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On uniform distribution of sequences \((a_nx)_1^\infty\)


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ON UNIFORM DISTRIBUTION OF SEQUENCES $\left( a_n x \right)_1^\infty$

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INTRODUCTION

There are two possible approaches to the study of uniform distribution (mod 1) of sequences

$$(1) \quad (a_n x)_1^\infty$$

where $a_n \in \mathbb{R}$ ($n = 1, 2, \ldots$) and $x \in \mathbb{R}$. The first such approach is the study of (1) with a fixed sequence $(a_n)_1^\infty$, $x$ running over real numbers, the second is the study of (1) with a fixed $x \in \mathbb{R}$ and $(a_n)_1^\infty$ running over a class of sequences of real numbers. The second approach leads to the concept of $\alpha$-good sequences (cf. [1]).

In the first part of the paper we will apply the first and in the second part the second approach to the investigation of sequences (1).

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1. Uniform distribution (mod 1) of sequences $(a_n x)_1^\infty$ with fixed $(a_n)_1^\infty$

In this part we restrict ourselves to the study of (1) with a fixed sequence $(a_n)_1^\infty$ of real numbers. Denote by $H(a_1, a_2, \ldots)$ the set of all $x \in \mathbb{R}$ for which the sequence (1) is uniformly distributed (mod 1) (shortly: u.d. mod 1). It is well known that if $a_n \in \mathbb{N}$ ($n = 1, 2, \ldots$), $a_i \neq a_j$ for $i \neq j$, then the set $H(a_1, a_2, \ldots)$ has full measure (i.e. the set $\mathbb{R} \setminus H(a_1, a_2, \ldots)$ is a null set—cf. [2], [4] pp. 32–33). This results evokes the question what is the Baire category of the set $H(a_1, a_2, \ldots)$. We will show that the “topological magnitude” of $H(a_1, a_2, \ldots)$ depends on the sequence $(a_n)_1^\infty$. Indeed, if we choose $a_n = n$ or more generally $a_n = a + nd$ ($n = 1, 2, \ldots$), $d \geq 1$, $a, d$ integers, then by Weyl’s criterion (cf. [4] pp. 7–8) the sequence $(a_n x)_1^\infty$ is u.d. mod 1 for each irrational $x$. Hence $H(a_1, a_2, \ldots)$ contains in this case all irrational numbers and so it is a residual set. In what follows we will give a class of sequences $(a_n)_1^\infty$ of positive integers such that $H(a_1, a_2, \ldots)$ is a set of the first category.

**Theorem 1.1.** Let $(q_k)_1^\infty$ be a sequence of positive integers greater than 1. Put

$$a_n = q_1 q_2 \ldots q_n \quad (n = 1, 2, \ldots).$$

Then $H(a_1, a_2, \ldots)$ is a set of the first Baire category in $\mathbb{R}$.

**Proof.** For $x \in \mathbb{R}$ we put

$$S(m, x) = \frac{1}{m} \sum_{n=1}^{m} e^{2\pi i a_n x} \quad (m = 1, 2, \ldots).$$

Then by Weyl’s criterion we have

$$H(a_1, a_2, \ldots) \subseteq H_0(a_1, a_2, \ldots),$$

where $H_0(a_1, a_2, \ldots) = \left\{ x \in \mathbb{R} : \lim_{m \to \infty} S(m, x) = 0 \right\}$. Denote by $C(a_1, a_2, \ldots)$ the set of all $x \in \mathbb{R}$ for which there exists $\lim_{m \to \infty} S(m, x) = S(x)$. Then evidently

$$H(a_1, a_2, \ldots) \subseteq H_0(a_1, a_2, \ldots) \subseteq C(a_1, a_2, \ldots).$$

Each of these sets has the full measure. By (2) it suffices to prove that $C = C(a_1, a_2, \ldots)$ is a set of the first category in $\mathbb{R}$. We prove it in the following.

Obviously each of the functions $S(m, x)$ ($m = 1, 2, \ldots$) is continuous on $C$ (i.e. the restrictions $S(m, x)|C$ are continuous on $C$). Hence the function $S(x) = \lim_{m \to \infty} S(m, x)$ defined on $C$ is in the first Baire class on $C$. But then the set of all
discontinuity points of $S$ is a set of the first category in $C$ (cf. [8] p. 185), and so it is a set of the first category in $\mathbb{R}$, as well.

To complete the proof it suffices to show that the function $S$ is discontinuous at every $x \in C$. For this it suffices to show that each of the sets

$$M_0 = \{ x \in C : S(x) = 0 \}, \quad M_1 = \{ x \in C : S(x) = 1 \}$$

is dense in $C$.

The density of $M_0$ follows from the fact that $M_0 \subseteq C$ and $M_0$ has the full measure.

We prove that $M_1$ is dense in $C$. It is wellknown that every $x \in \mathbb{R}$ has the Cantor series expansion

$$x = c_0 + \sum_{j=1}^{\infty} \frac{c_j}{q_1 q_2 \cdots q_j} = c_0 + \sum_{j=1}^{\infty} \frac{c_j}{a_j},$$

where $c_j$ are integers, $0 \leq c_j < q_j$, $a_j = q_1 q_2 \cdots q_j$ ($j = 1, 2, \ldots$).

Denote by $A_k$ the set of all $x \in \mathbb{R}$ of the form

$$x = c_0 + \sum_{j=1}^{k} \frac{c_j}{a_j},$$

where $k \in \mathbb{N}$, $c_0$ is an integer and $0 \leq c_j < q_j$ ($j = 1, 2, \ldots, k$). If $x \in A_k$, then $a_n x$ is an integer for $n > k$. Thus for $m > k$ we have

$$S(m, x) = \frac{1}{m} \sum_{n=1}^{k} + \frac{1}{m} \sum_{n=k+1}^{m} 1 = O(1) + \frac{m-k}{m} \to 1 \quad \text{if} \quad m \to \infty.$$

Put $A = \bigcup_{k=1}^{\infty} A_k$. Then $A \subseteq M_1 \subseteq C$ and $A$ is obviously dense in $C$. The density of $M_1$ in $C$ follows. This completes the proof. \hfill \square

We give the following simple observation.

**Proposition 1.1.** Let $(a_j)_{j=1}^{\infty}$ be an arbitrary sequence of real numbers. Then $H(a_1, a_2, \ldots)$ is an $F_{\sigma \delta}$-set in $\mathbb{R}$.

**Proof.** Using Weyl’s criterion we can easily check that

$$H(a_1, a_2, \ldots) = \bigcap_{h \neq 0} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A(n,h,k),$$

where

$$A(n,h,k) = \left\{ x \in \mathbb{R} : \left| \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h a_j x} \right| \leq \frac{1}{k} \right\}.$$
Since \( \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h a_j x} \) \((n = 1, 2, \ldots)\) are continuous functions, we see that \( A(n, h, k) \) is a closed set (if \( n, h, k \) are fixed) and so \( H(a_1, a_2, \ldots) \) is an \( F_{\sigma\delta} \)-set in \( \mathbb{R} \). \( \square \)

**Remark 1.1.** a) If \((a_n)_{1}^{\infty}\) is a sequence of distinct integers then by [2] and Proposition 1.1 the set \( H(a_1, a_2, \ldots) \) is an \( F_{\sigma\delta} \)-set of the full measure.

b) For some particular choices of \((a_n)_{1}^{\infty}\) the set \( H(a_1, a_2, \ldots) \) can belong to lower Borel classes. For instance if \( a_n = a \in \mathbb{R}, \ (n = 1, 2, \ldots) \), then the set \( H(a_1, a_2, \ldots) \) is empty while it coincides with the set \( \mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q} \) of all irrational numbers if \( a_n = n \ (n = 1, 2, \ldots) \).

2. **Uniform distribution (mod 1) of sequences \((a_n x)_{1}^{\infty}\)** with fixed \( x \)

Let \( \alpha \) be an irrational number. A sequence \( a_1 < a_2 < \ldots \) of positive integers is said to be \( \alpha \)-good provided the sequence \((a_n, \alpha)_{1}^{\infty}\) is uniformly distributed \( \pmod{1} \) (cf. [1]).

The sequence \( 1 < 2 < \ldots < n < \ldots \) and the sequence \( p_1 < p_2 < \ldots < p_n < \ldots \) of all prime numbers are \( \alpha \)-good for each irrational \( \alpha \) (cd. [1], [4] p. 22).

For \( \alpha \in \mathbb{Q}' \) (\( \mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q} \)) denote by \( D(\alpha) \) the set of all \( \alpha \)-good sequences. Note that every infinite sequence \( a_1 < a_2 < \ldots < a_n < \ldots \) of positive integers belongs to \( D(\alpha) \) for almost all \( \alpha \in \mathbb{Q}' \) (cf. [4] p. 32, Theorem 4.1).

We will investigate the properties of the classes \( D(\alpha) \) for \( \alpha \in \mathbb{Q}' \). We will show that these classes have several common properties (for all \( \alpha \in \mathbb{Q}' \)).

It seems to be interesting to deal with the question about magnitude of classes \( D(\alpha) \) \( (\alpha \in \mathbb{Q}' \) ). This “magnitude” will be studied from the point of view of dyadic numbers of sets \( A \subseteq \mathbb{N} \).

Denote by \( U \) the class of all infinite sets

\[ A = \{a_1 < a_2 < \ldots < a_n < \ldots\} \subseteq \mathbb{N}. \]

In what follows we identify the set \( A \) with the sequence \( a_1 < a_2 < \ldots < a_n < \ldots \) Put

\[ \varrho(A) = \sum_{k=1}^{\infty} 2^{-a_k} \in (0, 1] \]

for each \( A \subseteq U \). Then \( \varrho \) is a one-to-one mapping of \( U \) onto \((0, 1]\). If \( S \) is a class of infinite subsets of \( \mathbb{N} \), then we put \( \varrho(S) = \{\varrho(A) : A \in S\} \). The set \( \varrho(S) \) “measures” the magnitude of the class \( S \) (cf. [5] p. 17).

We will investigate metric and topological properties of the sets \( \varrho(D(\alpha)) \).
Recall that a measurable set $M \subseteq (0, 1]$ is called homogeneous if there is a real number $d \in [0, 1]$ such that for every interval $I \subseteq (0, 1]$ we have

$$\frac{\lambda(I \cap M)}{\lambda(I)} = d,$$

$\lambda$ being the Lebesgue measure (cf. [9], [10]).

**Theorem 2.1.** For each $\alpha \in \mathbb{Q}'$ the set $\varrho(D(\alpha))$ is a homogeneous $F_{\sigma \delta}$-set in $(0, 1]$.

**Proof.** According to Weyl’s criterion a sequence $a_1 < a_2 < \ldots$ of positive integers belongs to $D(\alpha)$ if and only if

$$(\forall h \in \mathbb{Z}, h \neq 0): \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} e^{2\pi i h a_n \alpha} = 0.$$ 

This condition is equivalent to the condition

$$(4) \quad (\forall |h| \geq 1)(\forall k \geq 1)(\exists v \in \mathbb{N})(\forall m \geq v): \left| \frac{1}{m} \sum_{n=1}^{m} e^{2\pi i h a_n \alpha} \right| \leq \frac{1}{k}.$$ 

From (4) we get

$$\varrho(D(\alpha)) = \bigcap_{|h| \geq 1} \bigcap_{k=1}^{\infty} \bigcup_{v=1}^{\infty} \bigcap_{m=v}^{\infty} M(m, h, k),$$

where

$$(6) \quad M(m, h, k) = \left\{ x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0, 1]: \left| \frac{1}{m} \sum_{n=1}^{m} e^{2\pi i h a_n \alpha} \right| \leq \frac{1}{k} \right\}.$$ 

Construct the functions

$$f_{m, h}(x) = \frac{1}{m} \sum_{n=1}^{m} e^{2\pi i h a_n \alpha} \quad (m = 1, 2, \ldots; \; h \in \mathbb{Z}, h \neq 0),$$

where $x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0, 1]$. These functions are defined for each $x \in (0, 1]$. We will verify that their restrictions to $\mathbb{Q}' \cap (0, 1]$ are continuous on $\mathbb{Q}' \cap (0, 1]$.

Let $x_0 \in \mathbb{Q}' \cap (0, 1], x_0 = \sum_{j=1}^{\infty} 2^{-b_j} \; (b_1 < b_2 < \ldots)$ be the dyadic expansion of $x_0$.

Fix the number $m$. Notice that the set of all numbers of the form $x = \sum_{j=1}^{\infty} 2^{-a_j}$,
$a_j = b_j \ (j = 1, 2, \ldots, m)$ fills up an interval $I_m$ containing $x_0$, the left-hand endpoint of which is the rational number $\sum_{j=1}^{m} 2^{-b_j}$ and the right-hand endpoint is

$$\sum_{j=1}^{m} 2^{-b_j} + \sum_{j=b_m+1}^{\infty} 2^{-j} = \sum_{j=1}^{m} 2^{-b_j} + 2^{-b_m}.$$ 

Obviously the function $f_{m,h} | \mathbb{Q}' \cap (0,1]$ is constant on $I_m$ and so it is continuous at $x_0$.

From the continuity of functions $f_{m,h} | \mathbb{Q}' \cap (0,1]$ the closedness of the sets $M(m, h, k)$ in $\mathbb{Q}' \cap (0,1]$ follows (see (6)). But then by (5) the set $\mathbb{Q}' \cap \varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$. Notice that

$$\varrho(D(\alpha)) = [\mathbb{Q}' \cap \varrho(D(\alpha))] \cup [\mathbb{Q} \cap \varrho(D(\alpha))],$$

the second “summand” on the right-hand side being countable. From this we see that $\varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$.

The homogeneity of the set $\varrho(D(\alpha))$ can be proved by using a result from [7] (cf. [7], Lemma 1, pp. 255–256). We will use the following special case of Lemma 1 from [7]:

(T) Let $B \subseteq (0,1]$ be a measurable set. Suppose that for each $n = 1, 2, \ldots$ and $k, k' \in \{0, 1, \ldots, 2^n - 1\}$ we have

$$\lambda(B \cap i_n^{(k)}) = \lambda(B \cap i_n^{(k')}),$$

where

$$i_n^{(v)} = \left(\frac{v}{2^n}, \frac{v+1}{2^n}\right], \quad v \in \{0, 1, \ldots, 2^n - 1\}.$$ 

Then $B$ is a homogeneous set in $(0,1]$.

If now $a_1 < a_2 < \ldots < a_n < \ldots$ is an $\alpha$-good sequence and a sequence $d_1 < d_2 < \ldots < \ldots$ differs from $a_1 < a_2 < \ldots < a_n < \ldots$ only in a finite number of terms, then evidently also $d_1 < d_2 < \ldots < \ldots$ is an $\alpha$-good sequence. Hence the assumptions in (T) are satisfied and so by (T) the set $\varrho(D(\alpha))$ is homogeneous in $(0,1]$.

It is well known that the Lebesgue measure of a homogeneous set $A \subseteq (0,1]$ is 0 or 1 (cf. [9], [10]). Hence by Theorem 2.1 we have $\lambda(\varrho(D(\alpha))) = 0$ or $\lambda(\varrho(D(\alpha))) = 1$ for each $\alpha \in \mathbb{Q}'$. We will show that this measure is equal to 1 for each $\alpha \in \mathbb{Q}'$. \qed

**Theorem 2.2.** For each $\alpha \in \mathbb{Q}'$ we have $\lambda(\varrho(D(\alpha))) = 1$.

**Proof.** Let $\alpha \in \mathbb{Q}'$. Then the sequence $(n\alpha)_{n=1}^{\infty}$ is u.d. mod 1 (cf. [4] pp. 7–8). By a theorem of Peterson (cf. [6]), if $(v_n)_{1}^{\infty}$ is a u.d. mod 1 sequence then for almost all
\[ x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0, 1] \text{ the sequence } (v_{a_j})_{j=1}^{\infty} \text{ (subsequence of } (v_n)_{n=1}^{\infty}) \text{ is u.d. mod 1 as well. Hence for almost all } x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0, 1] \text{ the sequence } (a_j\alpha)_{j=1}^{\infty} \text{ is u.d. mod 1. But this means that almost all } x \in (0, 1] \text{ belong to the set } g(D(\alpha)). \] □

We now will investigate the magnitude of sets \( g(D(\alpha)) \) from the topological point of view. We prove the following universal theorem.

**Theorem 2.3.** For every \( \alpha \in \mathbb{Q}' \) the set \( g(D(\alpha)) \) is a dense \( F_{\sigma\delta} \)-set of the first Baire category in \( (0, 1] \).

**Proof.** Let \( \alpha \in \mathbb{Q}' \). By Theorem 2.1 the set \( g(D(\alpha)) \) is an \( F_{\sigma\delta} \)-set in \( (0, 1] \).

Further, the set \( D(\alpha) \) is non-empty (and such is also the set \( g(D(\alpha)) \)) since the sequence \( 1 < 2 < \ldots < n \ldots \) belongs to \( D(\alpha) \). The density of \( g(D(\alpha)) \) follows from the above mentioned fact that together with \( 1 < 2 < \ldots < n \ldots \) the class \( D(\alpha) \) contains every sequence \( a_1 < a_2 < \ldots < a_n < \ldots \) which differs from \( 1 < 2 < \ldots < n \ldots \) only in a finite number of terms.

We prove that \( g(D(\alpha)) \) is a set of the first Baire category. For \( t = g(A), A = a_1 < a_2 < \ldots < a_n < \ldots \) we put

\[
g_m(t) = \frac{1}{m} \sum_{n=1}^{m} s^{2\pi i a_n} \quad (m = 1, 2, \ldots).
\]

Denote by \( M \) the set of all \( t \in (0, 1] \) \( (t = g(A), A = a_1 < a_2 < \ldots) \) for which there exists \( \lim_{m \to \infty} g_m(t) = g(t) \). By Weyl’s criterion we get

\[
(7) \quad g(D(\alpha)) \subseteq M.
\]

It is easy to verify that the functions \( g_m|\mathbb{Q'} \cap (0, 1] \) are continuous on \( \mathbb{Q'} \cap (0, 1] \) (and so they are continuous on \( M \cap \mathbb{Q'} \subseteq \mathbb{Q'} \cap (0, 1] \) as well). This can be proved in an analogous way as the continuity of the functions \( f_{m,h}|\mathbb{Q'} \cap (0, 1] \) in the proof of Theorem 2.1. Thus the function \( g|\mathbb{M} \cap \mathbb{Q'} \) is in the first Baire class on \( \mathbb{M} \cap \mathbb{Q'} \). This implies that the set of discontinuity points of \( g|\mathbb{M} \cap \mathbb{Q'} \) is a set of the first category in \( \mathbb{M} \cap \mathbb{Q'} \) (cf. [8] p. 185).

We will show that the function \( g \) is discontinuous at every point of \( \mathbb{M} \cap \mathbb{Q'} \). To show this it suffices to prove that each of the sets

\[ M_0 = \{ x \in \mathbb{M} \cap \mathbb{Q'}: g(x) = 0 \} , M_1 = \{ x \in \mathbb{M} \cap \mathbb{Q'}: g(x) = 1 \} \]

is dense in \( \mathbb{M} \cap \mathbb{Q'} \).
In the first place we prove the density of $M_0$ in $M \cap \mathbb{Q}'$. If $p_1 < p_2 < \ldots < p_n < \ldots$ is the sequence of all primes then $x_0 = \sum_{k=1}^{\infty} 2^{-p_k}$ belongs to $M_0$ (cf. [1], [4] p. 22). Together with $x_0$ each $\varrho(A)$ belongs to $M_0$, where $A$ is an infinite set of positive integers which differs from $\{p_1 < p_2 < \ldots < p_n < \ldots\}$ only in a finite number of elements. From this the density of $M_0$ in $M \cap \mathbb{Q}'$ follows.

For the proof of density of $M_1$ it suffices to construct a sequence $A_0 = a_1 < a_2 < \ldots < a_n < \ldots$ such that $y_0 = \varrho(A_0)$ is an irrational number with $g(y_0) = 1$. Such a sequence can be obtained by the following procedure:

Take into account the continued fraction of $\alpha$. It is well known that if $\frac{p_n}{q_n} (n = 1, 2, \ldots)$ are convergents of this continued fraction, then

$$|q_n \alpha - p_n| < \frac{1}{q_n} \quad (n = 1, 2, \ldots)$$

(cf. [3] p. 27). Further, if $n$ is even then $\frac{p_n}{q_n} < \alpha$ (cf. [3] p. 22). But then for such even $n$ we have $0 < q_n \alpha - p_n < \frac{1}{q_n}$, thus $\{q_n \alpha\} = q_n \alpha - [q_n \alpha] = q_n \alpha - p_n < \frac{1}{q_n}$. So we get

$$\{q_n \alpha\} = \frac{q_n' \alpha}{q_n}, \quad 0 < q_n' < 1.$$

Choose a set $N_2 = \{k_1 < k_2 < \ldots < k_n < \ldots\}$ of even numbers such that

$$\lim_{n \to \infty} (q_{k_{n+1}} - q_{k_n}) = +\infty$$

and put $q_n' = q_{k_n} (n = 1, 2, \ldots)$. Then $y_0 = \sum_{n \in N_2} 2^{-q_n'}$ belongs to $\mathbb{Q}'$ since the condition (8) guarantees that the dyadic expansion of $y_0$ is not periodic. Further,

$$e^{2\pi i q_n'} = e^{2\pi i \{q_n' \alpha\} + \{q_n' \alpha\}} = e^{2\pi i \{q_n' \alpha\}} = e^{\frac{2\pi i q_n'}{q_n'}} (0 < \vartheta_n' < 1, \ n \in N_2).$$

For all sufficiently large $n$‘s (e.g. for $n > n_0$) we have

$$0 < \frac{2\pi \vartheta_n'}{q_n'} < \frac{1}{2}.$$

So we get

$$g_m(y_0) = \frac{1}{m} \sum_{n=1}^{n_0} + \frac{1}{m} \sum_{n=n_0+1}^{m} e^{2\pi i \{q_n' \alpha\}} = O(1)$$

$$+ \left( \frac{1}{m} \sum_{n=n_0+1}^{m} \cos \frac{2\pi \vartheta_n'}{q_n'} + i \frac{1}{m} \sum_{n=n_0+1}^{m} \sin \frac{2\pi \vartheta_n'}{q_n'} \right).$$

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Note that
\[ \left| \sin \frac{2\pi \vartheta_n'}{q_n'} \right| \leq \frac{2\pi \vartheta_n'}{q_n'} . \]

Since \( q_n' \to \infty \) \((n \to \infty)\), we have
\[ \left| \frac{1}{m} \sum_{n=n_0+1}^{m} \sin \frac{2\pi \vartheta_n'}{q_n'} \right| \leq \frac{1}{m} \sum_{n=1}^{m} \frac{2\pi}{q_n'} \to 0 \]
(for \( m \to \infty \)) (Cesàro means).

Further, by the inequality
\[ \cos x > 1 - \frac{x^2}{2} \quad (x \in (0, 1)) \]
we get (for \( n > n_0 \))
\[ \cos \frac{2\pi \vartheta_n'}{q_n'} > 1 - \frac{1}{2} \left( \frac{2\pi \vartheta_n'}{q_n'} \right)^2 > 1 - \frac{2\pi^2}{q_n'^2} . \]

Therefore we have
\[ \frac{1}{m} \sum_{n=n_0+1}^{m} \cos \frac{2\pi \vartheta_n'}{q_n'} > \frac{1}{m} \sum_{n=n_0+1}^{m} \left( 1 - \frac{2\pi^2}{q_n'^2} \right) \]
\[ = \frac{1}{m} \sum_{n=n_0+1}^{m} 1 - \frac{2\pi^2}{m} \sum_{n=n_0+1}^{m} \frac{1}{n'^2} . \]

The second summand on the right-hand side has the limit 0 if \( m \to \infty \) while the first tends to 1. Hence \( \lim_{m \to \infty} g_m(y_0) = 1. \)

So we have proved that \( g \) is a function in the first Baire class on \( M \cap \mathbb{Q}' \), discontinuous at every point of \( M \cap \mathbb{Q}' \). Therefore \( M \cap \mathbb{Q}' \) is a set of the first category in \( M \cap \mathbb{Q}' \) (cf. [8] p. 185) and so of the first category in \((0, 1)\) as well. Since \( M \cap \mathbb{Q} \) is a countable set, we see that \( M = (M \cap \mathbb{Q}) \cup (M \cap \mathbb{Q}') \) is a set of the first category in \((0, 1)\). On account of (7) we get that \( \varrho(D(\alpha)) \) is a set of the first category in \((0, 1)\). This completes the proof. \( \square \)
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