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A REPRESENTATION THEOREM FOR PROBABILISTIC
METRIC SPACES IN GENERAL

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Abstract. In this paper, we present a representation theorem for probabilistic metric spaces in general.

Keywords: probabilistic metric space

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K. Menger, B. Schweizer, A. Sklar [1], H. Sherwood [3] and R. Stevens [5] investigated the relationship between probabilistic metrics and numerical metrics. Using a collection of ordinary metrics, R. Stevens presented a representation theorem of a class for probabilistic metric spaces:

Theorem A (cf. [5], p. 267). *If (S, F) is a Menger space under the t -norm $T = \text{Min}$ and if each distance distribution function $F_{pq}(x)$ ($p, q \in S$, $p \neq q$) is continuous, then (S, F) is a metrically generated PM space.*

Since Min is the strongest possible t -norm, one conjectures that Theorem A admits a considerable improvement (cf. [5], p. 267). In this paper, we thoroughly improve Theorem A and give a representation theorem for probabilistic metric spaces in general ($\sup_{a < 1} T(a, a) = 1$).

Definition 1. Let S be a nonempty set and Ω an index set. Let $\{d_t : t \in \Omega\}$ be a collection of mappings from $S \times S$ into $[0, +\infty)$. Then $\{d_t : t \in \Omega\}$ is a collection of semi-metrics on S if it satisfies the following conditions:

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(SM-1) For any t in Ω and all p, q in S , $d_t(p, q) = 0$ if and only if $p = q$;

(SM-2) For all t in Ω and all p, q in S , $d_t(p, q) = d_t(q, p)$;

(SM-3) For every t in Ω , there exists a τ in Ω such that $d_t(p, r) \leq d_\tau(p, q) + d_\tau(q, r)$ for all p, q and r in S .

Definition 2. A PM space (S, F) is semi-metrically generated if and only if there exist a probability space (Ω, B, P) and a collection of semi-metrics $\{d_t : t \in \Omega\}$ on S such that

(SMG-1) for every real number x and every pair p, q of points in S , the set $\{t \in \Omega : d_t(p, q) < x\}$ is a B -measurable set;

(SMG-2) for every real number x and every pair p, q of points belonging to S we have $F(p, q) = F_{pq}$, where F_{pq} is the distribution function defined by

$$(1) \quad F_{pq}(x) = P\{t \in \Omega : d_t(p, q) < x\}.$$

The correctness of Definition 2 follows immediately from the following Theorem 1.

Theorem 1. Let (Ω, B, P) be a probability space and $\{d_t : t \in \Omega\}$ a collection of semi-metrics on S . If $\{d_t : t \in \Omega\}$ satisfies the condition (SMG-1) in Definition 2 and F is defined by (1), then (S, F) is a PM space.

Proof. Theorem 1 can be proved by using the properties of probability measures. □

Theorem 2. If (S, F, T) is a Menger space with $\sup_{a < 1} T(a, a) = 1$, then each distance distribution function $F_{pq}(x)$ ($p, q \in S, p \neq q$) is right-continuous at zero if and only if (S, F) is a semi-metrically generated PM space.

Proof. Necessity: Suppose that B denotes the family of all Borel sets in the open interval $(0, 1)$. Let L be the Lebesgue measure on $(0, 1)$. Then $((0, 1), B, L)$ is a probability space. For any t in $(0, 1)$ and any pair p, q of points in S , we define

$$(2) \quad d_t(p, q) = L\{x \geq 0 : F_{pq}(x) < t\}.$$

Then $\{d_t : t \in (0, 1)\}$ is a collection of mappings from $S \times S$ into $[0, +\infty)$.

For any pair $p, q \in S$ of points with $p \neq q$, by the hypothesis, the distance distribution function $F_{pq}(x)$ is right-continuous at zero. Consequently, it is not hard to show that

$$d_t(p, q) = L\{x \geq 0 : F_{pq}(x) < t\} > 0$$

for all t in $(0, 1)$. Therefore it is easily seen that $\{d_t : t \in (0, 1)\}$ satisfies the condition (SM-1) in Definition 1.

It is clear that $\{d_t: t \in (0, 1)\}$ satisfies the condition (SM-2) in Definition 1. We now prove that $\{d_t: t \in (0, 1)\}$ satisfies the condition (SM-3) in Definition 1. In fact, for every t in $(0, 1)$, by $\sup_{a < 1} T(a, a) = 1$, it follows that there exists a τ in $(0, 1)$ such that $T(\tau, \tau) > t$. Hence for any positive integer n and all p, q, r in S , by (2), we have $F_{pq}(d_\tau(p, q) + 1/n) \geq \tau$ and $F_{qr}(d_\tau(q, r) + 1/n) \geq \tau$. Therefore

$$\begin{aligned} & F_{pr}(d_\tau(p, q) + d_\tau(q, r) + 2/n) \\ & \geq T(F_{pq}(d_\tau(p, q) + 1/n), F_{qr}(d_\tau(q, r) + 1/n)) \\ & \geq T(\tau, \tau) > t. \end{aligned}$$

Consequently, it follows from (2) that $d_t(p, r) \leq d_\tau(p, q) + d_\tau(q, r) + 2/n$. Letting $n \rightarrow \infty$, we obtain $d_t(p, r) \leq d_\tau(p, q) + d_\tau(q, r)$.

From (2), it is easy to see that for every pair p, q of points in S , $d_t(p, q)$ is a nondecreasing function of t on $(0, 1)$. Therefore it can be readily seen that for any pair p, q of points in S and any real number x , the set $\{t \in (0, 1): d_t(p, q) < x\}$ is Borel-measurable, that is, $\{d_t: t \in (0, 1)\}$ satisfies the condition (SMG-1) in Definition 2. Now we show that the condition (SMG-2) in Definition 2 is satisfied. Indeed, for each pair p, q of points in S , it follows from the definition of the PM space that $F_{pq}(x)$ is a nondecreasing, left-continuous function of x . Therefore, by (2) and Proposition 2 in [4], we have

$$F_{pq}(x) = L\{t \in (0, 1): d_t(p, q) < x\}$$

for all real numbers x . From the above argument it follows that (S, F) is a semi-metrically generated PM space.

Sufficiency: The proof proceeds in the same way as that of Theorem 2 from [5], and is therefore omitted. \square

Remark. Obviously the condition $\sup_{a < 1} T(a, a) = 1$ is much weaker than $T = \text{Min}$. Moreover, B. Morrel and J. Nagata [2] showed that no condition weaker than $\sup_{a < 1} T(a, a) = 1$ can guarantee that the ε, λ neighbourhoods induce a bona fide topology.

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