Václav Tryhuk

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ON GLOBAL TRANSFORMATIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Václav Tryhuk, Brno

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Abstract. The paper describes the general form of functional-differential equations of the first order with $m(m \geqslant 1)$ delays which allows nontrivial global transformations consisting of a change of the independent variable and of a nonvanishing factor. A functional equation

$$f(t, uv, u_1v_1, \dots, u_mv_m) = f(x, v, v_1, \dots, v_m)g(t, x, u, u_1, \dots, u_m)u + h(t, x, u, u_1, \dots, u_m)v$$

for $u \neq 0$ is solved on $\mathbb{R}$ and a method of proof by J. Aczél is applied.

Keywords: functional differential equations, ordinary differential equations, global transformations, functional equations in a single variable, functional equations in several variables

MSC 2000: Primary: 34-02, Secondary: 34K15, 39B40

1. Introduction

The theory of global transformations converting any homogeneous linear differential equation of the n-th order into another equation of the same kind and order on the whole interval of their definition, was developed in the monograph of F. Neuman [8] (see historical remarks, definitions, results and some applications). The most general form of global pointwise transformations for homogeneous linear differential equations of the n-th order $(n \geqslant 2)$ is

$$z(t) = L(t)y(\varphi(t)),$$

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where φ is a bijection of an interval J onto an interval I ($J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$) and $L(t)$ is a nonvanishing function on J, i.e. this global transformation consists of a change of the independent variable and of a nonvanishing factor L. The form of the most general pointwise transformation of homogeneous linear differential equations with deviating arguments was derived in [3, 9, 10, 11]. This form coincides for an arbitrary order with the form considered for linear differential equations of the n-th ($n \geqslant 2$) order without deviation.

In this paper we derive, similarly to Aczél [1], the general form of differential equations of the first order with deviating arguments

$$y'(x) = f(x, y(x), y(\xi_1(x)), \ldots, y(\xi_m(x)))$$

which allows the transformation $z(t) = L(t)y(\varphi(t))$.

2. Notation, basic definitions

Consider two functional-differential equations with $m (m \geqslant 1)$ deviating arguments

$$y'(x) = f(x, y(x), y(\xi_1(x)), \ldots, y(\xi_m(x))) \tag{1}$$

and

$$z'(t) = f^*(t, z(t), z(\eta_1(t)), \ldots, z(\eta_m(t))) \tag{2}$$

defined on I and J respectively.

Definition. We say that (1) is globally transformable into (2) if there exist two functions φ, L such that

– the function L is of the class $C^1(J)$ and is nonvanishing on J;

– the function φ is a C^1 diffeomorphism of the interval J onto I,

and the function

$$z(t) = L(t)y(\varphi(t)) \tag{3}$$

is a solution of (2) whenever $y(x)$ is a solution of (1).

We say that (3) is a *stationary transformation* if it globally transforms the equation (1) into itself on I, i.e. if $L \in C^1(I), L(x) \neq 0$ on I, φ is a C^1 diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = L(x)y(\varphi(x))$ is a solution of

$$z'(x) = f(x, z(x), z(\xi_1(x)), \ldots, z(\xi_m(x)))$$

whenever y is a solution of

$$y'(x) = f(x, y(x), y(\xi_1(x)), \dots, y(\xi_m(x))), x \in I.$$

Hence, if (3) is a stationary transformation of the equation (1) with a solution $y(x)$, then $L(x)y(\varphi(x))$ is also a solution of the same differential equation.

If (1) is globally transformable into (2), then (see [3, 5, 7, 9, 10, 11])

$$\xi_j(\varphi(t)) = \varphi(\eta_j(t)) \tag{4}$$

is satisfied on J for deviations ξ_j, η_j; $j = 1, 2, \dots, m$, and we say that (1), (2) *are equivalent equations.*

Observation 1. Every homogeneous linear functional-differential equation of the first order is a particular case of the equation (1). Consider two functional-differential equations

(a) $y'(x) = f(x, y(x), y(\xi_1(x)), \dots, y(\xi_m(x))) = p_0(x)y(x) + \sum\limits_{j=1}^{m} p_j(x)y(\xi_j(x))$, $x \in I$,

(b) $z'(t) = f^*(t, z(t), z(\eta_1(t)), \dots, z(\eta_m(t))) = q_0(t)z(t) + \sum\limits_{j=1}^{m} q_j(t)z(\eta_j(t))$, $t \in J$.

If (a) is globally transformable into (b), then the functions φ, L satisfy $L'(t) = (p_0(\varphi(t))\varphi'(t) - q_0(t))L(t)$ and $p_j(\varphi(t))\varphi'(t) = q_j(t)L(\eta_j(t))/L(t)$, $j = 1, 2, \dots, m$, on J. Thus φ' is a function depending on φ, L, $L(\eta_1), \dots, L(\eta_m)$, i.e.

$$\varphi'(t) = g(t, \varphi(t), L(t), L(\eta_1(t)), \dots, L(\eta_m(t)))$$

and

$$\begin{aligned} L'(t) &= h_1(t, \varphi(t), \varphi'(t), L(t)) \\ &= h_1(t, \varphi(t), g(t, \varphi(t), L(t), L(\eta_1(t)), \dots, L(\eta_m(t))), L(t)) \\ &= h(t, \varphi(t), L(t), L(\eta_1(t)), \dots, L(\eta_m(t))) \end{aligned}$$

on J.

Assumption. For transformations of homogeneous functional-differential equations of the first order we assume that there exist two differential equations such that

$$\begin{aligned} \varphi'(t) &= g(t, \varphi(t), L(t), L(\eta_1(t)), \dots, L(\eta_m(t))), \\ L'(t) &= h(t, \varphi(t), L(t), L(\eta_1(t)), \dots, L(\eta_m(t))) \end{aligned}$$

on J.

Lemma 1. *The transformation* (3) *is a stationary transformation of the equation* (1) *if and only if* $\xi_j(\varphi(t)) = \varphi(\xi_j(t))$ *on* $I = \varphi(I)$ *and the real functions* f, g, h *satisfy a functional equation in several variables*

$$f(t, uv, u_1v_1, \dots, u_mv_m) = f(x, v, v_1, \dots, v_m)g(t, x, u, u_1, \dots, u_m)u + h(t, x, u, u_1, \dots, u_m)v \tag{5}$$

for t, x, u, $u_1, \dots, u_m$, v, $v_1, \dots, v_m \in \mathbb{R}$, $u \neq 0$; $j = 1, 2, \dots, m$.

Proof. There exists a global stationary transformation of the equation (1) if and only if (1) is globally transformable into the equation $z'(t) = f(t, z(t), z(\xi_1(t)), \dots, z(\xi_m(t)))$, $t \in I = \varphi(I)$, by means of (3) and $\xi_j(\varphi(t)) = \varphi(\xi_j(t))$, $j = 1, 2, \dots, m$, $t \in I$. According to the definition of a global transformation,

$$\begin{aligned}
&\varphi(\xi_j(t)) = \xi_j(\varphi(t)),\\
&z(\xi_j(t)) = L(\xi_j(t))y(\varphi(\xi_j(t))) = L(\xi_j(t))y(\xi_j(\varphi(t))) = L(\xi_j(t))y(\xi_j(x)),
\end{aligned}$$

$i = 1, 2, \dots, m$; $t \in J$.

We denote $z_j = z(\xi_j)$, $x = \varphi$, $u_j = L(\xi_j)$, $v = y(\varphi)$, $v_j = y(\xi_j(\varphi)) = y(\varphi(\xi_j))$, $j = 1, 2, \dots, m$. Then $z' = L'y(\varphi) + L\dot{y}(\varphi)\varphi' \quad (\cdot = \mathrm{d}/\mathrm{d}\varphi)$ implies that

$$\begin{aligned}
z'(t) &= f(t, z, z_1, \dots, z_m) = f(t, uv, u_1v_1, \dots, u_mv_m)\\
&= f(x, v, v_1, \dots, v_m)g(t, x, u, u_1, \dots, u_m)u + h(t, x, u, u_1, \dots, u_m)v
\end{aligned}$$

and we obtain the functional equation (5). □

Theorem 1. *The general continuous solution of the functional equation* (5) *is of the form*

$$\begin{aligned}
f(t, v, v_1, \dots, v_m) &= a(t)b(v)\delta(v_1, \dots, v_m) + q(t)v,\\
g(t, x, u, u_1, \dots, u_m) &= a(t)b(u)\delta(u_1, \dots, u_m)/(a(x)u),\\
h(t, x, u, u_1, \dots, u_m) &= u[q(t) - q(x)g(t, x, u, u_1, \dots, u_m)],
\end{aligned}$$

where a, q *are arbitrary functions on* $J \subseteq \mathbb{R}$ $(a(x) \neq 0)$ *and functions* b, δ *are continuous solutions of Cauchy's power equations*

$$b(uv) = b(u)b(v)$$

and

$$\delta(u_1v_1,\dots,u_mv_m)=\delta(u_1,\dots,u_m)\delta(v_1,\dots,v_m)$$

respectively.

P r o o f. We have the functional equation (5), i.e.

$$f(t,uv,u_1v_1,\dots,u_mv_m)=f(x,v,v_1,\dots,v_m)g(t,x,u,u_1,\dots,u_m)u + h(t,x,u,u_1,\dots,u_m)v$$

on the domain as above, moreover let $u\neq 0$. Choosing $u_i=1 (i=1,2,\dots,m)$ and $x=1$ we have

$$f(t,uv,v_1,\dots,v_m)=f(1,v,v_1,\dots,v_m)\tilde g(t,u)u+\tilde h(t,u)v \tag{6}$$

where $\tilde g(t,u)=g(t,1,u,1,\dots,1), \tilde h(t,u)=h(t,1,u,1,\dots,1)$. Then (6) and $v=1$ give

$$f(t,u,v_1,\dots,v_m)=\delta^*(v_1,\dots,v_m)\tilde g(t,u)u+\tilde h(t,u), \tag{7}$$

$\delta^*(v_1,\dots,v_m)=f(1,1,v_1,\dots,v_m)$. If we combine (7) with (6) we get

$$\begin{aligned}\delta^*(v_1,\dots,v_m)&\tilde g(t,uv)uv+\tilde h(t,uv)\\ &=[\delta^*(v_1,\dots,v_m)\tilde g(1,v)v+\tilde h(1,v)]\tilde g(t,u)u+\tilde h(t,u)v,\end{aligned}$$

and

$$\tilde g(t,uv)=\tilde g(1,v)\tilde g(t,u), \tag{8}$$

$$\tilde h(t,uv)=\tilde h(1,v)\tilde g(t,u)u+\tilde h(t,u)v \tag{9}$$

are satisfied because $\delta^*(v_1,\dots,v_m)$ is independent of v.

First we solve the functional equation (8). For $u=1$ we have

$$\tilde g(t,v)=a(t)\tilde b(v) \tag{10}$$

where $a(t)=\tilde g(t,1)$ and $\tilde b(v)=\tilde g(1,v)$. Equations (8) and (10) imply that $a(t)\tilde b(uv)=a(t)\tilde b(u)\tilde b(v)$ and we obtain Cauchy's power equation

$$\tilde b(uv)=\tilde b(u)\tilde b(v). \tag{11}$$

Then the second functional equation (9) is of the form

$$\tilde h(t,uv)=a(t)\tilde b(u)ud(v)+\tilde h(t,u)v, \tag{12}$$

where $d(v) = \tilde{h}(1, v)$ and the function $\tilde{b}$ satisfies (11). Choosing $u = 1$ we obtain

$$\tilde{h}(t, v) = a(t)\tilde{b}(1)d(v) + c(t)v, \tag{13}$$

$c(t) = \tilde{h}(t, 1)$. Substituting (13) into (12) we have

$$\tilde{b}(1)d(uv) = \tilde{b}(u)ud(v) + \tilde{b}(1)d(u)v. \tag{14}$$

Hence $\tilde{b}(1)d(uv) = \tilde{b}(1)d(vu)$ implies $d(v)(\tilde{b}(u) - \tilde{b}(1))u = d(u)(\tilde{b}(v) - \tilde{b}(1))v$ and

$$d(v) = c(\tilde{b}(v) - \tilde{b}(1))v \tag{15}$$

($c \in \mathbb{R}$ is an arbitrary constant) for a nonconstant solution of the equation (11). Thus, in view of (7), (10), (13) and (15), we have

$$f(t, v, v_1, \dots, v_m) = a(t)b(v)\delta(v_1, \dots, v_m) + q(t)v, \tag{16}$$

where $\delta(v_1, \dots, v_m) = \delta^*(v_1, \dots, v_m) + c\tilde{b}(1)$, $q(t) = c(t) - ca(t)\tilde{b}(1)^2$ and $b(v) = \tilde{b}(v)v$ satisfies Cauchy's power equation (11). According to (5) and (16) we have

$$\begin{aligned} &a(t)b(uv)\delta(u_1v_1, \dots, u_mv_m) + q(t)uv \\ &\quad = h(t, x, u, u_1, \dots, u_m)v + [a(x)b(v)\delta(v_1, \dots, v_m) + q(x)v]g(t, x, u, u_1, \dots, u_m)u \end{aligned}$$

and for the function $\delta(v_1, \dots, v_m)$ we have two conditions

a) $q(t)uv = q(x)g(t, x, u, u_1, \dots, u_m)uv + h(t, x, u, u_1, \dots, u_m)v$, i.e.

$$h(t, x, u, u_1, \dots, u_m) = [q(t) - q(x)g(t, x, u, u_1, \dots, u_m)]u \tag{17}$$

and

b) $a(t)b(uv)\delta(u_1v_1, \dots, u_mv_m) = a(x)b(v)\delta(v_1, \dots, v_m)g(t, x, u, u_1, \dots, u_m)u$, i.e.

$$g(t, x, u, u_1, \dots, u_m)u = a(t)b(u)\delta(u_1, \dots, u_m)/(a(x)K) \tag{18}$$

using $v_i = 1$ $(i = 1, 2, \dots, m)$, $b(uv) = b(u)b(v)$, $K = \delta(1, \dots, 1)$.

Now we substitute the functions f, g, h [i.e. (16), (17) and (18)] into (5). Then

$$K\delta(u_1v_1, \dots, u_mv_m) = \delta(u_1, \dots, u_m)\delta(v_1, \dots, v_m). \tag{19}$$

Without loss of generality, we may take $K = 1$ because $\tilde{\delta} = \delta/K$ is a solution of

$$\tilde{\delta}(u_1v_1, \dots, u_mv_m) = \tilde{\delta}(u_1, \dots, u_m)\tilde{\delta}(v_1, \dots, v_m)$$

whenever δ is a solution of (19).

Summarizing (16), (17) and (18) we conclude that for a nonconstant function b

$$\begin{aligned} f(t,v,v_1,\dots,v_m) &= a(t)b(v)\delta(v_1,\dots,v_m)+q(t)v,\\ g(t,x,u,u_1,\dots,u_m) &= a(t)b(u)\delta(u_1,\dots,u_m)/(a(x)u),\\ h(t,x,u,u_1,\dots,u_m) &= [q(t)-q(x)g(t,x,u,u_1,\dots,u_m)]u, \end{aligned}$$

where a, q are arbitrary continuous functions ($a(x)\neq 0$) and the function $b(u)$ is a continuous solution of Cauchy's power equation $b(uv)=b(u)b(v)$ and δ is a continuous solution of Cauchy's power equation

$$\delta(u_1v_1,\dots,u_mv_m)=\delta(u_1,\dots,u_m)\delta(v_1,\dots,v_m)$$

in several variables.

We have $\tilde{b}(v)=\tilde{b}(1)=1$ in the case that $\tilde{b}(v)$ is a nonzero constant solution of (11). From (10), (13) and (14) one gets $\tilde{g}(t,v)=a(t)$, $\tilde{h}(t,v)=a(t)d(v)+c(t)v$, where the function $d(v)$ satisfies a functional equation of derivations $d(uv)=ud(v)+d(u)v$ (see Aczél [2], p. 23). Repeating the above arguments we get

$$\begin{aligned} f(t,v,v_1,\dots,v_m) &= a(t)v\delta(v_1,\dots,v_m)+a(t)d(v)+c(t)v,\\ g(t,x,u,u_1,\dots,u_m) &= a(t)\delta(u_1,\dots,u_m)/a(x),\\ h(t,x,u,u_1,\dots,u_m) &= [a(t)-a(x)g(t,x,u,u_1,\dots,u_m)]ud(v)/v\\ &\quad + [c(t)-c(x)g(t,x,u,u_1,\dots,u_m)]u+a(t)d(u), \end{aligned}$$

where $\delta(v_1,\dots,v_m)=\delta^*(v_1,\dots,v_m)$ satisfies Cauchy's power equation in several variables. We also have $d(v)=kv$ ($k\in\mathbb{R}$) because the function h is independent of v. By virtue of the functional equation of derivations we obtain $k=0$, i.e. $d(v)=0$ and (15) is satisfied with $c=0$. In this case $\delta(v_1,\dots,v_m)=\delta^*(v_1,\dots,v_m)$, $q(t)=c(t)$ and $b(v)=\tilde{b}(v)v=v$ in accordance with (16) and the assertion of Theorem 1 holds in all cases. □

Observation 2. For every $k\geqslant 1$, let f_i, g_i, h_i ($i=1,2,\dots,k$) be different continuous solutions of the functional equation (5), i.e.

$$\begin{aligned} f_i(t,uv,u_1v_1,\dots,u_mv_m) &= f_i(x,v,v_1,\dots,v_m)g_i(t,x,u,u_1,\dots,u_m)u\\ &\quad + h_i(t,x,u,u_1,\dots,u_m)v \end{aligned}$$

for $i\in\{1,2,\dots,k\}$. Then there exist functions

$$f=F(f_1,f_2,\dots,f_k),\quad g=G(g_1,g_2,\dots,g_k),\quad h=H(h_1,h_2,\dots,h_k) \tag{20}$$

satisfying the functional equation (5) if and only if the identities

$$\begin{aligned}&F(f_1(t,uv,u_1v_1,\dots,u_mv_m),\dots,f_k(t,uv,u_1v_1,\dots,u_mv_m))\\&=F(f_1g_1u+h_1v,\dots,f_kg_ku+h_kv)\\&=F(f_1,f_2,\dots,f_k)G(g_1,g_2,\dots,g_k)u+H(h_1,h_2,\dots,h_k)v\end{aligned}$$

hold, where $f_i=f_i(x,v,v_1,\dots,v_m)$, $g_i=g_i(t,x,u,u_1,\dots,u_m)$ and $h_i=h_i(t,x,u,u_1,\dots,u_m)$, $i=1,2,\dots,k$.

Theorem 2. *For $k\geqslant 1$, let f_i, g_i, h_i $(i=1,2,\dots,k)$ be different continuous solutions of the functional equation* (5). *Then the general continuous solution of the functional equation*

$$F(f_1g_1u+h_1v,\dots,f_kg_ku+h_kv)=F(f_1,\dots,f_k)G(g_1,\dots,g_k)u+H(h_1,\dots,h_k)v \tag{21}$$

defined on $\mathbb{R}$ is of the form

$$\begin{aligned}F(f_1,f_2,\dots,f_k)&=\sum_{i=1}^k c_if_i,\\H(h_1,h_2,\dots,h_k)&=\sum_{i=1}^k c_ih_i,\\G(g_1,g_2,\dots,g_k)&=g_1=g_2=\dots=g_k,\end{aligned}$$

where $c_i\in\mathbb{R}$ $(i=1,2,\dots,k)$ are arbitrary constants.

P r o o f. We consider an arbitrary fixed k $(k\geqslant 1)$. Choosing $v=0$ in (21) one gets

$$F(f_1g_1u,\dots,f_kg_ku)=F(f_1,\dots,f_k)G(g_1,\dots,g_k)u \tag{22}$$

and substituting (22) into (21) we obtain

$$F(f_1g_1u+h_1v,\dots,f_kg_ku+h_kv)=F(f_1g_1u,\dots,f_kg_ku)+H(h_1,\dots,h_k)v. \tag{23}$$

Then $h_i=0$ $(i=1,2,\dots,k)$ gives

$$H(0,0,\dots,0)=0. \tag{24}$$

Similarly, $f_i=0$ $(i=1,2,\dots,k)$ in (21) implies that

$$F(h_1v,\dots,h_kv)=F(0,\dots,0)G(g_1,\dots,g_k)u+H(h_1,\dots,h_k)v. \tag{25}$$

Substituting $h_i = 0$ $(i = 1, 2, \ldots, k)$ into (25) we obtain

$$F(0, \ldots, 0) = F(0, \ldots, 0)G(g_1, \ldots, g_k)u + H(0, \ldots, 0)v$$

and by (24) we have

$$F(0, 0, \ldots, 0) = 0. \tag{26}$$

Hence (25) implies that

$$F(h_1 v, \ldots, h_k v) = H(h_1, \ldots, h_k)v$$

and choosing $v = 1$ we obtain

$$F(h_1, \ldots, h_k) = H(h_1, \ldots, h_k). \tag{27}$$

If we now define $u_i = f_i g_i u$ $(i = 1, 2, \ldots, k)$ and put $v = 1$ in (23), then we obtain Cauchy's functional equation

$$F(u_1 + h_1, \ldots, u_k + h_k) = F(u_1, \ldots, u_k) + F(h_1, \ldots, h_k) \tag{28}$$

according to (27). The general continuous solution of (28) is of the form

$$F(u_1, \ldots, u_k) = \sum_{i=1}^{k} c_i u_i, \tag{29}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, k)$ are arbitrary constants (see [2]). Thus, by using (27),

$$H(h_1, \ldots, h_k) = \sum_{i=1}^{k} c_i h_i. \tag{30}$$

From (22) and (29) we have

$$u \sum_{i=1}^{k} c_i f_i g_i = uG(g_1, \ldots, g_k) \sum_{i=1}^{k} c_i f_i,$$

i.e.

$$\sum_{i=1}^{k} c_i f_i g_i = \sum_{i=1}^{k} c_i f_i G(g_1, \ldots, g_k).$$

We choose successively k-tuples $(c_1^j, c_2^j, \ldots, c_k^j)$ such that $c_i^j = 1$ for $j = i$ and $c_i^j = 0$ for $j \neq i$; $i, j \in \{1, 2, \ldots, k\}$. So we prove that

$$G(g_1, g_2, \ldots, g_k) = g_1 = g_2 = \ldots = g_k. \tag{31}$$

The functions (29), (30), (31) are solutions of the functional equation (21) and the assertion of Theorem 2 is proved. $\square$

Remark 1. Cauchy's power equation is of the form

$$g(xy) = g(x)g(y), \tag{32}$$

where $g\colon \mathbb{R}^* \to \mathbb{R}$, $\mathbb{R}^* = \mathbb{R} - \{0\}$. The general solutions in the class of functions continuous at a point are given by

$$g(x) = 0, \quad g(x) = |x|^c, \quad g(x) = |x|^c \operatorname{sign} x, \tag{33}$$

$c \in \mathbb{R}$ being an arbitrary constant (see Aczél [2]). Moreover, $g(1) = 1$ and $g(x) = g(\frac{xy}{y}) = g(\frac{x}{y})g(y)$ if $y \neq 0$ for a nontrivial solution g. If we consider Cauchy's power equation in several variables

$$F(x_1y_1, x_2y_2, \dots, x_my_m) = F(x_1, x_2, \dots, x_m)F(y_1, y_2, \dots, y_m), \tag{34}$$

$F\colon (\mathbb{R}^*)^m \to \mathbb{R}$, then

$$\begin{aligned}&F(x_1y_1, x_2y_2, \dots, x_my_m)\\ &= F(x_1, 1, \dots, 1)F(1, x_2, \dots, 1)\dots F(1, 1, \dots, x_m)F(y_1, y_2, \dots, y_m)\end{aligned}$$

holds and we have

$$F(x_1, x_2, \dots, x_m) = \prod_{i=1}^{m} \delta_i(x_i) = \delta_1(x_1)\delta_2(x_2)\dots\delta_m(x_m), \tag{35}$$

$\delta_i(x_i) = F(1, \dots, 1, x_i, 1, \dots, 1)$. Moreover, $x_i = 1$ $(i = 1, 2, \dots, m)$ implies that $F(y_1, y_2, \dots, y_m) = F(1, 1, \dots, 1)F(y_1, y_2, \dots, y_m)$, thus $F(1, 1, \dots, 1) = 1$ and we obtain

$$\delta_i(1) = F(1, 1, \dots, 1) = 1 \tag{36}$$

for $i \in \{1, 2, \dots, m\}$. Using (34) and (35) one gets $\prod\limits_{i=1}^{m} \delta_i(x_iy_i) = \prod\limits_{i=1}^{m} \delta_i(x_i)\delta_i(y_i)$ and choosing $x_i = y_i = 1$ for $i \neq j$, j being fixed, we get $\delta_j(x_jy_j) = \delta_j(x_j)\delta_j(y_j)$, $i, j \in \{1, 2, \dots, m\}$, with regard to (36). Consequently, the general solutions of the equation (34) are of the form (35), where δ_i $(i = 1, 2, \dots, m)$ are the general solutions (33) of Cauchy's power equation.

Theorem 3. *If* (3) *is a stationary transformation of the equation* (1) *then*

$$f(t, v, v_1, \dots, v_m) = \sum_{i=1}^{k} a_i(t)b_i(v) \prod_{j=1}^{m} \delta_{ij}(v_j) + q(t)v \tag{37}$$

holds for arbitrary functions a_i, q on the interval J and an arbitrary k ($k \geqslant 1$), and

$$g(t,x,u,u_1,\ldots,u_m) = \frac{a_i(t)b_i(u)\prod\limits_{j=1}^{m}\delta_{ij}(u_j)}{a_i(x)u} \tag{38}$$

for every $i \in \{1,2,\ldots,k\}$;

$$h(t,x,u,u_1,\ldots,u_m) = [q(t) - q(x)g(t,x,u,u_1,\ldots,u_m)]u, \tag{39}$$

where functions b_i, δ_{ij} are continuous solutions of Cauchy's power equations.

Proof. The assertion follows from Lemma 1, Theorem 1 and Theorem 2 with respect to Observation 2 and Remark 1. We have used $q(t) = \sum\limits_{i=1}^{k} q_i(t)$ in Theorem 3. □

Remark 2. In the case $k \leqslant m$ we obtain from (37) a linear functional-differential equation

$$y'(x) = q(x)y(x) + a_1(x)y(\xi_1(x)) + \ldots + a_k(x)y(\xi_k(x)) \tag{40}$$

choosing $b_i(x) = |y(x)|^0 = 1$ for every $i \in \{1,2,\ldots,k\}$; $\delta_{ij}(y(\xi_j)) = |y(\xi_j)|^0 = 1$ if $j \neq i$ and $\delta_{ii}(y(\xi_i)) = |y(\xi_i)| \cdot \operatorname{sign} y(\xi_i)$, $i,j \in \{1,2,\ldots,k\}$, according to Observation 1.

Theorem 4. *The transformation of the form $z(t) = L(t)y(\varphi(t))$ is the most general transformation converting any equation*

$$y'(x) = \sum_{i=1}^{k} a_i(x)b_i(y(x))\prod_{j=1}^{m}\delta_{ij}(y(\xi_j(x))) + q(x)y(x) \tag{41}$$

defined on I into another equation

$$z'(t) = \sum_{i=1}^{k} A_i(t)b_i(z(t))\prod_{j=1}^{m}\delta_{ij}(z(\eta_j(t))) + Q(t)z(t) \tag{42}$$

defined on J. Moreover, (41) is globally transformable into (42) if and only if the functions L, φ satisfy the relations

$$\xi_j(\varphi(t)) = \varphi(\eta_j(t)), \quad j = 1,2,\ldots,m;\ \varphi(J) = I, \tag{43}$$

$$Q(t) = \frac{L'(t)}{L(t)} + q(\varphi(t))\varphi'(t), \quad A_i(t) = \frac{a_i(\varphi(t))\varphi'(t)L(t)}{b_i(L(t))\prod\limits_{j=1}^{m}\delta_{ij}(L(\eta_j(t)))} \tag{44}$$

on J. Here the functions b_i, δ_{ij} $(i = 1, \dots, k;\ j = 1, \dots, m)$ are continuous solutions of Cauchy's power equation.

P r o o f. We prove that the transformation $z(t) = L(t)y(\varphi(t))$ converts any equation (41) into another equation (42). We have

$$z(t) = L(t)y(\varphi(t)) = L(t)y(x), \quad L(t) \neq 0,$$
$$z(\eta_j(t)) = L(\eta_j(t))y(\varphi(\eta_j(t))) = L(\eta_j(t))y(\xi_j(\varphi(t))) = L(\eta_j(t))y(\xi_j(x))$$

$(j = 1, 2, \dots, m;\ i = 1, 2, \dots, k)$, similarly to the proof of Lemma 1. We have the identities

$$\begin{aligned} z'(t) = L'(t)y(\varphi(t)) + L(t)\dot{y}(\varphi(t))\varphi'(t) = \left(\frac{L'(t)}{L(t)} + q(\varphi(t))\varphi'(t)\right) z(t) \\ + \sum_{i=1}^{k} \frac{a_i(\varphi(t))\varphi'(t)L(t)}{b_i(L(t)) \prod\limits_{j=1}^{m} \delta_{ij}(L(\eta_j(t)))} b_i(z(t)) \prod_{j=1}^{m} \delta_{ij}(z(\eta_j(t))) \\ = \sum_{i=1}^{k} A_i(t) b_i(z(t)) \prod_{j=1}^{m} \delta_{ij}(z(\eta_j(t))) + Q(t)z(t), \quad (\cdot = \mathrm{d}/\mathrm{d}\varphi) \end{aligned}$$

and (44) are valid on J. According to Remark 2, every linear differential equation is a particular case of the equation (41). The most general pointwise transformation for the linear differential equation (see [3, 10]) is of the form $z(t) = L(t)y(\varphi(t))$. This fact implies that (3) is the most general transformation for the equation (41). Moreover, the transformation (3) converts (41) into (42) if and only if (43) and (44) hold on J. $\square$

Corollary 1. *The transformation* (3) *is a stationary transformation of the equation* (41) *if and only if φ is a simultaneous solution of*

$$\xi_j(\varphi(x)) = \varphi(\xi_j(x)), \quad j = 1, 2, \dots, m;\ \varphi(I) = I$$

(see [12, 13]*) and*

$$a_i(\varphi(x))\varphi'(x)L(x) = a_i(x)b_i(L(x)) \prod_{j=1}^{m} \delta_{ij}(L(\xi_j(x))); \quad i = 1, \dots, k;$$
$$L'(x) = (q(x) - q(\varphi(x))\varphi'(x))L(x), \quad x \in I.$$

Here the functions b_i, δ_{ij}, $i \in \{1, 2, \dots, k\}$, $j \in \{1, 2, \dots, m\}$, are continuous solutions of Cauchy's power equation.

Remark 3. The equation (41) involves the equations

$$y'(x) + p(x)|y(\tau(x))|^\lambda \operatorname{sign} y(\tau(x)) = 0, \quad \lambda \geqslant 0;$$
$$y'(x) = \frac{(x-1)^3}{x^2(x-2)^2} y(x-1)^3, \ x \geqslant 3;$$
$$y'(x) = 2^{1-x} y(2x)^{1/3} y(3x) y(4x)^{1/3};$$
$$y'(x) = \frac{|y(x+\sin x)|^{\alpha_1} \operatorname{sign} y(x+\sin x) |y(x+\cos x)|^{\alpha_2} \operatorname{sign} y(x+\cos x)}{x^\beta |\ln(x+\sin x)|^{\alpha_1} |\ln(x+\cos x)|^{\alpha_2}},$$

$x \geqslant 2\pi$, $\alpha_i > 0$, $\alpha = \alpha_1 + \alpha_2 > 1$, $\beta < 1$;

$$\cdots$$

considered in [4].

Remark 4 (see [6] and [9], pp. 355, 357). In a situation when the deviating arguments in equation (41) are constant deviations

$$\xi_j(x) = x - c_j, \ c_j \in \mathbb{R} - \{0\}; \ j = 1, 2, \dots, m,$$

the condition $\xi_j(\varphi(t)) = \varphi(\eta_j(t))$ becomes a system of the Abel equations

$$\varphi(\eta_j(t)) = \varphi(t) - c_j; \ j = 1, 2, \dots, m.$$

When the deviating arguments in (42) are

$$\eta_j(t) = t - d_j, \quad d_j \in \mathbb{R} - \{0\},$$

then we get

$$\varphi(t - d_j) = \varphi(t) - c_j; \quad j = 1, 2, \dots, m.$$

If we require that the delayed arguments be converted into delayed ones (or the advanced into advanced), then we need $\varphi'(t) > 0$, $t \in J$. Let d_j/d_k be irrational for a pair $j, k \in \{1, 2, \dots, m\}$. Then for each fixed $j \in \{1, 2, \dots, m\}$, the Abel equation $\varphi(t - d_j) = \varphi(t) - c_j$ has a general solution $\varphi \in C^1(J)$, $\varphi'(t) > 0$, $\varphi(J) = I$, of the form

$$\varphi(t) = \frac{c_j}{d_j} t + k, \quad k \in \mathbb{R}.$$

For the existence of a simultaneous solution φ it is then sufficient and necessary to have $\varrho = c_j/d_j$ (a constant not depending on j) for all $j \in \{1, 2, \dots, m\}$.

Example. Consider two equations

$$\begin{aligned} y'(x) &= a_1 \exp\{\lambda_1 x\}|y(x)|^\alpha \operatorname{sign} y(x) y(x-c_1) \\ &\qquad + b_1 \exp\{\mu_1 x\}|y(x-c_1)|^\beta y(x-c_2) + q_1 y(x), \end{aligned}$$

$x \in I = [a, \infty)$,

$$\begin{aligned} z'(t) &= a_2 \exp\{\lambda_2 t\}|z(t)|^\alpha \operatorname{sign} z(t) z(t-d_1) \\ &\qquad + b_2 \exp\{\mu_2 t\}|z(t-d_1)|^\beta z(t-d_2) + q_2 z(t), \end{aligned}$$

$t \in J = [b, \infty)$, with constant deviations, $a_i, b_i, c_i, d_i, q_i, \lambda_i, \mu_i, \alpha, \beta \in \mathbb{R}$; $i = 1, 2$; $\frac{d_1}{d_2}$ being irrational, $\frac{c_1}{d_1} > 0$. These two equations are of the form (41) and (42), respectively. We have

$$\begin{gathered} b_1(u) = |u|^\alpha \operatorname{sign} u,\ \delta_{11}(u) = u,\ \delta_{12}(u) = 1, \\ b_2(u) = 1,\ \delta_{21}(u) = |u|^\beta,\ \delta_{22}(u) = u,\ u \in \mathbb{R} \end{gathered}$$

and the coefficients

$$\begin{gathered} a_1(x) = a_1 \exp\{\lambda_1 x\}, \quad a_2(x) = b_1 \exp\{\mu_1 x\}, \quad q(x) = q_1,\ x \in I, \\ A_1(t) = a_2 \exp\{\lambda_2 t\}, \quad A_2(t) = b_2 \exp\{\mu_2 t\}, \quad Q(t) = q_2,\ t \in J. \end{gathered}$$

Due to Theorem 4 the equations are equivalent if and only if

$$\begin{aligned} &\varphi(t-d_j) = \varphi(t) - c_j, \quad \varphi(J) = I,\ j = 1, 2; \\ &\frac{L'(t)}{L(t)} = Q(t) - q(\varphi(t))\varphi'(t) = q_2 - q_1 \varphi'(t), \\ &A_1(t) = \frac{a_1(\varphi(t))\varphi'(t)L(t)}{b_1(L(t))\delta_{11}(L(t-d_1))\delta_{12}(L(t-d_2))} = \frac{a_1(\varphi(t))\varphi'(t)L(t)}{|L(t)|^\alpha \operatorname{sign} L(t) L(t-d_1)} \\ &\qquad = \frac{a_1(\varphi(t))\varphi'(t)}{|L(t)|^{\alpha-1} L(t-d_1)}, \\ &A_2(t) = \frac{a_2(\varphi(t))\varphi'(t)L(t)}{b_2(L(t))\delta_{21}(L(t-d_1))\delta_{22}(L(t-d_2))} = \frac{a_2(\varphi(t))\varphi'(t)L(t)}{|L(t-d_1)|^\beta L(t-d_2)} \end{aligned}$$

on J.

In accordance with Remark 4 we have $\varphi(t) = \varrho t + k$, $\varrho = \frac{c_1}{d_1} = \frac{c_2}{d_2} > 0$, $k \in \mathbb{R} - \{0\}$ and $\varphi(t) = \varrho(t-b) + a$ for $\varphi(J) = I$. Then $\frac{L'(t)}{L(t)} = q_2 - q_1 \varrho$ and we obtain $L(t) = c \exp\{kt\}$, $k = q_2 - q_1 \varrho$, $c \in \mathbb{R} - \{0\}$. From the last two conditions we get

$$\begin{aligned} a_2 \exp\{\lambda_2 t\} &= \frac{a_1 \varrho \exp\{\lambda_1 \varrho t\} \exp\{(a - \varrho b)\lambda_1\}}{c|c|^{\alpha-1} \exp\{(\alpha-1)kt\} \exp\{kt\} \exp\{-kd_1\}}, \\ b_2 \exp\{\mu_2 t\} &= \frac{c b_1 \varrho \exp\{\mu_1 \varrho t\} \exp\{(a - \varrho b)\mu_1\} \exp\{kt\}}{c|c|^\beta \exp\{k\beta t\} \exp\{-k\beta d_1\} \exp\{kt\} \exp\{-kd_2\}}, \quad t \in J. \end{aligned}$$

Thus the equations are equivalent with respect to the transformation $z(t) = L(t)y(\varphi(t))$ if and only if there exist $c \in \mathbb{R} - \{0\}$ and $\varrho \in \mathbb{R}$, $\varrho > 0$, such that

$$\begin{aligned} c_j &= \varrho d_j \quad (j = 1, 2), & q_2 - q_1 \varrho &= k, \\ a_2 |c|^\alpha \operatorname{sign} c &= \varrho a_1 \exp\{(a - \varrho b)\lambda_1\} \exp\{k d_1\}, & \lambda_2 - \varrho\lambda_1 + k\alpha &= 0, \\ b_2 |c|^\beta &= \varrho b_1 \exp\{(a - \varrho b)\mu_1\} \exp\{k\beta d_1\} \exp\{k d_2\}, & \mu_2 - \varrho\mu_1 + k\beta &= 0. \end{aligned}$$

For c, k we get the transformation functions $L(t) = c\exp\{kt\}$, $\varphi(t) = (t - b)\varrho + a$.

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Author's address: Department of Mathematics, Faculty of Civil Engineering, Technical University of Brno, Žižkova 17, 602 00 Brno, Czech Republic, e-mail: mdtry@fce.vutbr.cz.