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ON UNIFORM DISTRIBUTION OF SEQUENCES $(a_n x)_1^\infty$

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Introduction

There are two possible approaches to the study of uniform distribution (mod 1) of sequences

$$(a_n x)_1^\infty \tag{1}$$

where $a_n \in \mathbb{R}$ $(n = 1, 2, \ldots)$ and $x \in \mathbb{R}$. The first such approach is the study of (1) with a fixed sequence $(a_n)_1^\infty$, x running over real numbers, the second is the study of (1) with a fixed $x \in \mathbb{R}$ and $(a_n)_1^\infty$ running over a class of sequences of real numbers. The second approach leads to the concept od α-good sequences (cf. [1]).

In the first part of the paper we will apply the first and in the second part the second approach to the investigation of sequences (1).

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1. Uniform distribution (mod 1) of sequences $(a_n x)_1^\infty$ with fixed $(a_n)_1^\infty$

In this part we restrict ourselves to the study of (1) with a fixed sequence $(a_n)_1^\infty$ of real numbers. Denote by $H\ (a_1, a_2, \ldots)$ the set of all $x \in \mathbb{R}$ for which the sequence (1) is uniformly distributed (mod 1) (shortly: u.d. mod 1). It is wellknown that if $a_n \in \mathbb{N}$ $(n = 1, 2, \ldots)$, $a_i \neq a_j$ for $i \neq j$, then the set $H(a_1, a_2, \ldots)$ has full measure (i.e. the set $\mathbb{R} \setminus H(a_1, a_2, \ldots)$ is a null set—cf. [2], [4] pp. 32–33). This results evokes the question what is the Baire category of the set $H(a_1, a_2, \ldots)$. We will show that the "topological magnitude" of $H(a_1, a_2, \ldots)$ depends on the sequence $(a_n)_1^\infty$. Indeed, if we choose $a_n = n$ or more generally $a_n = a + nd$ $(n = 1, 2, \ldots)$, $d \geqslant 1$, a, d integers, then by Weyl's criterion (cf. [4] pp. 7–8) the sequence $(a_n x)_1^\infty$ is u.d. mod 1 for each irrational x. Hence $H(a_1, a_2, \ldots)$ contains in this case all irrational numbers and so it is a residual set. In what follows we will give a class of sequences $(a_n)_1^\infty$ of positive integers such that $H(a_1, a_2, \ldots)$ is a set of the first category.

Theorem 1.1. *Let $(q_k)_1^\infty$ be a sequence of positive integers greater than* 1. *Put*

$$a_n = q_1 q_2 \ldots q_n \quad (n = 1, 2, \ldots).$$

Then $H(a_1, a_2, \ldots)$ is a set of the first Baire category in $\mathbb{R}$.

Proof. For $x \in \mathbb{R}$ we put

$$S(m, x) = \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi \mathrm{i} a_n x} \quad (m = 1, 2, \ldots).$$

Then by Weyl's criterion we have

$$H(a_1, a_2, \ldots) \subseteq H_0(a_1, a_2, \ldots),$$

where $H_0(a_1, a_2, \ldots) = \left\{x \in \mathbb{R}\colon \lim\limits_{m\to\infty} S(m, x) = 0\right\}$. Denote by $C(a_1, a_2, \ldots)$ the set of all $x \in \mathbb{R}$ for which there exists $\lim\limits_{m\to\infty} S(m, x) = S(x)$. Then evidently

$$H(a_1, a_2, \ldots) \subseteq H_0(a_1, a_2, \ldots) \subseteq C(a_1, a_2, \ldots). \tag{2}$$

Each of these sets has the full measure. By (2) it suffices to prove that $C = C(a_1, a_2, \ldots)$ is a set of the first category in $\mathbb{R}$. We prove it in the following.

Obviously each of the functions $S(m, x)$ $(m = 1, 2, \ldots)$ is continuous on C (i.e. the restrictions $S(m, x)\big|C$ are continuous on C). Hence the function $S(x) = \lim\limits_{m\to\infty} S(m, x)$ defined on C is in the first Baire class on C. But then the set of all discontinuity points of S is a set of the first category in C (cf. [8] p. 185), and so it is a set of the first category in $\mathbb{R}$, as well.

To complete the proof it suffices to show that the function S is discontinuous at every $x \in C$. For this it suffices to show that each of the sets

$$M_0 = \{x \in C\colon\ S(x) = 0\}, \quad M_1 = \{x \in C\colon\ S(x) = 1\}$$

is dense in C.

The density of M_0 follows from the fact that $M_0 \subseteq C$ and M_0 has the full measure.

We prove that M_1 is dense in C. It is wellknown that every $x \in \mathbb{R}$ has the Cantor series expansion

$$x = c_0 + \sum_{j=1}^{\infty} \frac{c_j}{q_1 q_2 \ldots q_j} = c_0 + \sum_{j=1}^{\infty} \frac{c_j}{a_j},$$

where c_j are integers, $0 \leqslant c_j < q_j$, $a_j = q_1 q_2 \ldots q_j$ $(j = 1, 2, \ldots)$.

Denote by A_k the set of all $x \in \mathbb{R}$ of the form

(3) $$x = c_0 + \sum_{j=1}^{k} \frac{c_j}{a_j},$$

where $k \in \mathbb{N}$, c_0 is an integer and $0 \leqslant c_j < q_j$ $(j = 1, 2, \ldots, k)$. If $x \in A_k$, then $a_n x$ is an integer for $n > k$. Thus for $m > k$ we have

$$S(m, x) = \frac{1}{m} \sum_{n=1}^{k} + \frac{1}{m} \sum_{n=k+1}^{m} 1 = O(1) + \frac{m-k}{m} \to 1 \quad \text{if } m \to \infty.$$

Put $A = \bigcup\limits_{k=1}^{\infty} A_k$. Then $A \subseteq M_1 \subseteq C$ and A is obviously dense in C. The density of M_1 in C follows. This completes the proof. □

We give the following simple observation.

Proposition 1.1. *Let $(a_j)_1^{\infty}$ be an arbitrary sequence of real numbers. Then $H(a_1, a_2, \ldots)$ is an $F_{\sigma\delta}$-set in $\mathbb{R}$.*

P r o o f. Using Weyl's criterion we can easily check that

$$H(a_1, a_2, \ldots) = \bigcap_{h \neq 0} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A(n, h, k),$$

where

$$A(n, h, k) = \left\{x \in \mathbb{R}\colon\ \left| \frac{1}{n} \sum_{j=1}^{n} \mathrm{e}^{2\pi \mathrm{i} h a_j x} \right| \leqslant \frac{1}{k} \right\}.$$

Since $\frac{1}{n}\sum\limits_{j=1}^{n} \mathrm{e}^{2\pi i h a_j x}$ $(n = 1, 2, \ldots)$ are continuous functions, we see that $A(n, h, k)$ is a closed set (if n, h, k are fixed) and so $H(a_1, a_2, \ldots)$ is an $F_{\sigma\delta}$-set in $\mathbb{R}$. $\square$

Remark 1.1. a) If $(a_n)_1^\infty$ is a sequence of distinct integers then by [2] and Proposition 1.1 the set $H(a_1, a_2, \ldots)$ is an $F_{\sigma\delta}$-set of the full measure.

b) For some particular choices of $(a_n)_1^\infty$ the set $H(a_1, a_2, \ldots)$ can belong to lower Borel classes. For instance if $a_n = a \in \mathbb{R}$, $(n = 1, 2, \ldots)$, then the set $H(a_1, a_2, \ldots)$ is empty while it coincides with the set $\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers if $a_n = n$ $(n = 1, 2, \ldots)$.

2. Uniform distribution (mod 1) of sequences $(a_n x)_1^\infty$ with fixed x

Let α be an irrational number. A sequence $a_1 < a_2 < \ldots$ of positive integers is said to be α-good provided the sequence $(a_n\alpha)_1^\infty$ is uniformly distributed (mod 1) (cf. [1]). The sequence $1 < 2 < \ldots < n < \ldots$ and the sequence $p_1 < p_2 < \ldots < p_n < \ldots$ of all prime numbers are α-good for each irrational α (cd. [1], [4] p. 22).

For $\alpha \in \mathbb{Q}'$ ($\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$) denote by $D(\alpha)$ the set of all α-good sequences. Note that every infinite sequence $a_1 < a_2 < \ldots < a_n < \ldots$ of positive integers belongs to $D(\alpha)$ for almost all $\alpha \in \mathbb{Q}'$ (cf. [4] p. 32, Theorem 4.1).

We will investigate the properties of the classes $D(\alpha)$ for $\alpha \in \mathbb{Q}'$. We will show that these classes have several common properties (for all $\alpha \in \mathbb{Q}'$).

It seems to be interesting to deal with the question about magnitude of classes $D(\alpha)$ ($\alpha \in \mathbb{Q}'$). This "magnitude" will be studied from the point of view of dyadic numbers of sets $A \subseteq \mathbb{N}$.

Denote by U the class of all infinite sets

$$A = \{a_1 < a_2 < \ldots < a_n < \ldots\} \subseteq \mathbb{N}.$$

In what follows we identify the set A with the sequence $a_1 < a_2 < \ldots < a_n < \ldots$ Put

$$\varrho(A) = \sum_{k=1}^{\infty} 2^{-a_k} \in (0, 1]$$

for each $A \subseteq U$. Then ϱ is a one-to-one mapping of U onto $(0, 1]$. If S is a class of infinite subsets of $\mathbb{N}$, then we put $\varrho(S) = \{\varrho(A) \colon A \in S\}$. The set $\varrho(S)$ "measures" the magnitude of the class S (cf. [5] p. 17).

We will investigate metric and topological properties of the sets $\varrho(D(\alpha))$.

Recall that a measurable set $M \subseteq (0,1]$ is called homogeneous if there is a real number $d \in [0,1]$ such that for every interval $I \subseteq (0,1]$ we have

$$\frac{\lambda(I \cap M)}{\lambda(I)} = d,$$

λ being the Lebesgue measure (cf. [9], [10]).

Theorem 2.1. *For each $\alpha \in \mathbb{Q}'$ the set $\varrho(D(\alpha))$ is a homogeneous $F_{\sigma\delta}$-set in $(0,1]$.*

Proof. According to Weyl's criterion a sequence $a_1 < a_2 < \ldots$ of positive integers belongs to $D(\alpha)$ if and only if

$$(\forall h \in Z, h \neq 0)\colon \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi \mathrm{i} h a_n \alpha} = 0.$$

This condition is equivalent to the condition

$$(\forall |h| \geqslant 1)(\forall k \geqslant 1)(\exists v \in \mathbb{N})(\forall m \geqslant v)\colon \left| \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi \mathrm{i} h \alpha a_n} \right| \leqslant \frac{1}{k}. \tag{4}$$

From (4) we get

$$\varrho(D(\alpha)) = \bigcap_{|h|\geqslant 1} \bigcap_{k=1}^{\infty} \bigcup_{v=1}^{\infty} \bigcap_{m=v}^{\infty} M(m,h,k),$$

where

$$M(m,h,k) = \left\{ x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0,1]\colon \left| \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi \mathrm{i} h \alpha a_n} \right| \leqslant \frac{1}{k} \right\}. \tag{6}$$

Construct the functions

$$f_{m,h}(x) = \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi \mathrm{i} h \alpha a_n} \quad (m = 1,2,\ldots;\ h \in Z, h \neq 0),$$

where $x = \sum\limits_{j=1}^{\infty} 2^{-a_j} \in (0,1]$. These functions are defined for each $x \in (0,1]$. We will verify that their restrictions to $\mathbb{Q}' \cap (0,1]$ are continuous on $\mathbb{Q}' \cap (0,1]$.

Let $x_0 \in \mathbb{Q}' \cap (0,1]$, $x_0 = \sum\limits_{j=1}^{\infty} 2^{-b_j}$ $(b_1 < b_2 < \ldots)$ be the dyadic expansion of x_0. Fix the number m. Notice that the set of all numbers of the form $x = \sum\limits_{j=1}^{\infty} 2^{-a_j}$,

$a_j = b_j$ $(j = 1, 2, \ldots, m)$ fils up an interval I_m containing x_0, the left-hand endpoint of which is the rational number $\sum_{j=1}^{m} 2^{-b_j}$ and the right-hand endpoint is

$$\sum_{j=1}^{m} 2^{-b_j} + \sum_{j=b_m+1}^{\infty} 2^{-j} = \sum_{j=1}^{m} 2^{-b_j} + 2^{-b_m}.$$

Obviously the function $f_{m,h}|\mathbb{Q}' \cap (0,1]$ is constant on I_m and so it is continuous at x_0.

From the continuity of functions $f_{m,h}|\mathbb{Q}' \cap (0,1]$ the closedness of the sets $M(m,h,k)$ in $\mathbb{Q}' \cap (0,1]$ follows (see (6)). But then by (5) the set $\mathbb{Q}' \cap \varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$. Notice that

$$\varrho(D(\alpha)) = [\mathbb{Q}' \cap \varrho(D(\alpha))] \cup [\mathbb{Q} \cap \varrho(D(\alpha))]\,,$$

the second "summand" on the right-hand side being countable. From this we see that $\varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$.

The homogeneity of the set $\varrho(D(\alpha))$ can be proved by using a result form [7] (cf. [7], Lemma 1, pp. 255–256). We will use the following special case of Lemma 1 from [7]:

(T) Let $B \subseteq (0,1]$ be a measurable set. Suppose that for each $n = 1, 2, \ldots$ and $k, k' \in \{0, 1, \ldots, 2^n - 1\}$ we have

$$\lambda(B \cap i_n^{(k)}) = \lambda(B \cap i_n^{(k')}),$$

where

$$i_n^{(v)} = \left(\frac{v}{2^n}, \frac{v+1}{2^n}\right], \quad v \in \{0, 1, \ldots, 2^n - 1\}.$$

Then B is a homogeneous set in $(0,1]$.

If now $a_1 < a_2 < \ldots < a_n < \ldots$ is an α-good sequence and a sequence $d_1 < d_2 < \ldots < \ldots$ differs from $a_1 < a_2 < \ldots < a_n < \ldots$ only in a finite number of terms, then evidently also $d_1 < d_2 < \ldots < \ldots$ is an α-good sequence. Hence the assumptions in (T) are satisfied and so by (T) the set $\varrho(D(\alpha))$ is homogeneous in $(0,1]$.

It is wellknown that the Lebesgue measure of a homogeneous set $A \subseteq (0,1]$ is 0 or 1 (cf. [9], [10]). Hence by Theorem 2.1 we have $\lambda(\varrho(D(\alpha))) = 0$ or $\lambda(\varrho(D(\alpha))) = 1$ for each $\alpha \in \mathbb{Q}'$. We will show that this measure is equal to 1 for each $\alpha \in \mathbb{Q}'$. □

Theorem 2.2. *For each $\alpha \in \mathbb{Q}'$ we have $\lambda(\varrho(D(\alpha))) = 1$.*

Proof. Let $\alpha \in \mathbb{Q}'$. Then the sequence $(n\alpha)_{n=1}^{\infty}$ is u.d. mod 1 (cf. [4] pp. 7–8). By a theorem of Peterson (cf. [6]), if $(v_n)_1^{\infty}$ is a u.d. mod 1 sequence then for almost all

$x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0,1]$ the sequence $(v_{a_j})_{j=1}^{\infty}$ (subsequence of $(v_n)_1^{\infty}$) is u.d. mod 1 as well. Hence for almost all $x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0,1]$ the sequence $(a_j\alpha)_{j=1}^{\infty}$ is u.d. mod 1. But this means that almost all $x \in (0,1]$ belong to the set $\varrho(D(\alpha))$. □

We now will investigate the magnitude of sets $\varrho(D(\alpha))$ from the topological point of view. We prove the following universal theorem.

Theorem 2.3. *For every $\alpha \in \mathbb{Q}'$ the set $\varrho(D(\alpha))$ is a dense $F_{\sigma\delta}$-set of the first Baire category in* $(0,1]$.

Proof. Let $\alpha \in \mathbb{Q}'$. By Theorem 2.1 the set $\varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$.

Further, the set $D(\alpha)$ is non-empty (and such is also the set $\varrho(D(\alpha))$ since the sequence $1 < 2 < \ldots < n \ldots$ belongs to $D(\alpha)$. The density of $\varrho(D(\alpha))$ follows from the above mentioned fact that together with $1 < 2 < \ldots < n \ldots$ the class $D(\alpha)$ contains every sequence $a_1 < a_2 < \ldots < a_n < \ldots$ which differs from $1 < 2 < \ldots < n \ldots$ only in a finite number of terms.

We prove that $\varrho(D(\alpha))$ is a set of the first Baire category. For $t = \varrho(A)$, $A = a_1 < a_2 < \ldots < a_n < \ldots$ we put

$$g_m(t) = \frac{1}{m} \sum_{n=1}^{m} s^{2\pi i \alpha a_n} \quad (m = 1, 2, \ldots).$$

Denote by M the set of all $t \in (0,1]$ $(t = \varrho(A),\ A = a_1 < a_2 < \ldots)$ for which there exists $\lim_{m\to\infty} g_m(t) = g(t)$. By Weyl's criterion we get

$$\varrho(D(\alpha)) \subseteq M. \tag{7}$$

It is easy to verify that the functions $g_m\big|\mathbb{Q}' \cap (0,1]$ are continuous on $\mathbb{Q}' \cap (0,1]$ (and so they are continuous on $M \cap \mathbb{Q}' \subseteq \mathbb{Q}' \cap (0,1]$ as well). This can be proved in an analogous way as the continuity of the functions $f_{m,h}\big|\mathbb{Q}' \cap (0,1]$ in the proof of Theorem 2.1. Thus the function $g\big|M \cap \mathbb{Q}'$ is in the first Baire class on $M \cap \mathbb{Q}'$. This implies that the set of discontinuity points of $g\big|M \cap \mathbb{Q}'$ is a set of the first category in $M \cap \mathbb{Q}'$ (cf. [8] p. 185).

We will show that the function g is discontinuous at every point of $M \cap \mathbb{Q}'$. To show this it suffices to prove that each of the sets

$$M_0 = \{x \in M \cap \mathbb{Q}' :\ g(x) = 0\}\,, M_1 = \{x \in M \cap \mathbb{Q}' :\ g(x) = 1\}$$

is dense in $M \cap \mathbb{Q}'$.

In the first place we prove the density of M_0 in $M \cap \mathbb{Q}'$. If $p_1 < p_2 < \ldots < p_n < \ldots$ is the sequence of all primes then $x_0 = \sum_{k=1}^{\infty} 2^{-p_k}$ belongs to M_0 (cf. [1], [4] p. 22). Together with x_0 each $\varrho(A)$ belongs to M_0, where A is an infinite set of positive integers which differs from $\{p_1 < p_2 < \ldots < p_n < \ldots\}$ only in a finite number of elements. From this the density of M_0 in $M \cap \mathbb{Q}'$ follows.

For the proof of density of M_1 it suffices to construct a sequence $A_0 = a_1 < a_2 < \ldots < a_n < \ldots$ such that $y_0 = \varrho(A_0)$ is an irrational number with $g(y_0) = 1$. Such a sequence can be obtained by the following procedure:

Take into account the continued fraction of α. It is wellknown that if $\frac{p_n}{q_n}$ $(n = 1, 2, \ldots)$ are convergents of this continued fraction, then

$$|q_n\alpha - p_n| < \frac{1}{q_n} \quad (n = 1, 2, \ldots)$$

(cf. [3] p. 27). Further, if n is even then $\frac{p_n}{q_n} < \alpha$ (cf. [3] p. 22). But then for such even n we have $0 < q_n\alpha - p_n < \frac{1}{q_n}$, thus $\{q_n\alpha\} = q_n\alpha - [q_n\alpha] = q_n\alpha - p_n < \frac{1}{q_n}$. So we get

$$\{q_n\alpha\} = \frac{\vartheta_n}{q_n}, \quad 0 < \vartheta_n < 1.$$

Choose a set $N_2 = \{k_1 < k_2 < \ldots < k_n < \ldots\}$ of even numbers such that

$$\lim_{n\to\infty} (q_{k_{n+1}} - q_{k_n}) = +\infty \tag{8}$$

and put $q'_n = q_{k_n}$ $(n = 1, 2, \ldots)$. Then $y_0 = \sum_{n\in N_2} 2^{-q'_n}$ belongs to $\mathbb{Q}'$ since the condition (8) guarantees that the dyadic expansion of y_0 is not periodic. Further,

$$\mathrm{e}^{2\pi\mathrm{i}\alpha q'_n} = \mathrm{e}^{2\pi\mathrm{i}([\alpha q'_n]+\{\alpha q'_n\})} = \mathrm{e}^{2\pi\mathrm{i}\{\alpha q'_n\}} = \mathrm{e}^{2\pi\mathrm{i}\frac{\vartheta'_n}{q'_n}} \quad (0 < \vartheta'_n < 1,\ n \in N_2).$$

For all sufficiently large n's (e.g. for $n > n_0$) we have

$$0 < \frac{2\pi\vartheta'_n}{q'_n} < \frac{1}{2}.$$

So we get

$$g_m(y_0) = \frac{1}{m}\sum_{n=1}^{n_0} + \frac{1}{m}\sum_{n=n_0+1}^{m} \mathrm{e}^{2\pi\mathrm{i}\{\alpha q'_n\}} = O(1)$$
$$+ \left(\frac{1}{m}\sum_{n=n_0+1}^{m} \cos\frac{2\pi\vartheta'_n}{q'_n} + i\frac{1}{m}\sum_{n=n_0+1}^{m} \sin\frac{2\pi\vartheta'_n}{q'_n}\right).$$

Note that

$$\left|\sin \frac{2\pi\vartheta'_n}{q'_n}\right| \leqslant \frac{2\pi\vartheta'_n}{q'_n}.$$

Since $q'_n \to \infty$ $(n \to \infty)$, we have

$$\left|\frac{1}{m}\sum_{n=n_0+1}^{m} \sin \frac{2\pi\vartheta'_n}{q'_n}\right| \leqslant \frac{1}{m}\sum_{n=1}^{m} \frac{2\pi}{q'_n} \to 0$$

(for $m \to \infty$) (Cesàro means).

Further, by the inequality

$$\cos x > 1 - \frac{x^2}{2} \quad (x \in (0,1))$$

we get (for $n > n_0$)

$$\cos \frac{2\pi\vartheta'_n}{q'_n} > 1 - \frac{1}{2}\left(\frac{2\pi\vartheta'_n}{q'_n}\right)^2 > 1 - \frac{2\pi^2}{{q'_n}^2}.$$

Therefore we have

$$\begin{aligned}\frac{1}{m}\sum_{n=n_0+1}^{m} \cos \frac{2\pi\vartheta'_n}{q'_n} &> \frac{1}{m}\sum_{n=n_0+1}^{m}\left(1 - \frac{2\pi^2}{{q'_n}^2}\right)\\ &= \frac{1}{m}\sum_{n=n_0+1}^{m} 1 - \frac{2\pi^2}{m}\sum_{n=n_0+1}^{m}\frac{1}{{q'_n}^2}.\end{aligned}$$

The second summand on the right-hand side has the limit 0 if $m \to \infty$ while the first tends to 1. Hence $\lim\limits_{m\to\infty} g_m(y_0) = 1$.

So we have proved that g is a function in the first Baire class on $M \cap \mathbb{Q}'$, discontinuous at every point of $M \cap \mathbb{Q}'$. Therefore $M \cap \mathbb{Q}'$ is a set of the first category in $M \cap \mathbb{Q}'$ (cf. [8] p. 185) and so of the first category in $(0,1]$ as well. Since $M \cap \mathbb{Q}$ is a countable set, we see that $M = (M \cap \mathbb{Q}) \cup (M \cap \mathbb{Q}')$ is a set of the first category in $(0,1]$. On account of (7) we get that $\varrho(D(\alpha))$ is a set of the first category in $(0,1]$. This completes the proof. $\square$

References

[1] *D. Carlson*: Good sequences of integers. J. Number Theory *7* (1975), 91–104.
[2] *H. Davenport, P. Erdös, W. J. Le Veque*: On Weyl's criterion for uniform distribution. Michigan Math. J. *10* (1963), 311–314.
[3] *A. Khintchine*: Continued Fractions. Gos. Izd. Mat. Lit., Moscow, 1961. (In russian.)
[4] *L. Kuipers, H. Niederreiter*: Uniform Distribution of Sequences. John Wiley, New York-London-Sydney-Toronto, 1974.
[5] *H. H. Ostmann*: Additive Zahlentheorie I.. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.
[6] *G. M. Petersen, M. T. Mc. Gregor*: On the structure of well distributed sequences (II.). Indag. Math. *XVI* (1964), 477–487.
[7] *T. Šalát*: Eine metrische Eigenschaft der Cantorschen Entwicklungen der reellen Zahlen und Irrationalitätskriterien. Czechoslovak Math. J. *14 (89)* (1964), 254–266.
[8] *R. Sikorski*: Real Functions I. (Polish). PWN, Warszawa, 1958.
[9] *A. Simmons*: An "Archimedean" paradox. Amer. Math. Monthly *89* (1982), 114–115.
[10] *C. Visser*: The law of nought-or-one. Studia Math. *7* (1938), 143–159.

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