\bar{a}_{n+1}(t) + a_{n+1n+1}(t)\bar{p}(\varphi(t)) - A^T(t)\bar{q}(t), \bar{y}(\varphi(t))) \\
&+ \sum_{j=1}^m (a_{n+1n+1}(t)\bar{p}_j(\varphi(t)) - A^T(\eta_j(t))\bar{q}_j(t), \bar{y}(\xi_j(\varphi(t)))) = 0, \quad t \in J.
\end{aligned}$$

Thus

$$\begin{aligned}
\bar{a}_{n+1}(t) &= A^T(t)\bar{q}(t) - a_{n+1n+1}(t)\bar{p}(\varphi(t)), \\
A^T(\eta_j(t))\bar{q}_j(t) &= a_{n+1n+1}(t)\bar{p}_j(\varphi(t))
\end{aligned}$$

on J and the assertion of Theorem 2 is proved. \square

Corollary 3. *If (6) is globally transformable into (7) by means of the transformation (3), then the $(n+1)$ -st derivatives ($n \in \mathbb{N}$) of the functions L, φ satisfy the*

differential equations

$$\begin{aligned}
 L^{(n+1)}(t) &= h(t, \varphi(t), \varphi'(t), L(t), \dots, L^{(n)}(t)) \\
 &= q_0(t)L(t) + \dots + q_n(t)L^{(n)}(t) - L(t)\varphi'(t)^{n+1}p_0(\varphi(t)); \\
 \varphi^{(n+1)}(t) &= g(t, \varphi(t), \dots, \varphi^{(n)}(t), L(t), \dots, L^{(n)}(t)) \\
 &= \frac{1}{L(t)} \sum_{k=1}^n (a_{k1}(t)q_k(t) - \binom{n+1}{k} L^{(k)}(t)\varphi^{(n+1-k)}(t)) - p_1(\varphi(t))\varphi'(t)^{n+1}, \\
 a_{k1}(t) &= a_{k1}(L, \dots, L^{(k-1)}, \varphi', \dots, \varphi^{(k)})(t); \quad t \in J.
 \end{aligned}$$

Moreover,

$$\varphi'(t) = G(t, \varphi(t), L(t), [L^{(i)}(\eta_j(t))]_{j=1, \dots, m}^{i=0, \dots, n}), \quad t \in J.$$

We obtain the assertion using the first two equations in (9) and the first equation in (10) together with the definitions of the functions $a_{ij}(t)$ in Lemma 1.

Corollary 4. *The transformation (3) is a stationary transformation of (6) if and only if the functions $L, \varphi \in C^{n+1}(I)$ satisfy the conditions*

$$\begin{aligned}
 \xi_j(\varphi(x)) &= \varphi(\xi_j(x)), \varphi(I) = I, \\
 \vec{a}_{n+1}(x) &= A^T(x)\vec{p}(x) - a_{n+1n+1}(x)\vec{p}(\varphi(x)), \\
 A^T(\xi_j(x))\vec{p}_j(x) &= a_{n+1n+1}(x)\vec{p}_j(\varphi(x)), \quad x \in I,
 \end{aligned}$$

where the functions a_{ij} are defined by Lemma 1 and \vec{p}_j denotes the j -th column of the matrix P ; $i, j \in \{0, 1, \dots, n+1\}, x \in I$.

Remark 2 (see [7]; pp. 355, 357.). In a situation when the deviating arguments in equation (6) are constant deviations

$$\xi_j(x) = x - c_j, \quad c_j \in \mathbb{R} - \{0\}; \quad j = 1, 2, \dots, m,$$

the condition $\xi_j(\varphi(t)) = \varphi(\eta_j(t))$ become a system of Abel equations

$$\varphi(\eta_j(t)) = \varphi(t) - c_j; \quad j = 1, 2, \dots, m.$$

When the deviating arguments in (7) are

$$\eta_j(t) = t - d_j, \quad d_j \in \mathbb{R} - \{0\},$$

then we get

$$\varphi(t - d_j) = \varphi(t) - c_j; \quad j = 1, 2, \dots, m.$$

If we require that the delayed arguments be converted into delayed ones (or the advanced into advanced), then we need $\varphi'(t) > 0$, $t \in J$. Let d_j/d_k be irrational for a pair $j, k \in \{1, 2, \dots, m\}$. Then for each fixed $j \in \{1, 2, \dots, m\}$, the Abel equation $\varphi(t - d_j) = \varphi(t) - c_j$ has a general solution $\varphi \in C^{n+1}(J)$, $\varphi'(t) > 0$, $\varphi(J) = I$, of the form

$$\varphi(t) = \frac{c_j}{d_j}t + k, \quad k \in \mathbb{R}.$$

For the existence of a simultaneous solution φ it is then sufficient and necessary to have $\varrho = c_j/d_j$ (a constant not depending on φ) for all $j \in \{1, 2, \dots, m\}$.

Example. Consider two equations

$$y''(x) = \left(\begin{bmatrix} p_0 \\ p_1 \end{bmatrix}, \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} \right) + \left(\begin{bmatrix} p_{01} & p_{02} \\ p_{11} & p_{12} \end{bmatrix}, \begin{bmatrix} y(x - c_1) & y(x - c_2) \\ y'(x - c_1) & y'(x - c_2) \end{bmatrix} \right), \\ x \in I = [a, \infty)$$

and

$$z''(t) = \left(\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}, \begin{bmatrix} z(t) \\ z'(t) \end{bmatrix} \right) + \left(\begin{bmatrix} q_{01} & q_{02} \\ q_{11} & q_{12} \end{bmatrix}, \begin{bmatrix} z(t - d_1) & z(t - d_2) \\ z'(t - d_1) & z'(t - d_2) \end{bmatrix} \right), \\ t \in J = [b, \infty)$$

with constant coefficients and deviations; $c_1, c_2, d_1, d_2 \in \mathbb{R} - \{0\}$, $\frac{c_1}{d_1} > 0$, $\frac{d_1}{d_2}$ being irrational. Due to Theorem 2 they are equivalent with respect to the transformation (3) if and only if

$$\begin{aligned} \varphi(t - d_j) &= \varphi(t) - c_j, \\ \begin{bmatrix} a_{20}(t) \\ a_{21}(t) \end{bmatrix} &= \begin{bmatrix} a_{00}(t) & a_{10}(t) \\ 0 & a_{11}(t) \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} - a_{22}(t) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}, \\ \begin{bmatrix} a_{00}(t - d_j) & a_{10}(t - d_j) \\ 0 & a_{11}(t - d_j) \end{bmatrix} \cdot \begin{bmatrix} q_{0j} \\ q_{1j} \end{bmatrix} &= a_{22}(t) \begin{bmatrix} p_{0j} \\ p_{1j} \end{bmatrix}, \end{aligned}$$

$j = 1, 2$, where $a_{00}(t) = L(t)$, $a_{10}(t) = L'(t)$, $a_{11}(t) = L(t)\varphi'(t)$, $a_{20}(t) = L''(t)$, $a_{21}(t) = 2L'(t)\varphi'(t) + L(t)\varphi''(t)$, $a_{22}(t) = L(t)\varphi'(t)^2$, $t \in J$. In accordance with Remark 2 we have $\varphi(t) = \varrho t + k$, $\varrho = \frac{c_1}{d_1} = \frac{c_2}{d_2} > 0$, $k \in \mathbb{R} - \{0\}$ and $\varphi(t) = \varrho \cdot (t - b) + a$ for $\varphi(J) = I$.

Then

$$\begin{aligned} L''(t) &= q_0L(t) + q_1L'(t) - p_0\varrho^2L(t), \\ 2L'(t) &= (q_1 - p_1\varrho)L(t), \\ q_{0j}L(t - d_j) + q_{1j}L'(t - d_j) &= p_{0j}\varrho^2L(t), \\ q_{1j}L(t - d_j) &= p_{1j}\varrho L(t), \quad j = 1, 2; t \in J. \end{aligned}$$

Using the second equation we obtain $L(t) = c \exp\{kt\}$, $k = \frac{q_1 - p_1 \varrho}{2}$, $c \in \mathbb{R} - \{0\}$, and

$$\begin{aligned} k^2 &= q_0 + q_1 k - p_0 \varrho^2, \\ q_{1j} &= p_{1j} \varrho \exp\{k d_j\}, \\ q_{0j} + q_{1j} k &= p_{0j} \varrho^2 \exp\{k d_j\}, \quad j = 1, 2. \end{aligned}$$

Thus the equations are equivalent with respect to the transformation $z(t) = L(t)y(\varphi(t))$ if and only if there exists $\varrho \in \mathbb{R}$, $\varrho > 0$, such that $c_j = \varrho d_j$,

$$\begin{aligned} q_1^2 + 4q_0 &= (p_1^2 + 4p_0) \varrho^2, \\ q_{0j} &= \varrho \left(p_{0j} \varrho - \frac{1}{2} p_{1j} (q_1 - p_1 \varrho) \right) \exp\left\{ \frac{q_1 - p_1 \varrho}{2} d_j \right\}, \\ q_{1j} &= p_{1j} \varrho \exp\left\{ \frac{q_1 - p_1 \varrho}{2} d_j \right\}, \quad j = 1, 2. \end{aligned}$$

For the functions L , φ we have

$$L(t) = c \exp\left\{ \frac{q_1 - p_1 \varrho}{2} t \right\}, \quad \varphi(t) = \varrho(t - b) + a; \quad c \in \mathbb{R} - \{0\}.$$

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