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ON TRANSFORMATIONS $z(t) = y(\varphi(t))$ OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The paper describes the general form of an ordinary differential equation of the order $n+1$ ($n \geqslant 1$) which allows a nontrivial global transformation consisting of the change of the independent variable. A result given by J. Aczél is generalized. A functional equation of the form

$$f\left(s, v, w_{11}v_1, \ldots, \sum_{j=1}^{n} w_{nj}v_j\right) = \sum_{j=1}^{n} w_{n+1j}v_j + w_{n+1n+1}f(x, v, v_1, \ldots, v_n),$$

where $w_{ij} = a_{ij}(x_1, \ldots, x_{i-j+1})$ are given functions, $w_{n+11} = g(x, x_1, \ldots, x_n)$, is solved on $\mathbb{R}$.

Keywords: ordinary differential equations, linear differential equations, transformations, functional equations

MSC 2000: 34A30, 34A34, 39B40

1. Introduction

The theory of global pointwise transformations $z(t) = L(t)y(\varphi(t))$ of homogeneous linear differential equations was developed in the monograph [5] by F. Neuman (see historical remarks, definitions, results and applications). The criterion of global equivalence of the second order linear equations was published by O. Borůvka [3],

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of the third and higher order linear equations by F. Neuman [5]. Transformations $z(t) = y(\varphi(t))$ were studied in [6] as a "motion" for n-th order linear differential equations. A general form

$$y''(x) = a(y(x))y'(x)^2 + p(x)y'(x),$$

where φ satisfies a differential equation $\varphi''(x) = p(x)\varphi'(x) - p(\varphi(x))\varphi'(x)^2$ and a, p are arbitrary functions, was derived by J. Aczél [2] for the second order differential equations (eliminating regularity conditions from [4]). This general form allows the transformation $z(t) = y(\varphi(t))$ and transforms the equation into itself on the whole interval of definition. This second order differential equation is generally nonlinear and the result was derived by means of functional equations. In this paper we derive, similarly to J. Aczél, a general form of ordinary differential equations of the order $n+1$ ($n \geqslant 1$) which allows transformations $z(t) = y(\varphi(t))$, and we generalize Aczél's result. We derive an effective criterion of equivalence with respect to transformations $z(t) = y(\varphi(t))$.

2. Notation, preliminary results

Denote by (f) and (f^*) respectively the ordinary differential equations

$$y^{(n+1)}(x) = f(x, y(x), \dots, y^{(n)}(x)), \quad x \in I \subseteq \mathbb{R},$$
$$z^{(n+1)}(t) = f^*(t, z(t), \dots, z^{(n)}(t)), \quad t \in J \subseteq \mathbb{R}$$

of the order $n+1$, $n \geqslant 1$.

Definition. We say that (f) is globally transformable into (f^*) with respect to the transformation $z(t) = y(\varphi(t))$ if the function φ is a C^{n+1} diffeomorphism of the interval J onto the interval I and the function

$$z(t) = y(\varphi(t)) \tag{1}$$

is a solution of the equation (f^*) whenever y is a solution of the equation (f).

If (f) is globally transformable into (f^*), then we say that (f), (f^*) are *equivalent equations*. We say that (1) is a *stationary transformation* if it globally transforms an equation (f) into itself on I, i.e. if φ is a C^{n+1} diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = y(\varphi(x))$ is a solution of $z^{(n+1)}(x) = f(x, z(x), \dots, z^{(n)}(x))$ whenever $y(x)$ is a solution of $y^{(n+1)}(x) = f(x, y(x), \dots, y^{(n)}(x))$, $x \in I$. Hence, if (1) is a stationary transformation of the equation (f) with a solution $y(x)$, then $y(\varphi(x))$ is also a solution of the equation (f).

We denote $y^{(i)}(\varphi(t)) = \mathrm{d}^i y(\varphi(t))/\mathrm{d}\varphi(t)^i$, $(y(\varphi(t)))^{(i)} = \mathrm{d}^i y(\varphi(t))/\mathrm{d}t^i$, $i \geqslant 0$.

Proposition 1 (a particular case of Lemma 1, [7]). *Let $n \in \mathbb{N}$ and let the following relation be satisfied:*

$$z(t) = y(\varphi(t)),$$

where real functions $y\colon I \to \mathbb{R}$, $z\colon J \to \mathbb{R}$ belong to the classes $C^{n+1}(I)$, $C^{n+1}(J)$ respectively, and φ is a C^r diffeomorphism of J onto I for some integer $r \geqslant n+1$. Then

$$z^{(i)}(t) = \sum_{j=1}^{i} a_{ij}(t) y^{(j)}(\varphi(t)), \quad i \in \{1, \ldots, n+1\},$$

where

$$\begin{aligned}
a_{11}(t) &= \varphi'(t);\\
a_{i1}(t) &= a'_{i-11}(t), \qquad i > 1;\\
a_{ij}(t) &= a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t), \qquad j \in \{2, \ldots, i-1\};\\
a_{ii}(t) &= a_{i-1i-1}(t)\varphi'(t), \qquad i > 1;\ i \in \{2, \ldots, n+1\}
\end{aligned}$$

and the real functions $a_{ij}(t) \in C^{r-(i-j)-1}(J)$. Moreover,

$$\begin{aligned}
a_{i1}(t) &= \varphi^{(i)}(t), \qquad i \geqslant 1;\\
&\ldots\\
a_{ij}(t) &= \binom{i}{j-1}\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t) + r_{ij}(\varphi'(t), \ldots, \varphi^{(i-j)}(t)), \qquad i > j > 2;\\
&\ldots\\
a_{ii-2}(t) &= \binom{i}{3}\varphi'(t)^{i-3}\varphi'''(t) + 3\binom{i}{4}\varphi'(t)^{i-4}\varphi''(t)^2, \qquad i \geqslant 2;\\
a_{ii-1}(t) &= \binom{i}{2}\varphi'(t)^{i-2}\varphi''(t), \qquad i \geqslant 2;\\
a_{ii}(t) &= \varphi'(t)^i, \qquad i \geqslant 1
\end{aligned}$$

and $a_{ij}(t) = a_{ij}(\varphi'(t), \ldots, \varphi^{(i-j+1)}(t))$, $j \in \{1, \ldots, i\}$, $i \in \{1, \ldots, n+1\}$.

Let $\mathbf{V}_n$ denote an n-dimensional vector space, $\vec{c} = [c_1, \ldots, c_n]^T = [c_i]_{i=1}^n \in \mathbf{V}_n$ being a vector of the space $\mathbf{V}_n$ written in the column form; ^{T} means the transposition. Denote by $\vec{o} = [0, \ldots, 0]^T$ the origin of $\mathbf{V}_n$ and by $\vec{e}_1, \ldots, \vec{e}_n$ an orthonormal basis in $\mathbf{V}_n$. Let $\mathbf{V}_n$ be equipped with the scalar product $(\vec{p}, \vec{q}) = \sum_{i=1}^{n} p_i q_i$ for any pair $\vec{p}, \vec{q}$ of its vectors. Let $\vec{p}_1, \ldots, \vec{p}_m$ be m vectors from $\mathbf{V}_n$. Notation $P =$

$[\vec{p}_1, \ldots, \vec{p}_m] = [p_{ij}]_{j=1,\ldots,m}^{i=1,\ldots,n}$ denotes a matrix and $(P, Q) = \sum_j^i p_{ij} q_{ij}$ the scalar product of two matrices of the same type. Similarly $P_{(j,\ldots,k)} = [\vec{p}_j, \ldots, \vec{p}_k]$ means a submatrix, $PQ = P_{(1,\ldots,n)} Q_{(1,\ldots,n)}$ is the matrix multiplication. For $y \in C^{n+1}(J)$ we denote $y_i(x) = y^{(i)}(x)$, $x \in I$, $i \in \{1, \ldots, n+1\}$. Then $\vec{y}(x) = [y_1(x), \ldots, y_n(x)]^T = [y'(x), \ldots, y^{(n)}(x)]^T \in \mathbf{V}_n$ for each $x \in I$.

Remark 1. Let the assumptions of Proposition 1 be satisfied. Then

$$\vec{z}(t) = A(t)\vec{y}(\varphi(t))$$

is true on J for $A(t) = [a_{ij}(t)]_{j=1,\ldots,n}^{i=1,\ldots,n}$ where $a_{ij}(t) = 0$ for $j > i$. Moreover,

$$z_{n+1}(t) = (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) y_{n+1}(\varphi(t)),$$

where $\vec{a}_{n+1}(t) = [a_{n+11}(t), \ldots, a_{n+1n}(t)]^T$, $t \in J$.

Observation 1 (A particular case of Corollary 1, [7])**.** *Every homogeneous linear differential equation of an order $n+1$ $(n \geqslant 1)$ is a particular case of the equation (f) and for two equivalent linear equations*

$$\begin{aligned} y_{n+1}(x) &= p_0(x)y_0(x) + p_1(x)y_1(x) + \ldots + p_n(x)y_n(x), \\ y_i(x) &= y^{(i)}(x), \quad x \in I; \\ z_{n+1}(t) &= q_0(t)z_0(t) + q_1(t)z_1(t) + \ldots + q_n(t)z_n(t), \\ z_i(t) &= z^{(i)}(t), \quad t \in J; \end{aligned}$$

$(i = 1, 2, \ldots, n)$ *there always exists a relation*

$$\varphi^{(n+1)}(t) = g(t, \varphi(t), \ldots, \varphi^{(n)}(t)) = \sum_{k=1}^{n} \varphi^{(k)}(t) q_k(t) - p_1(\varphi(t))\varphi'(t)^{n+1}$$

between the function φ and the coefficients of the linear differential equations.

Assumption. For transformations $z(t) = y(\varphi(t))$ of ordinary differential equations of an order $n+1$ $(n \geqslant 1)$ we assume that there exists a differential equation such that $\varphi^{(n+1)}(t) = g(t, \varphi(t), \ldots, \varphi^{(n)}(t))$, $t \in J$.

3. Results

Lemma 1. *Let $n, r \in \mathbb{N}$ and $r \geqslant n+1$. Let φ satisfy the assumptions of Proposition 1. Then (1) is a stationary transformation of the equation (f) if and only if $\varphi(I) = I$ and the real functions f, g satisfy the functional equation*

$$f(s, v, W\vec{v}) = (\vec{w}_{n+1}, \vec{v}) + w_{n+1n+1} f(x, v, \vec{v}), \tag{2}$$

where $W = [w_{ij}]_{j=1,\dots,n}^{i=1,\dots,n}$, $\vec{w}_{n+1} = [w_{n+11}, w_{n+12}, \dots, w_{n+1n}]^T$, $\vec{v} = [v_1, v_2, \dots, v_n]^T$ and $w_{ij} = a_{ij}(x_1, x_2, \dots, x_{i-j+1})$ are defined by

$$\begin{aligned}
w_{i1} &= x_i, \qquad i \geqslant 1;\\
&\dots\\
w_{ij} &= \binom{i}{j-1} x_1^{j-1} x_{i-j+1} + r_{ij}(x_1, \dots, x_{i-j}), \quad i > j > 2;\ w_{ij} = 0\ \textit{for}\ j > i;\\
&\dots\\
w_{ii-2} &= \binom{i}{3} x_1^{i-3} x_3 + 3\binom{i}{4} x_1^{i-4} x_2^2, \qquad i \geqslant 2;\\
w_{ii-1} &= \binom{i}{2} x_1^{i-2} x_2, \qquad i \geqslant 2;\\
w_{ii} &= x_1^i, \qquad i \geqslant 1
\end{aligned} \tag{3}$$

where $s, x, x_i, v, v_i \in \mathbb{R}$; r_{ij} are real functions, $j \in \{1, 2, \dots, i\}$, $i \in \{1, 2, \dots, n+1\}$, $x_{n+1} = g(s, x, x_1, \dots, x_n)$, $n \in \mathbb{N}$.

P r o o f. The transformation (1) is a global transformation of the equation (f) if and only if $\varphi(I) = I$ and at the same time the functions $y(x)$, $z(t) = y(\varphi(t))$ satisfy

$$\begin{gathered}
y^{(n+1)}(x) = y^{(n+1)}(\varphi(t)) = f(\varphi(t), y(\varphi(t)), \dots, y^{(n)}(\varphi(t))),\\
z^{(n+1)}(t) = f(t, z(t), \dots, z^{(n)}(t)), \quad t \in I = \varphi(I).
\end{gathered} \tag{4}$$

From (4) and Proposition 1 we get

$$\begin{aligned}
z^{(n+1)}(t) &= \sum_{j=1}^{n} a_{ij}(t) y^{(j)}(\varphi(t)) + a_{n+1n+1}(t) y^{(n+1)}(\varphi(t))\\
&= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f(\varphi(t), y(\varphi(t)), \dots, y^{(n)}(\varphi(t)))\\
&= f(t, z(t), \vec{z}(t)) = f(t, y(\varphi(t)), A(t)\vec{y}(\varphi(t))),
\end{aligned}$$

i.e.

$$f(t, y(\varphi(t)), A(t)\vec{y}(\varphi(t))) = (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f(\varphi(t), y(\varphi(t)), \vec{y}(\varphi(t))),$$

where the functions $a_{ij}(t)$ are defined by Proposition 1, $t \in J$. We denote $s = t$, $x = \varphi(t)$, $x_i = \varphi^{(i)}(t)$, $v = y(\varphi(t))$, $v_i = y^{(i)}(\varphi(t))$, $w_{ij} = a_{ij}(x_1, x_2, \dots, x_{i-j+1})$, $i \geqslant j \geqslant 1$. Using the definitions of a_{ij} we obtain the assertion of Lemma 1. Here $\varphi^{(n+1)}(t) = g(t, \varphi(t), \dots, \varphi^{(n)}(t))$, $t \in J$; i.e. $x_{n+1} = g(s, x, x_1, \dots, x_n)$ in accordance with Assumption. □

Theorem 1. *The general solution of the functional equation* (2) *is given by*

$$f(x, v, \vec{v}) = b(v)v_1^{n+1} + \sum_{i=1}^{n} p_i(x)v_i = b(v)v_1^{n+1} + (\vec{p}(x), \vec{v}), \tag{5}$$

$$w_{n+11} = g(s, x, x_1, \dots, x_n) = \sum_{i=1}^{n} p_i(s)x_i - p_1(x)x_1^{n+1},$$

$$\begin{aligned} w_{n+1j} &= g_j(s, x_1, \dots, x_{n-j+1}) \\ &= \sum_{k=j}^{n} p_k(s)w_{kj} - p_j(x)w_{n+1n+1}, \quad j \in \{2, \dots, n\}, \end{aligned}$$

where $b, p_1, p_2, \dots, p_n$ *are arbitrary functions and* $w_{ij} = a_{ij}(x_1, x_2, \dots, x_{i-j+1})$ *are defined by* (3), $n \geqslant i \geqslant j \geqslant 1$.

P r o o f. Consider the functional equation (2),

$$f(s, v, W\vec{v}) = \sum_{j=1}^{n} w_{n+1j}v_j + w_{n+1n+1}f(x, v, \vec{v})$$

and define functions $p_i(x) = f(x, 0, \vec{e}_i)$, $i \in \{1, 2, \dots, n\}$. Substituting $v = \vec{e}_i$ into (2) we obtain

$$w_{n+1i} = f(s, v, W\vec{e}_i) - w_{n+1n+1}f(x, v, \vec{e}_i), \quad i \in \{1, 2, \dots, n\} \tag{6}$$

and we can consider $v = 0$ because $w_{ij} = a_{ij}(x_1, \dots, x_{i-j+1})$ are independent of v, $\vec{v}$. So

$$w_{n+1i} = f(s, 0, W\vec{e}_i) - w_{n+1n+1}p_i(x), \quad i \in \{1, 2, \dots, n\}. \tag{7}$$

The functional equation (2) becomes

$$f(s, v, W\vec{v}) = w_{n+1n+1}\left(f(x, v, \vec{v}) - \sum_{j=1}^{n} p_j(x)v_j\right) + \sum_{i=1}^{n} f(s, 0, W\vec{e}_i)v_i. \tag{8}$$

We put $f(x,v,\vec{v}) - \sum_{j=1}^{n} p_j(x)v_j = f(x,v,\vec{v}) - (\vec{p}(x),\vec{v}) = \delta(v,\vec{v})$ since $w_{ij} = a_{ij}(x_1,\dots,x_{i-j+1})$ are independent of x. Hence

(9) $$f(x,v,\vec{v}) = (\vec{p}(x),\vec{v}) + \delta(v,\vec{v}); \quad x,v,v_1,\dots,v_n \in \mathbb{R}.$$

But (8) with $x=1$, $\vec{v}=\vec{e}_1$ gives

$$f(s,v,W\vec{e}_1) = w_{n+1n+1}(f(1,v,\vec{e}_1) - p_1(1)) + f(s,0,W\vec{e}_1),$$

i.e.

$$f(s,v,w_{11},\dots,w_{n1}) = w_{n+1n+1}b(v) + F(s,w_{11},\dots,w_{n1}),$$

where $b(v) = f(1,v,\vec{e}_1) - p_1(1)$, $F(s,w_{11},\dots,w_{n1}) = f(s,0,W\vec{e}_1)$. Now using (3) and (9) we have

(10)

$$f(s,v,x_1,\dots,x_n) = b(v)x_1^{n+1} + F(s,x_1,\dots,x_n) = \delta(v,x_1,\dots,x_n) + \sum_{j=1}^{n} p_j(s)x_j.$$

Comparison of the terms depending on v yields

$$\delta(v,x_1,\dots,x_n) = b(v)x_1^{n+1}$$

and further on

$$F(s,x_1,\dots,x_n) = \sum_{j=1}^{n} p_j(s)x_j.$$

Thus

(11) $$f(x,v,\vec{v}) = b(v)v_1^{n+1} + \sum_{j=1}^{n} p_j(x)v_j; \quad x,v,v_1,\dots,v_n \in \mathbb{R}.$$

Substituting (11) into (6) we get

$$\begin{aligned} w_{n+11} &= \sum_{j=1}^{n} p_j(s)w_{j1} + b(v)w_{11}^{n+1} - w_{n+1n+1}(b(v) + p_1(x)) \\ &= \sum_{j=1}^{n} p_j(s)x_j + b(v)x_1^{n+1} - x_1^{n+1}(b(v) + p_1(x)) \\ &= \sum_{j=1}^{n} p_j(s)x_j - p_1(x)x_1^{n+1} = g(s,x,x_1,\dots,x_n) = x_{n+1} \end{aligned}$$

and, in the same manner,

$$w_{n+1j} = \sum_{k=j}^{n} p_k(s) w_{kj} - p_j(x) w_{n+1n+1} = g_j(s, x, x_1, \ldots, x_{n-j+1}), \quad j \in \{2, \ldots, n\},$$

where $w_{ij} = a_{ij}(x_1, x_2, \ldots, x_{i-j+1})$ are defined by (3). The assertion of Theorem 1 is proved. □

Remark 2. If $n = 1$, then the functional equation (2) is of the form

$$f(s, v, w_{11} v_1) = w_{21} v_1 + w_{22} f(x, v, v_1),$$

where $w_{11} = x^1$, $w_{21} = x_2 = g(s, x, x_1)$, $w_{22} = x_1^2$ and we get

$$f(s, v, x_1 v_1) = g(s, x, x_1) v_1 + x_1^2 f(x, v, v_1); \quad s, v, v_1, x, x_1 \in \mathbb{R}.$$

The general solution of this functional equation is given by

$$f(s, v, v_1) = p_1(s) v_1 + b(v) v_1^2, \ g(s, x, x_1) = p_1(s) x_1 - p_1(x) x_1^2,$$

where b, p are arbitrary functions. This result was derived by J. Aczél [2].

Theorem 2. *Let n, $r \in \mathbb{N}$ and $r \geqslant n + 1$. Let φ satisfy the assumptions of Proposition* 1. *Then* (1) *is a stationary transformation of the equation*

(12)
$$y_{n+1}(x) = (\vec{p}(x), \vec{y}(x)) + b(y(x)) y_1(x)^{n+1}; \quad x \in I \subseteq \mathbb{R}, \ b \text{ being an arbitrary function}$$

if and only if $\varphi(I) = I$ and the real functions $p_1, \ldots, p_n$ satisfy

(13)
$$\varphi^{(n+1)}(t) = \sum_{k=1}^{n} p_k(t) \varphi^{(k)}(t) - p_1(\varphi(t)) \varphi'(t)^{n+1};$$
$$a_{n+1j}(t) = \sum_{k=j}^{n} p_k(t) a_{kj}(t) - p_j(\varphi(t)) \varphi'(t)^{n+1}, \quad j \in \{2, \ldots, n\},$$

where the functions a_{kj} are defined by Proposition 1, *$t \in I = \varphi(I)$, $n \in \mathbb{N}$.*

P r o o f. The assertion of Theorem 2 is a consequence of Lemma 2 and Theorem 1. We use $s = t$, $x = \varphi(t)$, $x_i = \varphi^{(i)}(t)$, $v = y(\varphi(t))$, $v_i = y^{(i)}(\varphi(t))$, $w_{ij} = a_{ij}(x_1, x_2, \ldots, x_{i-j+1}) = a_{ij}(t)$, $n \geqslant i \geqslant j \geqslant 1$, $t \in I$. □

Theorem 3. *Let n, $r \in \mathbb{N}$ and $r \geqslant n+1$. Let φ satisfy the assumptions of Proposition* 1 *and let $\varphi'(t) \neq 0$ on J. Then* (1) *transforms any equation* (12) *into an equation*

$$z_{n+1}(t) = (\vec{q}(t), \vec{z}(t)) + b(z(t))z_1(t)^{n+1}, \quad t \in J \subseteq \mathbb{R}. \tag{14}$$

P r o o f. According to Lemma 1, $z(t) = y(\varphi(t))$, $\vec{z}(t) = A(t)\vec{y}(\varphi(t))$, where $\det A(t) = \prod\limits_{j=1}^{n} a_{jj}(t) = \varphi'(t)^{\frac{n(n-1)}{2}} \neq 0$ on J. The equation (12) is transformable into (14) if and only if

$$\begin{aligned} z_{n+1}(t) &= (\vec{q}(t), \vec{z}(t)) + b(z(t))z_1(t)^{n+1} \\ &= (\vec{q}(t), A(t)\vec{y}(\varphi(t))) + b(y(\varphi(t))(a_{11}(t)y_1(\varphi(t)))^{n+1} \\ &= (A^T(t)\vec{q}(t), \vec{y}(\varphi(t))) + b(y(\varphi(t))y_1(\varphi(t))^{n+1}a_{11}(t)^{n+1} \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t)y_{n+1}(\varphi(t)), \quad t \in J, \end{aligned}$$

due to Proposition 1. Then $a_{11}(t)^{n+1} = a_{n+1n+1}(t) = \varphi'(t)^{n+1}$ and

$$(A^T(t)\vec{q}(t), \vec{y}(\varphi(t))) = (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + (\varphi'(t)^{n+1}\vec{p}(\varphi(t)), \vec{y}(\varphi(t))),$$

i.e.

$$A^T(t)\vec{q}(t) = \vec{p}(\varphi(t))\varphi'(t)^{n+1} + \vec{a}_{n+1}(t), \quad t \in J.$$

As a consequence, the inverse matrix $(A^T(t))^{-1}$ such that

$$\vec{q}(t) = (A^T(t))^{-1}(\vec{p}(\varphi(t))\varphi'(t)^{n+1} + \vec{a}_{n+1}(t)), \quad t \in J,$$

always exists and (14) is satisfied on J. □

Remark 3. Functions $y(x)$, $y(\varphi(x))$, where φ is a given function, are solutions of the equation (12) if and only if the transformation $z(x) = y(\varphi(x))$, $\varphi(I) = I$, is a stationary transformation of the equation (12), i.e. if and only if the relations from Theorem 2 are satisfied. In this sense, Theorem 2 is an effective (i.e. verifiable) criterion.

Example. Consider the equation

$$y'''(x) = \frac{1}{x^2}\left(\frac{a}{(\ln x)^2} + \frac{3b}{\ln x} - 1\right)y'(x) + \frac{3}{x}\left(\frac{b}{\ln x} - 1\right)y''(x) + b(y(x))y'(x)^3 \tag{15}$$

on $I=(1,\infty)$, a, b being real constants and $b(y)$ an arbitrary function. The equation (15) is of the form (12), $p_1(x)=\frac{1}{x^2}\bigl(\frac{a}{(\ln x)^2}+\frac{3b}{\ln x}-1\bigr)$, $p_2(x)=\frac{3}{x}\left(\frac{b}{\ln x}-1\right)$. Using Proposition 1 we have

$$\begin{aligned} a_{11}(t)&=\ \varphi'(t),\\ a_{21}(t)&=\ \varphi''(t),\qquad a_{22}(t)=\varphi'(t)^2,\\ a_{31}(t)&=\ \varphi'''(t),\quad a_{32}(t)=3\varphi'(t)\varphi''(t),\quad a_{33}(t)=\varphi'(t)^3,\qquad t\in I\end{aligned}$$

and from (13) we get

$$\begin{aligned}\varphi'''(t)&=p_1(t)\varphi'(t)+p_2(t)\varphi''(t)-p_1(\varphi(t))\varphi'(t)^3,\\ 3\varphi'(t)\varphi''(t)&=p_2(t)\varphi'(t)^2-p_2(\varphi(t))\varphi'(t)^3,\qquad t\in I.\end{aligned}\tag{16}$$

Solving (16), first the second equation for example, we obtain

$$\varphi(t)=\varphi_r(t)=t^r,\quad r\in\mathbb{R}.\tag{17}$$

The function $\varphi_r(t)$ is a C^3 diffeomorphism of I onto I if and only if $r>0$. Thus $y(x^r)$, $r\in\mathbb{R}^+$, is a solution of (15) whenever $y(x)$ is a solution of (15) with regard to Remark 3.

For $b(y)=0$, (15) is a linear differential equation with the general solution $y(x)=c_1+c_2\ln x+c_3\ln\ln x$ and $y(x^r)=(c_1+c_3\ln r)+c_2r\ln x+c_3\ln\ln x=d_1+d_2\ln x+d_3\ln\ln x$ on $I=(1,\infty)$.

We get $y'''(x)=p_1(x)y'(x)+p_2(x)y''(x)-p_1(y(x))y'(x)^3$ on $I=(1,\infty)$ for $b(y)=-p_1(y)$. In view of (17), the functions $y_r(x)=\varphi_r(x)=x^r$, $r\in\mathbb{R}^+$, are solutions of this differential equation. Indeed, if $s\in\mathbb{R}^+$, then $y_r(x^s)=(x^s)^r=x^{rs}=x^p=y_p(x)$ and $p=rs\in\mathbb{R}^+$.

References

[1] *J. Aczél*: Lectures on Functional Equations and Their Applications. Academic Press, New York, 1966.

[2] *J. Aczél*: Über Zusammenhänge zwischen Differential- und Funktionalgleichungen. Jahresber. Deutsch. Math.-Verein. *71* (1969), 55–57.

[3] *O. Borůvka*: Linear Differential Transformations of the Second Order. The English Univ. Press, London, 1971.

[4] *A. Moór, L. Pintér*: Untersuchungen Über den Zusammenhang von Differential- und Funktionalgleichungen. Publ. Math. Debrecen *13* (1966), 207–223.

[5] *F. Neuman*: Global Properties of Linear Ordinary Differential Equations. Mathematics and Its Applications (East European Series) 52, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991.

[6] *J. Posluszny, L. A. Rubel*: The motion of an ordinary differential equation. J. Differential Equations *34* (1979), 291–302.

[7] *V. Tryhuk*: Remark to transformations of linear differential and functional-differential equations. Czechoslovak Math. J. *50 (125)* (2000), 265–278.

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