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OSCILLATION OF CERTAIN DIFFERENCE EQUATIONS

S. R. Grace, Orman

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Abstract. Some new criteria for the oscillation of difference equations of the form

$$\Delta^2 x_n - p_n \Delta x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0$$

and

$$\Delta^i x_n + p_n \Delta^{i-1} x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0,\ i = 2, 3,$$

are established.

1. Introduction

In this paper we will discuss the oscillatory property of certain difference equations of the form

$$\Delta^2 x_n - p_n \Delta x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0 \tag{(E_1)}$$

and

$$\Delta^i x_n + p_n \Delta^{i-1} x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0,\ i = 2, 3, \tag{(E_i)}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers, $\{g_n\}$ is a sequence of integers, h is an integer and c is any positive real number, and $g_n \to \infty$ as $n \to \infty$.

The oscillation, nonoscillation and asymptotic behavior of Eq. (E_1) when $p_n = 0$ have been considered by many authors, we refer to [4–7, 9, 10, 12] and the references cited therein.

A real solution $\{x_n\}$, $n \geqslant 0$ of Eq. (E_1) (or Eq. (E_i), $i = 2, 3$) is said to be nonoscillatory if there exists $N \geqslant 0$ such that $x_n x_{n+1} > 0$ for all $n \geqslant N$, and is oscillatory otherwise. Eq. (E_i), $i = 1, 2$ or 3 is said to be almost oscillatory if every solution $\{x_n\}$ of Eq. (E_i), $i = 1, 2$ or 3 is oscillatory or $\{\Delta x_n\}$ is oscillatory for Eq. (E_i), $i = 1$ or 2, or $\{\Delta^2 x_n\}$ is oscillatory for Eq. (E_3).

Eq. (E_1) and Eq. (E_i), $i = 2, 3$ may be viewed as discrete analogues of the functional differential equations

$$x''(t) - p(t)x'(t-h) + q(t)|x(g(t))|^c \operatorname{sgn} x(g(t)) = 0 \tag{(F_1)}$$

and

$$x^{(i)}(t) + p(t)x^{(i-1)}(t-h) + q(t)|x(g(t))|^c \operatorname{sgn} x(g(t)) = 0,\ i = 2, 3 \tag{(F_i)}$$

respectively, where $g, p, q : [t_0, \infty) \to R$, $t_0 \geqslant 0$ are continuous, $g(t) \to \infty$ as $t \to \infty$, $p(t) \geqslant 0$ and $q(t) \geqslant 0$ eventually, c and h are real numbers and $c > 0$. In fact the results in this paper are motivated by similar results for Eq. (F_1) and Eq. (F_i), $i = 2, 3$, see [1–3].

The purpose of this paper is to establish some new criteria for the almost oscillation of Eq. (E_i), $i = 1, 2, 3$. In Section 2 we establish two criteria for the almost oscillation of Eq. (E_1) when $c > 0$ and $c > 1$. In Section 3 we deal with the oscillatory and asymptotic behavior of Eq. (E_2) and obtain sufficient conditions for any solution $\{x_n\}$ of Eq. (E_2) either to be oscillatory or else approach zero monotonically as $n \to \infty$. Also, we give sufficient conditions for all solutions of Eq. (E_2) to be almost oscillatory when $c = 1$. The final section presents two criteria for the almost oscillation of Eq. (E_3) when $c > 0$ and $c = 1$.

2. Almost oscillation of Eq. (E_1)

The following result is concerned with the oscillation of Eq. (E_1) for any $c > 0$.

Theorem 1. *Let h be any nonnegative integer and $\Delta p_n \geqslant 0$ for $n \geqslant n_0 \geqslant 0$. If*

$$\sum_{i=n_0}^{\infty} q_i = \infty \tag{1}$$

and

$$\sum_{j=n_0}^{\infty} a_{j+1} \sum_{i=n_0}^{j-1} q_i = \infty, \tag{2}$$

where

$$a_{j+1} = \prod_{i=n_0}^{j} (1 + p_i)^{-1}, \quad j \geqslant 1,$$

then Eq. (E_1) is almost oscillatory.

Proof. Assume for the sake of contradiction that Eq. (E_1) has a nonoscillatory solution $\{x_n\}$, which we may and will assume to be eventually positive. There exists a positive integer $n_1 \geqslant n_0$ such that $x_{g_n} > 0$ for $n \geqslant n_1$.

Next, we consider the following two cases:

(A) $\Delta x_n < 0$ eventually, (B) $\Delta x_n > 0$ eventually.

(A) Assume $\Delta x_n < 0$ eventually. From Eq. (E_1), we observe that $\Delta^2 x_n \leqslant 0$ eventually and hence one can easily see that $x_n \to -\infty$ as $n \to \infty$, a contradiction.

(B) Assume $\Delta x_n > 0$ eventually. There exist $N \geqslant n_2$ and a constant $c_1 > 0$ such that

$$x_{g_n} \geqslant c_1 \quad \text{for} \quad n \geqslant N. \tag{3}$$

Using (3) in Eq. (E_1) we have

$$\Delta^2 x_n - p_n \Delta x_{n-h} + b q_n \leqslant 0 \quad \text{for} \quad n \geqslant N, \tag{4}$$

where $b = c_1^c$. Summing both sides of (4) from N to $n-1 \geqslant N$, we get

$$\Delta x_n - \Delta x_N - \sum_{i=N}^{n-1} p_i \Delta x_{i-h} + b \sum_{i=N}^{n-1} q_i \leqslant 0,$$

or, using summation by part,

$$\Delta x_n - \Delta x_N - p_n x_{n-h} + p_N x_{N-h} + \sum_{i=N}^{n-1} x_{n-h+1} \Delta p_i + b \sum_{i=N}^{n-1} q_i \leqslant 0.$$

Using the fact that $\Delta p_n \geqslant 0$ and $x_n > 0$ for $n \geqslant n_2$, we have

$$\Delta x_n - \Delta x_N - p_n x_n + b \sum_{i=N}^{n-1} q_i \leqslant 0, \; n \geqslant N+1.$$

From (1), there exists $N_1 \geqslant N+1$ such that

$$\Delta x_N \leqslant \frac{1}{2} b \sum_{i=N}^{n-1} q_i \quad \text{for} \quad n \geqslant N_1 + 1.$$

Thus,

$$\Delta x_n - p_n x_n + \frac{1}{2} b \sum_{i=N}^{n-1} q_i \leqslant 0 \quad \text{for} \quad n \geqslant N_1 + 1. \tag{5}$$

Define a sequence $\{r_n\}$ by the recurrence relation

$$r_{n+1} = \frac{1}{1+p_n}, \quad n \geqslant n_0 \geqslant 0 \text{ and } r_{n_0} > 0.$$

Next, we multiply (5) by r_{n+1}, obtaining

$$\Delta(r_n x_n) + \frac{1}{2} b r_{n+1} \sum_{i=N}^{n-1} q_i \leqslant 0 \quad \text{for} \quad n \geqslant N_1 + 1. \tag{6}$$

Summing both sides of (6) from $N_1 + 1$ to $k \geqslant N_1 + 1$, we have

$$0 < r_{k+1} x_{k+1} \leqslant r_{N_1+1} x_{N_1+1} - \frac{1}{2} b \sum_{n=N_1+1}^{k} r_{n+1} \sum_{i=N}^{n-1} q_i \to -\infty \text{ as } k \to \infty,$$

a contradiction. This completes the proof. □

The following theorem deals with the almost oscillation of Eq. (E_1) when $g_n \geqslant n+2$, $n \geqslant n_0 \geqslant 0$ and $c > 1$.

Theorem 2. *Let h be a nonnegative integer, $c > 1$, $g_n \geqslant n+2$ for $n \geqslant n_0 \geqslant 0$, and assume that there exists a real sequence $\{z_n\}$, $n \geqslant n_0$ such that*

$$z_n > 0,\ \Delta z_n \geqslant 0,\ \Delta^2 z_n \leqslant 0 \text{ and } \Delta(z_n p_n) \leqslant 0 \text{ for } n \geqslant n_0. \tag{7}$$

If

$$\sum_{n=n_0}^{\infty} z_n q_n = \infty, \tag{8}$$

then Eq. (E_1) is almost oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_1), say $x_n > 0$ and $x_{g_n} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. As in the proof of Theorem 1, we consider the cases (A) and (B) and observe that case (A) is impossible. Next, we consider the case (B):

(B) Assume $\Delta x_n > 0$ for $n \geqslant N \geqslant n_1 + h$. Set

$$w_n = z_n \Delta x_n / x_{n+1}^c \quad \text{for} \quad n \geqslant N.$$

Then

$$\begin{aligned} \Delta w_n &= z_{n+1}(\Delta x_{n+1}/x_{n+2}^c) - z_n(\Delta x_n/x_{n+1}^c) \\ &= -z_n q_n (x_{g_n}/x_{n+2})^c + z_n p_n (\Delta x_{n-h}/x_{n+2}^c) + z_n \Delta x_{n+1}(x_{n+2}^{-c} - x_{n+1}^{-c}) \\ &\quad + \Delta z_n(\Delta x_{n+1}/x_{n+2}^c), \text{ and hence we see that} \\ \Delta w_n &\leqslant -z_n q_n + z_n p_n (\Delta x_{n-h}/x_{n+2}^c) + z_n(\Delta x_{n+1}/x_{n+2}^c),\ n \geqslant N. \end{aligned} \tag{9}$$

Summing both sides of (9) from N to $k-1 \geqslant N$, using (7) and the fact that $x_{n+2} \geqslant x_{n-h+1}$, $n \geqslant N$, we obtain

$$w_k - w_N \leqslant -\sum_{n=N}^{k-1} z_n q_n + z_N p_N \sum_{n=N}^{k-1} \Delta x_{n-h}/x_{n-h+1}^c + \Delta z_N \sum_{n=N}^{k-1} \Delta x_{n+1}/x_{n+2}^c.$$

As in the proof of Theorem 4.1 in [7], we have

$$\sum^{\infty} \Delta x_i/x_{i+1}^c < \infty,$$

and hence by (8), it follows that

$$0 < w_k \leqslant C - \sum_{n=N}^{k-1} z_n q_n \to -\infty \text{ as } k \to \infty,$$

where C is a constant, a contradiction. This completes the proof. □

Remark 1. One can easily observe that Theorems 1 and 2 are applicable to equations of type (E_1) when $h = 0$ or $p_n = 0$ for $n \geqslant 0$.

3. Oscillation and asymptotic behavior of Eq. (E_2)

Theorem 3. *Let h be any positive integer,*

$$\liminf_{n\to\infty} \sum_{k=n-h}^{n-1} p_k > \Big(\frac{h}{1+h}\Big)^{1+h}, \tag{10}$$

and assume that there exists a real sequence $\{z_n\}$ such that

$$z_n > 0, \Delta z_n \geqslant 0 \text{ and } \Delta(z_n p_n) \leqslant 0 \text{ for } n \geqslant n_0 \geqslant 0. \tag{11}$$

If condition (8) holds and

$$\sum_{i=n_0}^{\infty} 1/z_n = \infty, \tag{12}$$

then every solution $\{x_n\}$ of Eq. (E_2) is oscillatory or $\{\Delta x_n\}$ is oscillatory or else $x_n \to 0$ monotonically as $n \to \infty$.

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_2). There exists $n_1 \geqslant n_0 \geqslant 0$ such that $x_{g_n} > 0$ for $n \geqslant n_1$. Next, we consider the following two cases:

(A*) $\Delta x_n > 0$ eventually, (B*) $\Delta x_n < 0$ eventually.

(A*) Suppose $\Delta x_n > 0$ eventually. From Eq. (E_2) we see that

$$\Delta^2 x_n + p_n \Delta x_{n-h} = -q_n x_{g_n}^c \leqslant 0 \quad \text{eventually.}$$

Set $y_n = \Delta x_n > 0$ eventually. Then

$$\Delta y_n + p_n y_{n-h} \leqslant 0 \text{ eventually.} \tag{13}$$

In view of Theorem 3 in [11] and condition (10), inequality (13) has no eventually positive solution, which is a contradiction.

(B*) Suppose $\Delta x_n < 0$ for $n \geqslant N \geqslant n_2$. So, we have

$$x_n \to c_1 \geqslant 0 \quad \text{as} \quad n \to \infty.$$

Suppose that $c_1 > 0$ and consider the sequence $\{w_n\}$ defined by

$$w_n = z_{n-1} \Delta x_n \quad \text{for } n \geqslant N.$$

Then

$$\begin{aligned} \Delta w_n &= \Delta(z_{n-1}\Delta x_n) = z_n \Delta^2 x_n + \Delta z_{n-1} \Delta x_n \\ &\leqslant -b z_n q_n - z_n p_n \Delta x_{n-h} + \Delta z_{n-1} \Delta x_n \\ &\leqslant -b z_n q_n - z_n p_n \Delta x_{n-h}, \ n \geqslant N, \end{aligned}$$

where $b = c_1^c$. Summing both sides of the above inequality from N to $k-1 \geqslant N$, we obtain

$$w_k - w_N \leqslant -b \sum_{n=N}^{k-1} z_n q_n - \sum_{n=N}^{k-1} z_n p_n \Delta x_{n-h}.$$

Using (11), we have

$$w_k \leqslant -b \sum_{n=N}^{k-1} z_n q_n + z_N p_N (x_{N-h} - x_{k-h}) \leqslant C - b \sum_{n=N}^{k-1} z_n q_n,$$

where $C = z_N p_N x_{N-h}$. By condition (8), there exist $N^* \geqslant N$ and a constant $c^* > 0$ such that

$$w_k = z_{k-1} \Delta x_k \leqslant -c^*$$

or

$$\Delta x_k \leqslant -c^*/z_{k-1} \quad \text{for} \quad n \geqslant N^*.$$

Summing both sides of the above inequality from N^* to $m \geqslant N^* + 1$, letting $m \to \infty$ and using (12), we obtain a contradiction to the fact that $x_n > 0$ eventually. This complete the proof. $\square$

Theorem 4. *Let h be any integer and $\Delta p_n \leqslant 0$, $n \geqslant n_0 \geqslant 0$. If condition* (1) *holds, then the conclusion of Theorem* 3 *holds.*

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_2). As in the proof of Theorem 3, we see that $x_{g_n} > 0$ for $n \geqslant n_1$. Next, we consider the following two cases:

(A*) $\Delta x_n > 0$ eventually, (B*) $\Delta x_n < 0$ eventually.

(A*) Suppose $\Delta x_n > 0$ for $n \geqslant n_2 \geqslant n_1$. There exist constants $c_1 > 0$ and $N \geqslant n_2$ such that (3) holds for $n \geqslant N$. Now, from Eq. (E_2) we have

$$\Delta^2 x_n + bq_n \leqslant 0 \text{ for } n \geqslant N, \text{ where } b = c_1^c.$$

Summing both sides of the above inequality from N to $m \geqslant N+1$, letting $m \to \infty$ and using (1), we obtain a contradiction to the fact that $\Delta x_n > 0$ for $n \geqslant n_2$.

(B*) Suppose $\Delta x_n < 0$ eventually. The proof of this case is similar to the proof of Theorem 3 (B*) with $z_n = 1$, and hence is omitted.

The following result is concerned with the almost oscillation of Eq. (E_2). □

Theorem 5. *Let h be a nonpositive integer, $c = 1$ and $\Delta p_n \leqslant 0$ for $n \geqslant n_0 \geqslant 0$, and let condition* (1) *hold. Moreover, assume that there exists a sequence $\{k_n\}$ of positive integers such that $g_n \leqslant n - k_n$, $n \geqslant n_0$, $\{n - k_n\}$, $n \geqslant 0$ is increasing. If*

$$\liminf_{n\to\infty} \sum_{j=n-k_n}^{n-1} Q_j > \limsup_{n\to\infty} \Big(\frac{k_n}{(1+k_n)}\Big)^{1+k_n}, \tag{14}$$

where

$$Q_j = \sum_{i=j-k_j}^{j-1} q_i - p_{j-k_j} > 0, \ n-1 \geqslant j \geqslant n-k_n, \tag{15}$$

then Eq. (E_2) is almost oscillatory.

Proof. Let x_n be an eventually positive solution of Eq. (E_2). As in the proof of Theorem 3, we observe that $x_{g_n} > 0$ for $n \geqslant n_1$. Next, we consider the following two cases:

(A*) $\Delta x_n > 0$ eventually, (B*) $\Delta x_n < 0$ eventually.

(A*) Suppose $\Delta x_n > 0$ eventually. The proof of this case is similar to the proof of Theorem 4 (A*) and hence is omitted.

(B*) Suppose $\Delta x_n < 0$ for $n \geqslant N \geqslant n_2$. From Eq. (E_2) and the fact that $g_n \leqslant n - k_n$, $n \geqslant N$, we have

$$\Delta^2 x_n + p_n \Delta x_{n-h} + q_n x_{n-k_n} \leqslant 0 \quad \text{for} \quad n \geqslant N. \tag{16}$$

Summing both sides of (16) from $n-k_n$ to $n-1 \geqslant n-k_n$, $n \geqslant N$, we have

$$\Delta x_n - \Delta x_{n-k_n} + \sum_{j=n-k_n}^{n-1} p_j \Delta x_{j-h} + \sum_{j=n-k_n}^{n-1} q_j x_{j-k_j} \leqslant 0$$

or, using summation by parts,

$$\Delta x_n + \Big[p_n x_{n-h} - p_{n-k_n} x_{n-k_n} - \sum_{j=n-k_n}^{n-1} x_{n-h+1} \Delta p_j\Big] + \sum_{j=n-k_n}^{n-1} q_j x_{j-k_j} \leqslant 0, \; n \geqslant N.$$

Using the fact that $\Delta p_n \leqslant 0$ and $h < 0$, we obtain

$$\Delta x_n - p_{n-k_n} x_{n-k_n} + x_{n-k_n} \sum_{j=n-k_n}^{n-1} q_j \leqslant 0$$

or

$$\Delta x_n + Q_n x_{n-k_n} \leqslant 0 \quad \text{for} \quad n \geqslant N, \tag{17}$$

where Q_n is defined as in (15). But Theorem 3 in [11] and condition (14) imply that inequality (17) has no eventually solution, which is a contradiction. This completes the proof. □

Next, we consider the special case of Eq. (E_2), namely the equation

$$\Delta^2 x_n + p\Delta x_{n-h} + q x_{n-k} = 0, \tag{(L_2)}$$

where p and q are positive constants, h is a nonnegative integer and k is any positive integer.

The following corollary is a consequence of Theorem 5.

Corollary 1. *If*

$$kq - p > \frac{k^{k+1}}{(1+k)^{1+k}}, \tag{18}$$

then Eq. (L_2) *is almost oscillatory.*

Remark 2. (i) If we set $p_n = 0$, $n \geqslant 0$ in Theorems 3 and 4, we can easily check that Theorem 3 with $c > 1$ (or $0 < c < 1$) and Theorem 2.3 (or Theorem 2.4) in [4] are similar and Theorem 4 and Theorem 2.5 in [4] are the same and hence, we conclude that Eq. (E_2) with c as given above is oscillatory.

We note that the presence of the term—$p_n \Delta x_{n-h}$ makes the coexistence of oscillatory and monotonically decreasing positive (increasing negative) solutions for Eq. (E_2) possible.

(ii) We note that Theorem 5 is applicable to Eq. (E_2) when $p_n = 0$. Only condition (14) is disregarded.

4. Almost oscillation of Eq. (E_3)

Theorem 6. *Let h be a positive integer, $\Delta p_n \leqslant 0$ for $n \geqslant n_0 \geqslant 0$ and let conditions* (1) *and* (10) *hold. Then Eq.* (E_3) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_3), say $x_n > 0$ and $x_{g_n} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. Next, we consider the following two cases:
(A) $\Delta^2 x_n > 0$ eventually, (B) $\Delta^2 x_n < 0$ eventually.

(A) Suppose $\Delta^2 x_n > 0$ eventually. From Eq. (E_3) we have

$$\Delta^3 x_n + p_n \Delta^2 x_{n-h} = -q_n x_{g_n}^c \leqslant 0 \text{ eventually.}$$

Set $y_n = \Delta^2 x_n > 0$ eventually. Then

$$\Delta y_n + p_n y_{n-h} \leqslant 0 \text{ eventually.}$$

The rest of the proof is similar to that of Theorem 3 (A*) and hence is omitted.

(B) Suppose $\Delta^2 x_n < 0$ for $n \geqslant n_2 \geqslant n_1 + h$. It is easy to check that $\Delta x_n > 0$ for $n \geqslant n_1$ and there exist $N \geqslant n_2$ and a constant $c_1 > 0$ such that (3) holds for $n \geqslant N$. From Eq. (E_3) it follows that

$$\Delta^3 x_n + p_n \Delta^2 x_{n-h} + b q_n \leqslant 0, \tag{19}$$

where $b = c_1^c$. Summing both sides of (19) from N to $m-1 \geqslant N$, we have

$$\Delta^2 x_m - \Delta^2 x_N + \sum_{n=N}^{m-1} p_n \Delta^2 x_{n-h} + b \sum_{n=N}^{m-1} q_n \leqslant 0,$$

or

$$\Delta^2 x_m + \left[p_m \Delta x_{m-h} - p_N \Delta x_{N-h} - \sum_{n=N}^{m-1} \Delta p_n \Delta x_{n-h+1} \right] + b \sum_{n=N}^{m-1} q_n \leqslant 0.$$

Using $\Delta p_n \leqslant 0$ for $n \geqslant n_0$, we have

$$\Delta^2 x_m - p_N \Delta x_{N-n} + b \sum_{n=N}^{m-1} q_n \leqslant 0 \quad \text{for } m-1 \geqslant n \geqslant N.$$

From (1) it follows that there exist $N^* \geqslant N+1$ and $c^* > 0$ such that

$$\Delta^2 x_m \leqslant -c^* \quad \text{for } m \geqslant N^*,$$

and consequently

$$0 < \Delta x_j \to -\infty \quad \text{as} \quad j \to \infty,$$

a contradiction. This completes the proof. □

Theorem 7. *Let h be a nonpositive integer, $c = 1$, $\Delta p_n \leqslant 0$ and $g_n = n - k$, $n \geqslant n_0 \geqslant 0$ where k is a positive integer, and let condition* (1) *hold. If every bounded solution of*

$$\Delta^3 y_n + q_n y_{n-k} = 0 \tag{20}$$

is oscillatory and

$$\liminf_{n\to\infty} \Big(\frac{n-k}{2}\Big)\Big[\sum_{j=n-k}^{n-1} q_j - p_{n-k}\Big] > \Big(\frac{k}{1+k}\Big)^{1+k}, \tag{21}$$

then Eq. (E_3) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_3), say $x_n > 0$ and $x_{n-k} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. As in the proof of Theorem 6, we consider the following two cases:

(A) $\Delta^2 x_n > 0$ eventually, (B) $\Delta^2 x_n < 0$ eventually.

(A) Suppose $\Delta^2 x_n > 0$ eventually. Then there are two possibilities:

(A$_1$) $\Delta^2 x_n > 0$ and $\Delta x_n > 0$ eventually, (A$_2$) $\Delta^2 x_n > 0$ and $\Delta x_n < 0$ eventually.

(A$_1$) Suppose $\Delta^2 x_n > 0$ and $\Delta x_n > 0$ for $n \geqslant n_2 \geqslant n_1 + h$. There exist constants $c_1 > 0$ and $N \geqslant n_2$ such that (3) holds for $n \geqslant N$. From Eq. (E_3) and (3) we have

$$\Delta^3 x_n + c_1 q_n \leqslant 0 \quad \text{for } n \geqslant N. \tag{22}$$

Summing both sides of (22) from N to $m - 1 \geqslant N$, we have

$$0 < \Delta^2 x_m \leqslant \Delta^2 x_N - c_1 \sum_{n=N}^{m-1} q_n \to -\infty \quad \text{as} \quad m \to \infty,$$

a contradiction.

(A$_2$) Suppose $\Delta^2 x_n > 0$ and $\Delta x_n < 0$ eventually. From Eq. (E_3) we have

$$\Delta^3 x_m + q_n x_{n-k} \leqslant 0 \text{ eventually.} \tag{23}$$

But, by Theorem 1$'$ in [8], if (23) has an eventually positive solution, then (20) has an eventually positive solution as well, a contradiction.

(B) Suppose $\Delta^2 x_n < 0$ for $n \geqslant n_2 \geqslant n_1 + h$. Then $\Delta x_n > 0$ for $n \geqslant n_2$, and by Lemma 4.1 (d) in [7] there exists N sufficiently large, $N \geqslant 2n_2 + k$ such that

$$x_{n-k} \geqslant \Big(\frac{n-k}{2}\Big)\Delta x_{n-k} \quad \text{for} \quad n \geqslant N. \tag{24}$$

Using (24) in Eq. (E_3) and setting $y_n = \Delta x_n > 0$ for $n \geqslant N$, we have

$$\Delta^2 y_n + p_n \Delta y_{n-h} + \Big(\frac{n-k}{2}\Big) q_n y_{n-k} \leqslant 0 \quad \text{for} \quad n \geqslant N.$$

The rest of the proof is similar to the proof of Theorem 5 (B) and hence is omitted. This completes the proof. □

Next, we consider a special case of Eq. (E_3), namely the equations

$$((L_3)) \qquad \Delta^3 x_n + p\Delta^2 x_{n-h} q x_{n-k} = 0$$

and

$$((L_3^*)) \qquad \Delta^3 x_n + q x_{n-k} = 0,$$

where p and q are positive constants, h and k are nonnegative integers.

From Corollary 2 in [8] we obtain the following results:

Corollary 2. *All bounded solutions of Eq. (L_3^*) are oscillatory if one of the following conditions holds:*

(i) $k = 0$ *and* $q \geqslant 1$;

(ii) $k \geqslant 1$ *and* $q > \frac{27k^k}{(3+k)^{3+k}}$.

Now, from Theorem 6 *and* 7 *and Corollary* 2, *we obtain the following result:*

Corollary 3. *Eq. (L_3) is almost oscillatory if one of the following conditions holds:*

(I) $h > 0$ *is odd and* $p > \frac{h^h}{(1+h)^{1+h}}$;

(II) $h \leqslant 0$ *is odd and* $q > \frac{27k^k}{(3+k)^{3+k}}$.

Remark 3. (i) If we set $p_n = 0$ in Theorem 6, we see that condition (1) is not sufficient to allow every solution of Eq. (E_3) with $p_n = 0$ to oscillate. This can be shown by consider the equation

$$\Delta^3 x_n + \Big(1 - \frac{1}{e}\Big)^3 x_n = 0,$$

which has a nonoscillatory solution $x_n = e^{-n}$.

Therefore, we conclude that the presence of p_n in Eq. (E_3) generates oscillations.

(ii) We note that Theorem 7 is applicable to Eq. (E_3) when $p_n = 0$. Only condition (21) is disregarded.

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Author's address: Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12211, A.R. of Egypt.