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BARRELLEDNESS OF GENERALIZED SUMS OF NORMED SPACES

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Abstract. Let $(E_i)_{i\in I}$ be a family of normed spaces and λ a space of scalar generalized sequences. The λ-sum of the family $(E_i)_{i\in I}$ of spaces is

$$\lambda\{(E_i)_{i\in I}\} := \{(x_i)_{i\in I}, x_i \in E_i, \quad \text{and} \quad (\|x_i\|)_{i\in I} \in \lambda\}.$$

Starting from the topology on λ and the norm topology on each E_i, a natural topology on $\lambda\{(E_i)_{i\in I}\}$ can be defined. We give conditions for $\lambda\{(E_i)_{i\in I}\}$ to be quasi-barrelled, barrelled or locally complete.

Keywords: barrelled spaces, generalized sequences

MSC 2000: 46A08, 46A45, 46E40

1. Introduction and preliminary results

The barrelledness, and related topics, of spaces of vector-valued sequences and functions have been studied in several papers [1]–[6], [8] and [10]. In particular, Florencio, Paúl and Sáez, extending the work of Lurje [10] where the barrelledness of $\ell^p\{E_n\}$ had been studied, characterized the barrelledness of the λ-sum of a sequence of normed spaces in [6]. More recently, Kakol and Roelcke in [8] have studied the barrelledness of ℓ^p-direct sums of a family of seminormed spaces for $1 \leqslant p \leqslant \infty$. Drewnowski, Florencio and Paúl have studied the barrelledness of bounded vector functions defined on a set with certain restrictions on its cardinal in [5].

In this paper we continue this investigations using techniques similar to those used in [1]–[6], to obtain more general results in the setting of the locally convex sum of a family of normed spaces.

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Let us recall at this point some definitions and notation. In what follows we will consider a fixed index set I and a space λ of scalar families, or generalized sequences, on I, i.e., a linear subspace of the space of all real or complex functions defined on I.

We say that λ is solid (see [9, §30]) if whenever it contains $\beta = (\beta_i)_i$ it also contains all families $\alpha = (\alpha_i)_i$ with $|\alpha_i| \leqslant |\beta_i|$ for all $i \in I$. The Köthe-dual $\lambda^\times$ of the space λ is defined as for sequences spaces, i.e., $\lambda^\times$ consists of all generalized sequences $(\eta_i)_i$ such that $\sum |\alpha_i \eta_i| < \infty$ for every $(\alpha_i)_i$ in λ. We always consider on λ a normal locally convex Hausdorff topology in the sense of Rosier (see [12]). Such a topology can be given by a system Q of seminorms with the following properties:

(a) If $\alpha \leqslant \beta$ (i.e. $|\alpha_i| \leqslant |\beta_i|$ for all $i \in I$), then $q(\alpha) \leqslant q(\beta)$ for all $q \in Q$.
(b) For every η in the Köthe-dual space $\lambda^\times$ there exists a seminorm $q \in Q$ such that $|\langle \gamma, \eta \rangle| \leqslant q(\gamma)$ for all $\gamma \in \lambda$.

If $(E_i)_{i\in I}$ is a family of real or complex normed spaces, we define the λ-sum of $(E_i)_{i\in I}$ as

$$\lambda\{(E_i)_{i\in I}\} := \{(x_i)_{i\in I} \colon\ x_i \in E_i \text{ and } (\|x_i\|)_{i\in I} \in \lambda\}.$$

To ensure that $\lambda\{(E_i)\}$ is a linear space we must assume that λ be solid. Starting from the topology of λ and the norm topology on each E_i, we consider the locally convex topology on $\lambda\{(E_i)\}$ determined by the seminorms:

$$\sigma_q \colon\ x = (x_i)_i \in \lambda\{(E_i)\} \longrightarrow \sigma_q(x) := q\big((\|x_i\|)_i\big) \in \mathbb{R},$$

as q runs through Q.

In this paper we study the barrelledness of $\lambda\{(E_i)\}$. Recall that a locally convex space E is barrelled if and only if it is quasi-barrelled (every $\beta(E', E)$-bounded set in the dual of E is equicontinuous) and has the Banach-Mackey property (every $\sigma(E', E)$-bounded set in its dual is $\beta(E', E)$-bounded). We refer the reader to the monographs [7], [9] and [11] for the terminology in local convexity and barrelledness used here.

We start by lifting the property of quasi-barrelledness from the space λ to the space $\lambda\{(E_i)\}$.

Following a way similar to the proofs of [4, Theorem 1] or [6, Theorem 4] we can show the following

Theorem 1. *If λ is quasi-barrelled, then $\lambda\{(E_i)\}$ is quasi-barrelled.*

The next step will be to analyze when the space $\lambda\{(E_i)\}$ has the Banach-Mackey property. We will do this in two cases.

First, in Section 2, when the space λ is defined on an index set I that has nonmeasurable cardinal. Recall that a set I has nonmeasurable cardinal if there exists no countable additive measure $\mu\colon \mathscr{P}(I) \longrightarrow \{0,1\}$ such that $\mu(I)=1$ and $\mu(\{i\})=0$ for all $i \in I$ [11, Def. 6.2.21]. As we shall see, this concept will be strongly connected to the assumption that the space $\lambda\{(E_i)\}$ is not barrelled.

Secondly, in Section 3, we will deal with a space λ which has the property of convergence of sections without any restrictions on the cardinal of I.

Before to do this we need to prove a result about the local completeness of the space $\lambda\{(E_i)\}$. Its proof is standard but we include it for the sake of convenience.

Theorem 2. *If λ is locally complete and every E_i is a Banach space, then $\lambda\{(E_i)\}$ is locally complete.*

Proof. Taking into account [11, Prop. 5.1.6 and Prop. 3.2.3] we shall show that if B is a closed disc in $\lambda\{(E_i)\}$ and $(x^{(n)})_n$ is a sequence of elements of B, then the series $\sum_{n=1}^{\infty} 2^{-n}x^{(n)}$ converges to an element of B.

Let $M=\{(\|z_i\|)_i\colon\ z=(z_i)_i \in B\}$. Note that M is bounded in λ. Since λ is locally complete it follows that $D=\overline{acx}(M)$ is a Banach disc. Now $\big((\|x_i^{(n)}\|)_i\big)_n \subset D$ so that the series $\sum_{n=1}^{\infty} 2^{-n}\big(\|x_i^{(n)}\|\big)_i$ converges in λ to an element, say $\alpha=(\alpha_i)_i$. Coordinatewise we have that $\sum_{n=1}^{\infty} 2^{-n}\|x_i^{(n)}\|=\alpha_i$ for every $i \in I$.

From the boundedness of $(x_i^{(n)})_n$ and the completeness of each E_i it follows that $\sum_{n=1}^{\infty} 2^{-n}x_i^{(n)}$ converges to an element $x_i \in E_i$. Observe that $x=(x_i)_i$ is an element of $\lambda\{(E_i)\}$ since $\|x_i\| \leqslant \alpha_i$ for every $i \in I$.

We complete the proof by proving that x is the sum of the series $\sum_{n=1}^{\infty} 2^{-n}x^{(n)}$ and $x \in B$. Given an arbitrary seminorm q on λ we have

$$\begin{aligned}\sigma_q\bigg(x-\sum_{n=1}^{k} 2^{-n}x^{(n)}\bigg) &= q\bigg((\|x_i-\sum_{n=1}^{k} 2^{-n}x_i^{(n)}\|)_i\bigg)\\ &\leqslant q\bigg(\sum_{n\geqslant k+1} 2^{-n}(\|x_i^{(n)}\|)_i\bigg) \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

since $\sum_{n=1}^{\infty} 2^{-n}(\|x_i\|)_i$ is convergent in λ. Finally, note that $x \in B$ because B is a closed disc. □

2. When I has nonmeasurable cardinal

We start this section by studing when the space $\lambda\{(E_i)\}$ has the Banach-Mackey property. In the next theorem we will use the following notation about projections. If J is a subset of I and x is an element of $\lambda\{(E_i)\}$, then $P_J(x)$ is the generalized sequence that agrees with x on J and is null on $I \setminus J$.

Theorem 3. *If I has nonmeasurable cardinal, the space λ is locally complete and the normed spaces $(E_i)_i$ are all barrelled, then the space $\lambda\{(E_i)\}$ has the Banach-Mackey property.*

Proof. Suppose, on the contrary, that there exists G in the dual space $\lambda\{(E_i)\}'$ that is $\sigma(\lambda\{(E_i)\}', \lambda\{(E_i)\})$-bounded but is not a bounded set in $\beta(\lambda\{(E_i)\}', \lambda\{(E_i)\})$. Then there exists a bounded set A in $\lambda\{(E_i)\}$ such that

$$\sup\{|\langle a, u\rangle| \colon\ u \in G, \quad a \in A\} = +\infty.$$

The set $B := \bigcup_{J\subseteq I} P_J(A)$ is bounded because A is bounded and the set of projections $\{P_J \colon\ J \subseteq I\}$ is equicontinuous.

From the sets G and B let us consider the filter given on I by

$$\mathscr{F} = \{J \subseteq I \colon\ G^\circ \text{ absorbs } P_{I\setminus J}(B)\},$$

and let $\mathscr{U}$ be the ultrafilter generated by the filter $\mathscr{F}$. Denote by μ the standard finitely additive measure on $\mathscr{P}(I)$ associated with the ultrafilter $\mathscr{U}$. Since I has nonmeasurable cardinal, we have that μ is noncountable additive, hence there exists a sequence $(J_k)_k$ of pairwise disjoint subsets of I with $\mu(J_k) = 0$ for all k and $\mu\Big(\bigcup_{k=1}^{\infty} J_k\Big) = 1$.

If we put $I_n = \bigcup_{k\geqslant n} J_k$ for all $n = 1, 2, \ldots$, we have a decreasing sequence of subsets of I with empty intersection such that $\mu(I_n) = 1$ for all $n = 1, 2, \ldots$ because

$$\begin{aligned} 1 = \mu\Big(\bigcup_{k=1}^{\infty} J_k\Big) &= \mu\Big(J_1 \cup \ldots \cup J_{n-1} \cup \bigcup_{k=n}^{\infty} J_k\Big) \\ &= \mu(J_1) + \ldots + \mu(J_{n-1}) + \mu\Big(\bigcup_{k=n}^{\infty} J_k\Big) \\ &= \mu(I_n). \end{aligned}$$

It follows that each I_n is in $\mathscr{U}$ and therefore G° does not absorb $P_{I_n}(B)$, so there exist $z^{(n)} \in B$, supported in I_n ($P_{I_n}(z^{(n)}) = z^{(n)}$), and $u^{(n)} \in G$ such that

$$|\langle z^{(n)}, u^{(n)}\rangle| > n, \quad \text{for all } n = 1, 2, \ldots. \tag{1}$$

From the bounded sequence $(z^{(n)})_n$ we consider the set

$$D=\left\{\sum_{n=1}^{\infty}\alpha_n z^{(n)}\colon\ (\alpha_n)_n \text{ in the unit ball of } \ell^1\right\}.$$

Let us observe that for all $(\alpha_n)_n$ in ℓ^1 (the space of absolutely summable sequences) the series $\sum_{n=1}^{\infty}\alpha_n z^{(n)}$ converges in $\lambda\{(\widehat{E}_i)_{i\in I}\}$ to an element which we will denote by $z=(z_i)_{i\in I}$. This follows by applying Theorem 2 above to the space λ and the Banach spaces $(\widehat{E}_i)_i$. Moreover, as we shall see in a moment, each z_i is in E_i, so $\sum_{n=1}^{\infty}\alpha_n z^{(n)}$ really converges in $\lambda\{(E_i)\}$. Indeed, since $(I_n)_n$ is a decreasing sequence of subsets of I with empty intersection, for each $i\in I$ there are two possibilities:

1) If $i\notin I_1$, then $P_{\{i\}}(z^{(n)})=0$ for all $n=1,2,\ldots$, so $z_i=0$.
2) There exists a natural number p_i such that $i\in I_{p_i}$ but $i\notin I_k$ for all $k>p_i$. In this case we have

$$\begin{aligned} z_i=P_{\{i\}}(z)&=P_{\{i\}}\left(\sum_{n=1}^{\infty}\alpha_n z^{(n)}\right)=\sum_{n=1}^{\infty}\alpha_n P_{\{i\}}(z^{(n)})\\ &=\sum_{n=1}^{p_i}\alpha_n P_{\{i\}}(z^{(n)})\in E_i. \end{aligned}$$

Now, it follows from [1, Prop. p. 74] that D is a Banach disc in $\lambda\{(E_i)\}$. As barrels absorb every Banach disc [7, 8.3.3], we have that there exists a number $\varrho>0$ such that $D\subset\varrho G^\circ$. On the other hand, from (1), $z^{(n)}\notin nG^\circ$ for all $n\geqslant 1$. This contradiction completes the proof of the theorem. □

Our main result is the next

Theorem 4. *If λ is a locally complete and barrelled space of generalized sequences on a set I which has nonmeasurable cardinal, then $\lambda\{(E_i)\}$ is barrelled if and only if every E_i is barrelled.*

P r o o f. The direct implication follows from Theorems 1 and 3, and the inverse one follows from the fact that every E_i is complemented in $\lambda\{(E_i)\}$. □

Remark 1. This theorem implies, as important particular cases, the barrelledness of spaces $\ell^p_I\{(E_i)_{i\in I}\}$ with $1\leqslant p\leqslant\infty$, where I is a nonmeasurable set. Compare our Theorem 4 with the results of Kakol and Roelcke in [8] and Drewnowski, Florencio and Paúl in [5].

3. Spaces λ with the property of convergence of sections

Let us introduce some more notation in order to establish the barrelledness of $\lambda\{(E_i)\}$ in the setting of a space λ with the property of convergence of sections. The sections of an element of λ are defined to be its projections over finite subsets of I. This property allows us to represent the dual of $\lambda\{(E_i)\}$ as the space $\lambda^\times\{(E_i')_{i\in I}\}$. With similar arguments to those used in [6, Theorem 1] we can prove the equality

$$(2)\quad \lambda\{(E_i)_{i\in I}\}' = \lambda^\times\{(E_i')_{i\in I}\} = \Big\{(u_i)_i, u_i \in E_i', \sum_{i\in I} |\langle x_i, u_i\rangle| < \infty, \text{ for all } (x_i)_i \in \lambda\{(E_i)\}\Big\}.$$

Lemma 5. *If the space λ has the property of convergence of sections and $(E_i)_i$ is a family of normed spaces such that every E_i' is $\sigma(E_i', E_i)$-sequentially complete, then $\lambda\{(E_i)\}'$ is $\sigma(\lambda\{(E_i)\}', \lambda\{(E_i)\})$-sequentially complete. In particular, the space $\lambda\{(E_i)\}$ has the Banach-Mackey property.*

Proof. According to (2) let $(u^{(n)})_n$ be a $\sigma(\lambda\{(E_i)\}', \lambda\{(E_i)\})$-Cauchy sequence in $\lambda\{(E_i)\}'$. By using the natural inclusion of E_i in $\lambda\{(E_i)\}$ we see that for every $i \in I$ the sequence $(u_i^{(n)})_n$ is $\sigma(E_i', E_i)$-Cauchy in E_i'. Since every E_i' is $\sigma(E_i', E_i)$-sequentially complete, $(u_i^{(n)})_n$ is actually convergent to an element $u_i \in E_i'$. Put $u = (u_i)_{i\in I}$. We complete the proof by proving that u is in $\lambda\{(E_i)\}'$ and that $(u^{(n)})_n$ converges to u in the $\sigma(\lambda\{(E_i)\}', \lambda\{(E_i)\})$-topology.

If $x = (x_i)_i$ is in $\lambda\{(E_i)\}$, then $\alpha^{(n)} = (\langle u_i^{(n)}, x_i\rangle)_i$ is in ℓ_I^1 for all $n \geqslant 1$. Furthermore, $(\alpha^{(n)})_n$ is a Cauchy sequence in $\sigma(\ell_I^1, \ell_I^\infty)$ since $(u^{(n)})_n$ is a Cauchy sequence in the topology $\sigma(\lambda\{(E_i)\}', \lambda\{(E_i)\})$ and $\lambda\{(E_i)\}$ is solid. Now the Schur lemma [7, §10.5 Cor. 4] ensures that $(\alpha^{(n)})_n$ is norm convergent to an element $\alpha = (\alpha_i)_i$ of ℓ_I^1.

Coordinatewise we have that

$$\alpha_i = \lim_n \langle x_i, u_i^{(n)}\rangle = \langle x_i, u_i\rangle$$

for every $i \in I$. This yields that $(\langle x_i, u_i\rangle)_i$ is in ℓ_I^1, since $u \in \lambda\{(E_i)\}'$.

Now

$$|\langle x, u^{(n)} - u\rangle| \leqslant \sum_{i\in I} |\langle x_i, u_i^{(n)}\rangle - \langle x_i, u_i\rangle| = \sum_{i\in I} |\alpha_i^{(n)} - \alpha_i|$$

and taking into account that $\alpha^{(n)} \to \alpha$ in $(\ell_I^1, \|\cdot\|_1)$, it follows that $u = \lim_n u^{(n)}$ and the proof is complete. □

Theorem 6. *If λ has the property of convergence of sections and is barrelled, then $\lambda\{(E_i)\}$ is barrelled if and only if each E_i is barrelled.*

P r o o f. If every E_i is barrelled, then it follows from [9, §23.6 (4)] that E_i' is $\sigma(E_i', E_i)$-sequentially complete. The barrelledness of $\lambda\{(E_i)\}$ follows from Theorem 1 and Lemma 5. The inverse implication follows from the fact that every E_i is complemented in $\lambda\{(E_i)\}$. □

Remark 2. As spaces ℓ_I^p $(1 \leqslant p < \infty)$ and c_{0I} have the property of convergence of sections, the spaces $\ell_I^p\{(E_i)_{i\in I}\}$ and $c_{0I}\{(E_i)_{i\in I}\}$ are barrelled if each E_i is barrelled.

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