Stanislav Jendroľ; Heinz-Jürgen Voss
Light paths with an odd number of vertices in polyhedral maps

Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 3, 555–564

Persistent URL: `http://dml.cz/dmlcz/127593`

Czechoslovak Mathematical Journal, 50 (125) (2000), 555–564

LIGHT PATHS WITH AN ODD NUMBER OF VERTICES IN POLYHEDRAL MAPS

S. Jendroľ, Košice, and H. J. Voss, Dresden

(Received February 27, 1998)

Abstract. Let P_k be a path on k vertices. In an earlier paper we have proved that each polyhedral map G on any compact 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leqslant 0$ contains a path P_k such that each vertex of this path has, in G, degree $\leqslant k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$. Moreover, this bound is attained for $k = 1$ or $k \geqslant 2$, k even. In this paper we prove that for each odd $k \geqslant \frac{4}{3} \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor + 1$, this bound is the best possible on infinitely many compact 2-manifolds, but on infinitely many other compact 2-manifolds the upper bound can be lowered to $\left\lfloor (k-\frac{1}{3}) \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$.

Keywords: graphs, path, polyhedral map, embeddings

MSC 2000: 05C10, 05C38, 52B10

1. Introduction

In this paper all manifolds are compact 2-dimensional manifolds. If a graph G is embedded in a manifold $\mathbb{M}$ then the closures of the connected components of $\mathbb{M} - G$ are called the *faces* of G. If each face is a closed 2-cell and each vertex has valence at least three then G is called a *map* in $\mathbb{M}$. If, in addition, no two faces have a multiply connected union then G is called a *polyhedral map* in $\mathbb{M}$. This condition on the union of two faces is equivalent to saying that any two faces that meet, meet on a single vertex or a single edge. When two faces in a map meet in one of these two ways we say that they *meet properly.*

In the sequel let $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) surface of genus g (q, respectively). We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The *degree* of a face α of a polyhedral map is the number of edges incident to α. Vertices and faces of degree i are called i-*vertices* and i-*faces*, respectively. Let $v_i(G)$ and $p_j(G)$ denote the number of i-valent vertices and j-valent faces, respectively. For a polyhedral map G let $V(G)$, $E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$ if G is known from the context. A path and a cycle on k vertices is defined to be the k-*path* and the k-*cycle*, respectively. A k-path passing through vertices $A_1, A_2, \ldots, A_k$ is denoted by $[A_1, A_2, \ldots, A_k]$ provided $A_iA_{i+1} \in E(G)$ for any $i = 1, 2, \ldots, k-1$.

It is an old classical consequence of the famous Euler's formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig [Ko1, Ko2] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example, Ivančo [Ivan] has proved that every polyhedral map on $\mathbb{S}_g$ contains an edge with degree sum of their end vertices at most $2g+13$ if $0 \leqslant g \leqslant 3$ and at most $4g+7$, if $g \geqslant 4$. For other results in this topic see e.g. [FaJe, GrSh, Jen, JeVo1, Zaks].

Fabrici and Jendroľ [FaJe] have proved that every 3-connected planar graph G of maximum degree at least k contains a path P_k on k-vertices such that each vertex of this path has, in G, degree $\leqslant 5k$, the bound $5k$ being the best possible. In the same paper [FaJe] they have asked if an analogous result can be established for closed 2-manifolds other than the sphere. More precisely, they have asked the following

Problem 1. For a given connected graph H let $\mathcal{G}(H, \mathbb{M})$ be the family of all polyhedral maps on a closed 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic to H. What is the minimum integer $\varphi(H, \mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}(H, \mathbb{M})$ contains a subgraph K isomorphic to H for which

$$\deg_G(A) \leqslant \varphi(H, \mathbb{M}) \text{ for every vertex } A \in V(K)?$$

(If such minimum does not exist we write $\varphi(H, \mathbb{M}) = \infty$. If $\varphi(H, \mathbb{M}) < +\infty$ then the graph H is called *light* in $\mathcal{G}(H, \mathbb{M})$.)

The answer to this question for $\mathbb{S}_0$ is contained in

Theorem 1 ([FaJe]). *Let k be an integer, $k \geqslant 1$. Then*

$$\varphi(P_k, \mathbb{S}_0) = 5k \quad \text{for any } k \geqslant 1$$

and

$$\varphi(H, \mathbb{S}_0) = \infty \quad \text{for any } H \neq P_k.$$

A slight modification of the method used in [FaJe] yields

Theorem 2. *For any integer $k \geqslant 1$ we have*

$$\varphi(P_k, \mathbb{N}_1) = 5k.$$

For compact 2-manifolds of higher genera we obtained

Theorem 3 ([JeVo1]). *Let k be an integer, $k \geqslant 1$, and let $\mathbb{M}$ be a closed 2-manifold with Euler characteristic $\chi(\mathbb{M}) \notin \{1,2\}$. Then*

(i) $\varphi(P_1, \mathbb{M}) = \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$,

(ii) $2\lfloor \frac{k}{2} \rfloor \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor \leqslant \varphi(P_k, \mathbb{M}) \leqslant k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$, *$k \geqslant 2$, and*

(iii) $\varphi(H, \mathbb{M}) = \infty$ *for any $H \neq P_k$.*

In Theorem 3 the upper bound is sharp for even $k \geqslant 2$ and $k = 1$. In this paper we investigate for which odd $k \geqslant 3$ the upper bound is attained.

The precise bounds for the torus $\mathbb{S}_1$ and the Klein bottle $\mathbb{N}_2$ have already been determined.

Theorem 4 ([JeVo2]). *Let k be an integer, $k \geqslant 1$. Then*

$$\varphi(P_k, \mathbb{S}_1) = \varphi(P_k, \mathbb{N}_2) = \begin{cases} 6k & \text{if } k = 1 \text{ or } k \text{ is even} \\ 6k - 2 & \text{if } k \text{ is odd}, k \geqslant 3. \end{cases}$$

Let K_n and K_n^- denote the complete graph on n vertices with no or one edge missing, respectively. For each large odd k we can show:

(i) the upper bound in Theorem 3 is attained at an infinite sequence of orientable 2-manifolds and at an infinite sequence of nonorientable 2-manifolds, these sequences being characterized by the fact that each member of them is a triangular embedding of a K_n^- (Theorems 5 and 6);

(ii) the upper bound in Theorem 3 is not attained at an infinite sequence of orientable 2-manifolds and at an infinite sequence of nonorientable 2-manifolds, these sequences being characterized by the fact that each member of them is a triangular embedding of a K_n (Theorems 7 and 8).

If $n \equiv 2$, or $5 \pmod{12}$ then K_n^- has a triangular embedding into an orientable 2-manifold $\mathbb{S}_g$ of minimal genus g, where $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ and $g = 12t^2 \pm 3t$, see [Rin], [Jun].

Theorem 5. *Let k be an odd integer and $\mathbb{S}_g$ an orientable compact 2-manifold of genus $g = 12t^2 \pm 3t$, $t = 1, 2, \ldots$.*

If $k \geqslant \lfloor \frac{1}{2}(5 + \sqrt{1 + 48g}) \rfloor + 1$, then

$$\varphi(P_k, \mathbb{S}_g) = k \left\lfloor \frac{5 + \sqrt{1 + 48g}}{2} \right\rfloor .$$

If $n \equiv 2, 5$, or $11 \pmod{12}$ then K_n^- has a triangular embedding into a nonorientable 2-manifold $\mathbb{N}_q$ of minimal genus q, where $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ and $q = 24t^2 \pm 6t$, $t = 1, 2, \ldots$, or $n = 12t + 11$ and $q = 24t^2 + 30t + 9$, $t = 1, 2, \ldots$, see [Rin].

Theorem 6. *Let k be an odd integer and $\mathbb{N}_q$ a nonorientable compact 2-manifold of genus $q = 24t^2 \pm 6t$, $t = 1, 2, \ldots$, or $q = 24t^2 + 30t + 9$, $t = 1, 2, \ldots$.*

If $k \geqslant \lfloor \frac{1}{2}(5 + \sqrt{1 + 24q}) \rfloor + 1$, then

$$\varphi(P_k, \mathbb{N}_q) = k \left\lfloor \frac{5 + \sqrt{1 + 24q}}{2} \right\rfloor .$$

If $n \equiv 0, 3, 4$, or $7 \pmod{12}$ then K_n has a triangular embedding into an orientable 2-manifold $\mathbb{S}_g$ of minimal genus g, where $n = 12t + \frac{7}{2} \pm \frac{7}{2}$ and $g = 12t^2 \pm 7t + 1$, or $n = 12t + \frac{7}{2} \pm \frac{1}{2}$ and $g = 12t^2 \pm t$, $t = 1, 2, \ldots$, see [Rin].

Theorem 7. *Let k be an odd integer and $\mathbb{S}_g$ an orientable compact 2-manifold of genus $g = 12t^2 \pm 7t + 1$, $t = 1, 2, \ldots$, or $g = 12t^2 \pm t$, $t = 1, 2, \ldots$.*

If $k > \frac{4}{3} \, \frac{5+\sqrt{1+48g}}{2} - \frac{4}{3}$, then

$$\varphi(P_k, \mathbb{S}_g) \leqslant \left\lfloor \left(k - \frac{1}{3}\right) \frac{5 + \sqrt{1 + 48g}}{2} \right\rfloor =: m_k(\mathbb{S}_g).$$

Since K_7 has a triangular embedding into the torus $\mathbb{S}_1$, Theorem 7 is also true for the torus. It gives the bound $6k - 2$ already known by Theorem 4. Since K_7 has no embedding into the Klein bottle $\mathbb{N}_2$ the result of Theorem 4 for $\mathbb{N}_2$ cannot be deduced from Theorem 7.

If $n \equiv 0$, or $1 \pmod 3$, $6 \leqslant n \neq 7$ then K_n has a triangular embedding into a nonorientable 2-manifold $\mathbb{N}_q$ of minimal genus q, where $n = 3t$ and $q = \frac{1}{2}(3t^2 - 7t + 4)$, or $n = 3t + 1$ and $q = \frac{1}{2}(3t^2 - 5t + 2)$, $t = 2, 3, \ldots$, where $3t + 1 \neq 7$ [Rin].

Theorem 8. *Let k be an odd integer and $\mathbb{N}_q$ a nonorientable compact 2-manifold of genus $q = \frac{1}{2}(3t^2 - 7t + 4)$, $t = 2, 3, \ldots$, or $q = \frac{1}{2}(3t^2 - 5t + 2)$, $t = 3, 4, \ldots$.*

If $k > \frac{4}{3}\frac{5+\sqrt{1+24q}}{2} - \frac{4}{3}$, *then*

$$\varphi(P_k, \mathbb{N}_q) \leqslant \left\lfloor \left(k - \frac{1}{3}\right) \frac{5+\sqrt{1+24q}}{2} \right\rfloor =: m_k(\mathbb{N}_q).$$

2. Minimum degrees of graphs on $\mathbb{M}$

In this paper $\chi(\mathbb{M}) \leqslant 0$.

Let G be a graph embedded in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. If G is a map, i.e. each face is a 2-cell, then G fulfils Euler's formula

$$n - e + f = \chi(\mathbb{M}),$$

where

$$\chi(\mathbb{M}) = \begin{cases} 2(1-g) & \text{if } \mathbb{M} = \mathbb{S}_g, \\ 2 - q & \text{if } \mathbb{M} = \mathbb{N}_q. \end{cases}$$

If G contains a face F which is not a 2-cell then add an edge to its interior so that F is not subdivided. Add edges in this way until a 2-cell embedding is obtained. Let e^* denote the number of these edges, then Euler's formula is fulfilled with

$$n - (e + e^*) + f = \chi(\mathbb{M}),$$

where n, e and f denote the number of vertices, edges and faces of G, respectively. We summarize this in

Lemma 1. *Let G be the embedding of a graph in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the number of edges which can be added to G without changing the number of its faces. Then the Euler sum is*

$$n - e + f = \chi(\mathbb{M}) + e^*,$$

where n, e and f denote the number of vertices, edges and faces of G, respectively.

Lemma 2. *Let G be the embedding of a* simple *graph with minimum degree $\geqslant 2$ in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the maximum number of edges which can be added to G without changing the number of its faces. It is allowed that the new edges destroy the simplicity of G, i.e., the new edges can create loops or multiple edges. Then $p_0 = p_1 = p_2 = 0$, and the number of edges of G is*

$$e \leqslant 3(n + |\chi(\mathbb{M})| - e^*).$$

Equality holds if and only if all faces of the embedding of G are bounded by three edges.

P r o o f. By Lemma 1 we have

$$n - e + f = \chi(\mathbb{M}) + e^*. \tag{1}$$

On the boundary of each face F a vertex, say V, lies. Since $\delta(G) \geqslant 2$ and the graph G is simple, at least two edges incident with V belong to F. For the endvertices of these edges different from V the same is true. Hence F is bounded by at least three edges, $p_0 = p_1 = p_2 = 0$, and

$$3f \leqslant 2e, \tag{2}$$

where the equality holds if all faces of the embedding of G are bounded by three edges. The formulas (1) and (2) imply

$$3(\chi(\mathbb{M}) + e^*) = 3n - 3e + 3f \leqslant 3n - 3e + 2e$$

and

$$e \leqslant 3(n + |\chi(\mathbb{M})| - e^*).$$

□

Lemma 3. *Let G be the embedding of* a simple *graph with minimum degree $\geqslant 2$ in an orientable compact 2-dimensional manifold $\mathbb{S}_g$ of genus $g = 12t^2 \pm 7t + 1$, $t = 1, 2, \ldots$, or $g = 12t^2 \pm t$, $t = 1, 2 \ldots$. Then the minimum degree of G is $\delta(G) < \frac{5}{2} + \frac{1}{2}\sqrt{1 + 48g}$ or G is a triangulation of $\mathbb{S}_g$ which is a triangular embedding of K_n into $\mathbb{S}_g$ with $n = \frac{7}{2} + \frac{1}{2}\sqrt{1 + 48g}$.*

P r o o f. Let e^* denote the maximum number of edges which can be added to G without changing the number of its faces. Lemma 2 implies $p_0 = p_1 = p_2 = 0$, and the number e of edges of G is

$$e \leqslant 3(n + |\chi(\mathbb{M})| - e^*),$$

where the equality holds if and only if all faces are bounded by precisely 3 edges. From $2e \geqslant n \cdot \delta$ it follows that

$$n(\delta - 6) \leqslant 6|\chi(\mathbb{M})| - 6e^*$$

where the equality holds if and only if G is δ-regular and all faces are bounded by precisely 3 edges. If $\delta \leqslant 6$, then Lemma 3 is true.

Next, let $\delta > 6$. Then by $n \leqslant \delta + 1$ we have

$$(\delta + 1)(\delta - 6) \leqslant 6|\chi(\mathbb{M})| - 6e^*,$$

where the equality holds if and only if $n = \delta + 1$, G is δ-regular and all faces are bounded by precisely 3 edges. Hence

$$\delta \leqslant \frac{5 + \sqrt{49 - 24\chi(\mathbb{M}) - 24e^*}}{2}.$$

Consequently, $\delta \leqslant \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2}$, and the equality only holds if $e^* = 0$, i.e. all faces are 2-cells, and G is a triangular embedding of K_n in $\mathbb{S}_g$, $n = \frac{7+\sqrt{49-24\chi(\mathbb{M})}}{2}$.

By Ringel [Rin] a triangular embedding of K_n in $\mathbb{S}_g$ exists if $g = 12t^2 \pm 7t + 1$, $t = 1, 2, \ldots$, or $g = 12t^2 \pm t$, $t = 1, 2, \ldots$. From $\chi(\mathbb{M}) = 2 - 2g$ the validity of Lemma 3 follows. □

Similarly the following Lemma 4 can be proved.

Lemma 4. *Let G be the embedding of a* simple *graph with minimum degree $\geqslant 2$ in a nonorientable compact 2-dimensional manifold $\mathbb{N}_q$ of genus $q = 24t^2 \pm 6t$, $t = 1, 2, \ldots$, or $q = 24t^2 + 30t + 9$, $t = 1, 2, \ldots$. Then the minimum degree of G is $\delta(G) < \frac{5}{2} + \frac{1}{2}\sqrt{1 + 24q}$ or G is a triangulation of $\mathbb{N}_q$ which is a triangular embedding of K_n into S_g with $n = 12t + \frac{7}{2} \pm \frac{3}{2}$, $t = 1, 2, \ldots$, or $n = 12t + 11$, $t = 1, 2, \ldots$.*

3. Proof of Theorems 7 and 8—upper bounds

The proof follows the ideas of [FaJe]. First the orientable case is proved. Suppose that there is a counterexample to our Theorem 7 having n vertices. Let G be a counterexample with the maximum number of edges among all counterexamples having n vertices. A vertex A of the graph G is major (minor) if $\deg_A(A) > m_k(\mathbb{S}_g)$ ($\leqslant m_k(\mathbb{S}_g)$, respectively), where $m_k(\mathbb{S}_g) := \lfloor (k - \frac{1}{3}) \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \rfloor = \lfloor (k - \frac{1}{3}) \frac{5+\sqrt{1+48g}}{2} \rfloor$.

Lemma 5. *Every r-face α, $r \geqslant 4$, of G is incident only with minor vertices.*

Proof. Suppose there is a major vertex B incident with an r-face α, $r \geqslant 4$. Let C be a diagonal vertex on α with respect to B. Because G is a polyhedral map we can insert an edge BC into the r-face α. The resulting embedding is again a counterexample but with one edge more, a contradiction. □

Let $H(G) = H$ be the subgraph of G induced by the set of major vertices of G. By Lemma 3 there is in H either a vertex A such that

$$\deg_H(A) \leqslant \frac{3}{2} + \frac{\sqrt{49 - 24\chi(\mathbb{M})}}{2} = \frac{5 + \sqrt{1 + 48g}}{2} - 1,$$

or H is a triangulation of $\mathbb{S}_g$ on $n = \frac{7+\sqrt{1+48g}}{2}$ vertices, where H is isomorphic to K_n.

Case 1. Assume that H contains a vertex A of degree $\deg_H(A) \leqslant \frac{3+\sqrt{1+48g}}{2}$. On the other hand, A is a major vertex in G, so the degree of A in G is $\deg_G(A) \geqslant (k - \frac{1}{3})\frac{5+\sqrt{1+48g}}{2} - \frac{2}{3}$. Because of Lemma 5 the subgraph of G induced on the set of vertices consisting of A and its neighbours is a wheel of length $\deg_G(A)$. The major vertices of the cycle of the wheel partition the minor vertices of this cycle into $\deg_H(A) \leqslant \frac{3+\sqrt{1+48g}}{2}$ paths, and one of these paths has a length

$$\geqslant \frac{\deg_G(A) - \deg_H(A)}{\deg_H(A)} \geqslant \frac{(k - \frac{1}{3})\frac{5+\sqrt{1+48g}}{2} - \frac{2}{3} - (\frac{5+\sqrt{1+48g}}{2} - 1)}{\frac{5+\sqrt{1+48g}}{2} - 1}$$

$$\geqslant k - \frac{1}{3} + \frac{k - \frac{1}{3} - \frac{2}{3}}{\frac{5+\sqrt{1+48g}}{2} - 1} - 1 > k - \frac{1}{3} + \frac{4}{3} - 1 = k,$$

a contradiction! (Note that $k > \frac{4}{3}\frac{5+\sqrt{1+48g}}{2} - \frac{4}{3}$.) This contradiction completes the proof in Case 1.

Case 2. Assume that H is a triangulation of $\mathbb{S}_g$ on n vertices, where H is isomorphic to K_n. In Lemma 10 of [JeVo2] we studied precisely the properties of the components of the subgraph H' of G induced on the minor vertices of G.

Lemma 6 ([JeVo2]). *In each triangle D of H there exists a vertex A which is adjacent with only $\leqslant k - 2$ minor vertices of D.*

H has altogether n vertices and $\frac{n(n-1)}{3}$ faces (note that $3f = 2e = n(n-1)$). Therefore, one vertex B of H is incident with $\geqslant \lceil \frac{1}{n}(\frac{n(n-1)}{3}) \rceil = \lceil \frac{n-1}{3} \rceil$ faces F_i so that B has $\leqslant k - 2$ neighbours in the interior of F_i, $i = 1, 2, \ldots, \frac{n-1}{3}$. The number of neighbours of B in the interior of the other faces is $\leqslant k - 1$, $\deg_H(B) = n - 1$. Consequently, $\deg_G(B) \leqslant (n-1) + \lceil \frac{n-1}{3} \rceil (k-2) + (n - 1 - \lceil \frac{n-1}{3} \rceil)(k-1)$, and the major vertex B has a degree

$$\deg_G(B) \leqslant \Big(k - \frac{1}{3}\Big)(n-1).$$

This contradiction proves the theorem in Case 2.

The proof of Theorem 8 can be accomplished in a similar way. □

4. Proof of Theorems 5 and 6—lower bounds

The validity of the upper bounds follows from Theorem 3.

Here the lower bounds are shown by appropriate constructions.

Ringel [Rin] and Jungerman [Jun] presented a triangular embedding T_n of K_n^- in an orientable compact 2-manifold of genus g for $n \equiv 5 (\mathrm{mod}\, 12)$ or $n \equiv 2 (\mathrm{mod}\, 12)$, respectively. With the help of T_n they constructed a triangular embedding of K_n into $\mathbb{S}_{g+1}$, where $g+1$ is the smallest genus α such that K_n can be embedded into $\mathbb{S}_\alpha$. A consequence of Euler's formula reads

$$\sum_{j \geqslant 6} (6-j) v_j + 2 \sum_{j \geqslant 3} (3-j) p_j = 6\chi(\mathbb{M}).$$

Since T_n is a triangulation of $\mathbb{S}_g$ and except two vertices of valency $n-2$, all vertices have valency $n-1$, this formula implies

$$(6-(n-1))(n-2) + (6-(n-2))2 = 6\chi(\mathbb{M}),$$

and

$$n = \frac{1}{2}\left(7 + \sqrt{57 - 24\chi(\mathbb{M})}\right) = \frac{1}{2}\left(7 + \sqrt{9 + 48g}\right). \tag{3}$$

For $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ the genus of $\mathbb{S}_g$ is $g = 12t^2 \pm 3t$. Since T_n is a triangulation there is no embedding of K_n into $\mathbb{S}_g$. If a vertex of T_n of degree $n-2$ is deleted then an embedding of K_{n-1} into $\mathbb{S}_g$ is obtained.

By Ringel [Rin] we know: If K_s can be embedded into $\mathbb{S}_g$ but K_{s+1} has no embedding into $\mathbb{S}_g$ then $s = \lfloor \frac{1}{2}(7 + \sqrt{1+48g}) \rfloor$. Applied to K_{n-1}, this gives $n-1 = \lfloor \frac{1}{2}(5 + \sqrt{1+48g}) \rfloor + 1$, and $n = \lfloor \frac{5+\sqrt{1+48g}}{2} \rfloor + 2 =: m(\mathbb{S}_g) + 2$. Hence T_n contains two nonadjacent vertices of degree $m(\mathbb{S}_g)$ and $n-2$ vertices of degree $m(\mathbb{S}_g)+1$.

Our construction ends in the following way: Into every triangle $[A_1A_2A_3]$ of T_n we insert a generalized 3-star consisting of a central vertex Z and three paths starting in Z, one of length $\frac{k+1}{2}$ and the other of length $\frac{k-1}{2}$. (By the length of a path we mean the number of vertices on it.) Let the paths p_1, p_2, p_3 of this star be ordered in the same way as the vertices of the face $[A_1A_2A_3]$ are ordered. The construction continues by joining the vertex A_i to all vertices of the paths p_i and p_{i+1}, $i = 1, 2, 3$, indices being taken modulo 3. If $[A_1A_2A_3]$ contains a vertex of degree $m(\mathbb{S}_g)$ then let A_1 be this vertex and p_1 the path of length $\frac{k+1}{2}$. Let $D_1, D_2, \ldots, D_s$ denote the triangles of T_n incident with a vertex A. If A is a vertex of degree $\deg_{T_n}(A) = s = m(\mathbb{S}_g)$ then A is adjacent to $k-1$ vertices of the 3-star of D_i, $i = 1, 2, \ldots, s = m(\mathbb{S}_g)$. Hence for $k-1 \geqslant m(\mathbb{S}_g)$ we have

$$\deg_G(A) = (k-1)m(\mathbb{S}_g) + \deg_{T_n}(A) = (k-1)m(\mathbb{S}_g) + m(\mathbb{S}_g) = km(\mathbb{S}_g).$$

If A is a vertex of degree $\deg_{T_n}(A) = s = m(\mathbb{S}_g) + 1$ then A is adjacent to $\geqslant k-2$ vertices of the 3-star of D_i, $i = 1, 2, \ldots, s = m(\mathbb{S}_g) + 1$. Hence for $k - 1 \geqslant m(\mathbb{S}_g)$ we have

$$\begin{aligned}\deg_G(A) &\geqslant (k-2)(m(\mathbb{S}_g)+1) + \deg_{T_n}(A) = (k-2)(m(\mathbb{S}_g)+1) + (m(\mathbb{S}_g)+1)\\ &= (k-1)(m(\mathbb{S}_g)+1) = km(\mathbb{S}_g) - m(\mathbb{S}_g) + k - 1 \geqslant km(\mathbb{S}_g).\end{aligned}$$

This completes the proof in the orientable case.

The proof of the nonorientable case runs in a similar way. Ringel [Rin] presented a triangular embedding T_n of K_n^- in a nonorientable compact 2-manifold $\mathbb{N}_q$ of genus q if $n \equiv 2$, 5 or $11 \pmod{12}$. By formula (3) this implies

$$n = \frac{1}{2}(7 + \sqrt{57 - 24\chi(\mathbb{M})}) = \frac{1}{2}(7 + \sqrt{9 + 24q}),$$

and $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ and $q = 24t^2 \pm 6t$, or $n = 12t + 11$ and $q = 24t^2 + 30t + 9$. The rest of the construction can be accomplished as in the orientable case. □

Acknowledgement. Support of Slovak VEGA grant no. 1/4377/97 is acknowledged.

References

[FaJe] *I. Fabrici, S. Jendroľ*: Subgraphs with restricted degrees of their vertices in planar 3-connected graphs. Graphs Combin. *13* (1997), 245–250.

[GrSh] *B. Grünbaum, G. C. Shephard*: Analogues for tiling of Kotzig's theorem on minimal weights of edges. Ann. Discrete Math. *12* (1982), 129–140.

[Ivan] *J. Ivančo*: The weight of a graph. Ann. Discrete Math. *51* (1992), 113–116.

[Jen] *S. Jendroľ*: Paths with restricted degrees of their vertices in planar graphs. Czechoslovak Math. J. *49 (124)* (1999), 481–490.

[JeVo1] *S. Jendroľ, H.-J. Voss*: A local property of polyhedral maps on compact 2-dimensional manifolds. Discrete Math. *212* (2000), 111–120.

[JeVo2] *S. Jendroľ, H.-J. Voss*: Light paths with an odd number of vertices in large polyhedral maps. Ann. Comb. *2* (1998), 313–324.

[Jun] *M. Jungerman*: Ph. D. Thesis. Univ. of California. Santa Cruz, California 1974.

[Ko1] *A. Kotzig*: Contribution to the theory of Eulerian polyhedra. Math. Čas. SAV (Math. Slovaca) *5* (1955), 111–113.

[Ko2] *A. Kotzig*: Extremal polyhedral graphs. Ann. New York Acad. Sci. *319* (1979), 569–570.

[Rin] *G. Ringel*: Map Color Theorem. Springer-Verlag Berlin (1974).

[Zaks] *J. Zaks*: Extending Kotzig's theorem. Israel J. Math. *45* (1983), 281–296.

Authors' addresses: S. Jendroľ, Department of Geometry and Algebra, P. J. Šafárik University and Institute of Mathematics, Slovak Academy of Sciences, Jesenná 5, 041 54 Košice, Slovakia, e-mail: jendrol@kosice.upjs.sk; H. J. Voss, Department of Algebra, Technical University Dresden, Mommsenstrasse 13, D-01062 Dresden, Germany, e-mail: voss@math.tu-dresden.de.