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SIMPLE ZEROPOTENT PARAMEDIAL GROUPOIDS ARE BALANCED

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Abstract. This short note is a continuation of [1] and [2] and its purpose is to show that every simple zeropotent paramedial groupoid containing at least three elements is strongly balanced in the sense of [4].

Keywords: commutative semigroup, invariant congruence

MSC 2000: Primary 16Y60, secondary 13J25, 12K10, 20M14

1. Introduction

By a paramedial groupoid we mean a groupoid G such that $ax \cdot yb = bx \cdot ya$ for all $a, b, x, y \in G$ (see [1] for further details). Now, according to [2], simple paramedial groupoids can be divided into five principal subclasses, one of them being the class of simple zeropotent paramedial groupoids. In this short note, we are going to prove that every simple zeropotent paramedial groupoid containing at least three elements is strongly balanced (see [4] for definition and further details).

2. Main result

Theorem 2.1. *Let G be a simple zeropotent paramedial groupoid containing at least three elements. Then G is strongly balanced.*

P r o o f. Since $\operatorname{card}(G) \geqslant 3$ and G is simple, we have $G = GG$ and, by [3, Corollary 6.2], there exist a commutative semigroup $S(+)$ and automorphisms f,

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g of $S(+)$ such that $G \subseteq S$, $f^2 = g^2$, $xy = f(x) + g(y)$ for all $x, y \in G$ and $u + o = o = f(u) + g(u)$ for every $u \in S$. Now, denote by A the automorphism group of $S(+)$ generated by f, g. Then A is operating on S, S becomes an A-semimodule and we can assume that S, as a semimodule, is generated by the set G. Furthermore, considering a congruence r of the semimodule S maximal with respect to $r|G = \mathrm{id}_G$ and the factorsemimodule S/r, we may also assume that already $r = \mathrm{id}_S$.

The rest of the proof is divided into six parts:

(i) The semimodule S is simple.

– Let $s \neq \mathrm{id}_S$ be a congruence of S. Then $s|G \neq \mathrm{id}_G$ and, since G is a simple groupoid, we have $G \times G \subseteq s$. Now, since S is generated by G and $o \in G$, we conclude that $s = S \times S$.

(ii) Every subsemigroup of the group A is both left and right uniform.

– First, if P is a subset of A such that $a^n \neq b^m$ for all $a, b \in P$, $a \neq b$, and all $n \geqslant 1$, $m \geqslant 1$, then one may check easily that $\mathrm{card}(P) \leqslant 3$. Next, if B, C are subsemigroups of A such that $BC \subseteq C$ and $CB \subseteq B$ and if $b \in B$, $c \in C$, then the sets $(B \cup C)bc^k$, $k \geqslant 1$, are left ideals of the subsemigroup $B \cup C$ and consequently $a_1 bc^{k_1} = a_2 bc^{k_2}$ for some $a_1, a_2 \in B \cup C$ and $k_1 > k_2$. Consequently, $a_1 bc^{k_1 - k_2} = a_2 b \in B \cap C$ and $B \cap C \neq \emptyset$. We have proved that every subsemigroup of A is left uniform. None the less, the map $a \to a^{-1}$ is an antiautomorphism of A.

(iii) $S(+)$ is a semilattice.

– Assume first that $u + u = o$ for every $u \in S$. Since S is a simple semimodule, we have $S + S = S$, and so $x = y + z \neq o$ for some $x, y, z \in S$. Let B denote the set of $b \in A$ such that either $y = bx$ or $y = bx + u$ for some $u \in S$ and, similarly, let C be the set of $c \in A$ such that either $z = cx$ or $z = cx + v$ for some $v \in S$. Since S is simple, we have $B \neq \emptyset \neq C$ and one checks easily that $BB \cup BC \subseteq B$, $CC \cup CB \subseteq C$. Now, by (ii), we get $d \in B \cap C$ and $o \neq x = y + z = dx + dx + w = o$, a contradiction.

We have proved that $u + u \neq o$ for at least one $u \in S$. Since S is simple, the map $x \to 2x$ is an injective endomorphism of S. Now, define a relation t on S by $(x, y) \in t$ iff there exists $i \geqslant 0$ such that either $x = 2^i y$ or $x = 2^i y + u$ and either $y = 2^i x$ or $y = 2^i x + v$. Again, t is a congruence of S and $(2v, v) \in t$, $v \in S$. If $t = \mathrm{id}_S$, then $S(+)$ is idempotent, and hence assume that $t \neq \mathrm{id}_S$. Then $t = S \times S$ and, if $v \in S \setminus \{o\}$, then $2^i v = o$ for some $i \geqslant 0$, a contradiction with the fact that $x \to 2^i x$ is an injective endomorphism of S.

(iv) $x + y = o$ for all $x, y \in S$, $x \neq y$.

– Let, on the contrary, $x, y, z, w \in S$ be such that $w + x = o \neq w + z = y$ and $x \neq o$ (use the fact that S is simple). Now, denote by B the set of $b \in A$ such that either $z = bx$ or $z = bx + u$, $u \in S$, and by C the set of $c \in A$ such that either $z = cy$ or $z = cy + v, v \in S$. Then, since S is simple, we have $B \neq \emptyset \neq C$ and it is also easy to see that $BB \cup CB \subseteq B$ and $CC \cup BC \subseteq C$. By (ii), $B \cap C \neq \emptyset$ and, if $d \in B \cap C$, then $z = z + z = dx + dy + t = d(x + y) + t = o$, a contradiction.

(v) Let $x, y, u, v \in G$ be such that $xy = uv \neq o$. Then $x = u$ and $y = v$.

– We have $f(x) + g(y) = xy = uv = f(u) + g(v) \neq o$, and hence $f(x) = f(u) = g(y) = g(v)$ by (iv).

(vi) $I = \{x \in G;\ xG = o\} = \{o\} = \{y \in G;\ Gy = o\} = J$.

– Put $K = I \cup J$. Then $o \in I \cap J$ and, if $x \in I$, $w, z, u, v \in G$, $w = uv$, then also $xz = o$ and $w \cdot zx = uv \cdot zx = xv \cdot zu = o \cdot zu = o$. This means that $xz, zx \in J$ or $IG \cup GI \subseteq J$. Quite similarly, $JG \cup GJ \subseteq I$ and we have shown that K is an ideal of the groupoid G. Now, the situation is clear for $K = \{o\}$. However, if $K \neq \{o\}$, then $K = G$, since G is simple and, since $G = GG$, we have $xy \cdot uv \neq o$ for some $x, y, u, v \in G$. Then $xy \neq o$, $x \notin I$, $x \in J$ and $o = vy \cdot o = vy \cdot ux = xy \cdot uv \neq o$, a contradiction. □

Remark 2.2. Let G be a simple zeropotent paramedial groupoid.

(i) If $\mathrm{card}(G) = 2$, then G is apparently isomorphic to the following groupoid:

	0	1
0	0	0
1	0	0

(ii) If $\mathrm{card}(G) \geqslant 3$, then G is strongly balanced by 2.1, and hence there are two permutations f and g of the set $G^* = G \setminus \{o\}$ such that for $a, b \in G^*$ we have $ab \neq o$ if and only if $f(a) = g(b)$, and then $ab = f(a) = g(b)$. Now, denote by $\mathscr{G}$ the permutation group (on G^*) generated by f, g and by $\mathscr{T}$ the subsemigroup generated by f, g in $\mathscr{G}$. According to [4, Prop. 2.4], $\mathscr{T}$ acts transitively on G^*. Now, if $u \in G^*$ and $\mathscr{H} = \mathrm{Stab}_{\mathscr{G}}(u)$, then $\mathscr{H}$ is a corefree subgroup of $\mathscr{G}$ and for every $h \in \mathscr{G}$ there exist $k_1, k_2 \in \mathscr{T}$ such that $hk_1, k_2h \in \mathscr{H}$. Finally, since G is paramedial and zeropotent, we have $f^2 = g^2$ and $f(v) \neq g(v)$ for every $v \in G^*$ (in particular, $f \neq g$).

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