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CONVEXITIES OF NORMAL VALUED LATTICE ORDERED GROUPS

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Convexities of lattice ordered groups were investigated in [5]. Earlier, convexities of lattices and of d-groups had been dealt with in [3] or [4], respectively. Let us recall that the notation of convexity of lattices was introduced by Fried ([7], p. 225).

We denote by

$\mathcal{G}$—the class of all lattice ordered groups;

$\mathcal{A}$—the class of all abelian lattice ordered groups;

$\mathcal{N}$—the class of all normal valued lattice ordered groups;

X_0—the class of all one-element lattice ordered groups.

For $G \in \mathcal{G}$ we denote by $C(G)$ the convexity of lattice ordered groups which is generated by G. Let Z, Q and R be the additive group of all integers, rationals and reals, respectively, with the natural linear order.

If we consider a result on varieties of lattice ordered groups, torsion classes or radical classes, then we can ask whether a similar result holds for convexities.

The following result is well-known (cf., e.g., [1]):

(A) There exists a variety X_1 (namely, $X_1 = \mathcal{A}$) such that, whenever Y is a variety with $Y \neq X_0$, then $X_1 \subseteq Y$.

A result analogous to (A) holds neither for torsion classes nor for radical classes.

In the present paper we prove:

(B) There exists a convexity $Z_1 \neq X_0$ (namely, $Z_1 = C(R)$) such that, whenever Z is a convexity with $X_0 \neq Z \subseteq \mathcal{N}$, then $Z_1 \subseteq Z$.

Some further results are also proved.

Let us remark that the class $\mathcal{N}$ is large in the sense that whenever $\mathcal{V}$ is a variety with $\mathcal{V} \neq \mathcal{G}$, then $\mathcal{V} \subseteq \mathcal{N}$ (cf., e.g., [1]).

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1. The partially ordered classes $\mathscr{C}$ and $\mathscr{C}_{nv}$

For $X \subseteq \mathscr{G}$ we denote by

HX—the class of all homomorphic images of elements of X;

CX—the class of all isomorphic images of convex ℓ-subgroups of elements of X;

PX—the class of all direct products of elements of X.

1.1. Definition. A nonempty subclass of X of $\mathscr{G}$ is called a convexity if $HX \subseteq X$, $CX \subseteq X$ and $PX \subseteq X$.

The class of all convexities of lattice ordered groups will be denoted by $\mathscr{C}$; it is partially ordered by the class-theoretical inclusion. Also, each nonempty subclass of $\mathscr{C}$ is partially ordered by the induced partial order.

1.2. Lemma. (Cf. [5].) *Let $\emptyset \neq X \subseteq \mathscr{G}$. Then*

(i) *$HCPX \in \mathscr{C}$;*

(ii) *for each $Y \in C$ with $X \subseteq Y$ the relation $HCPX \subseteq Y$ is valid.*

In view of 1.2, the convexity $HCPX$ will be said to be generated by X. If $X = \{G\}$ is a one-element set, then we put $HCPX = C(G)$.

For direct products of lattice ordered groups we apply the same notation and conventions as in [5], Section 1.

From 1.2 we immediately obtain

1.3. Lemma. *Let $\{X_i\}_{i\in I}$ be a nonempty subclass of $\mathscr{C}$. Then $\bigcap_{i\in I} X_i \in \mathscr{C}$.*

Let X_0 be as above. It is obvious that X_0 is the least element of $\mathscr{C}$ and that $\mathscr{G}$ is the greatest element of $\mathscr{C}$. From this and from 1.3 we obtain

1.4. Lemma. *Let $\{X_i\}_{i\in I}$ be a nonempty subclass of $\mathscr{C}$. Then there exist Y_1 and Y_2 in $\mathscr{C}$ such that the relations $Y_1 = \inf\{X_i\}_{i\in I}$ and $Y_2 = \sup\{X_i\}_{i\in I}$ are valid in the partially ordered collection $\mathscr{C}$. Moreover, $Y_1 = \bigcap_{i\in I} X_i$.*

In [5] it was proved that the collection $\mathscr{C}$ is large in the sense that there exists an injective mapping of the class of all infinite cardinals into $\mathscr{C}$.

Nevertheless, in view of 1.4 we can apply for $\mathscr{C}$ the usual lattice-theoretical terminology and notation. Thus, if Y_1 and Y_2 are in 1.4, then we write

$$Y_1 = \bigwedge_{i\in I} X_i, \quad Y_2 = \bigvee_{i\in I} X_i.$$

1.5. Lemma. *Let $\{X_i\}_{i\in I}$ be a nonempty subclass of $\mathscr{C}$. Then*

$$\bigvee_{i\in I} X_i = HCP\Big(\bigcup_{i\in I} X_i\Big).$$

P r o o f. This is a consequence of 1.2. □

1.6. Proposition. *Let $X_1, X_2 \in \mathscr{C}$. Next, let Y be the set of all $G \in \mathscr{G}$ such that there exist $G_1 \in X_1$ and $G_2 \in X_2$ with $G = G_1 \times G_2$. Then $X_1 \vee X_2 = Y$.*

P r o o f. Let $G \in Y$. Then (under the notation as above) we have $G_1, G_2 \in X_1 \vee X_2$, whence $G \in X_1 \vee X_2$.

Conversely, let $G \in X_1 \vee X_2$. In view of 1.5 there exists a set $\{G_j\}_{j\in J} \subseteq X_1 \cup X_2$ such that

$$G \in HCP\{G_j\}_{j\in J}.$$

Hence there are $A_1, A_2 \in \mathscr{G}$ with

$$A_1 \in P\{G_j\}_{j\in J}, \quad A_2 \in C\{A_1\}, \quad G \in H\{A_2\}.$$

Therefore

$$A_1 = \Big(\prod_{j\in J(1)} G_j\Big) \times \Big(\prod_{j\in J(2)} G_j\Big),$$

where

$$\{G_j\}_{j\in J(1)} \subseteq X_1, \quad \{G_j\}_{j\in J(2)} \subseteq X_2.$$

Put

$$\prod_{j\in J(1)} G_j = G_1^1, \qquad \prod_{j\in J(2)} G_j = G_2^1.$$

Hence $G_1^1 \in X_1$ and $G_2^1 \in X_2$.

From the relation $A_1 = G_1^1 \times G_2^1$ and from Lemma 1.2 in [5] we obtain

$$A_2 = (A_2 \cap G_1^1) \times (A_2 \cap G_2^1).$$

Since $A_2 \cap G_i^1$ is a convex ℓ-subgroup of G_i^1 we get $A_2 \cap G_i^1 \in X_i$ $(i = 1, 2)$. Now it suffices to apply Lemma 1.3 from [5] to verify that $G \in Y$. □

1.7. Theorem. *The lattice $\mathscr{C}$ is distributive.*

P r o o f. Let $X_1, X_2, X_3 \in \mathscr{C}$. We have to verify that the relation

$$X_1 \wedge (X_2 \vee X_3) = (X_1 \wedge X_2) \vee (X_1 \wedge X_3)$$

is valid. Clearly $(X_1 \wedge X_2) \vee (X_1 \wedge X_3) \subseteq X_1 \wedge (X_2 \vee X_3)$. Let $G \in X_1 \wedge (X_2 \vee X_3)$. Thus $G \in X_1$ and $G \in X_2 \vee X_3$. According to 1.6 there are $G_2 \in X_2$ and $G_3 \in X_3$ such that $G = G_2 \times G_3$. Hence $G_2, G_3 \in C\{G\}$ and therefore $G_2, G_3 \in X_1$. We get

$$G_i \in X_1 \wedge X_i \quad (i = 2, 3)$$

and hence $G \in (X_1 \wedge X_2) \vee (X_1 \wedge X_3)$, completing the proof. □

A complete lattice is said to be infinitely distributive if it satisfies the identities

$$x \vee \left(\bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \vee y_i), \tag{1}$$

$$x \wedge \left(\bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} (x \wedge y_i). \tag{2}$$

The collection of all radical classes of lattice ordered groups satisfies identically the relation (2). The question whether $\mathscr{C}$ satisfies (1) or (2) remains open.

Let $G \in \mathscr{G}$ and $0 \neq g \in G$. The convex ℓ-subgroup of G generated by g will be denoted by $[g]$. Next, let $C_1(g)$ be the set of all convex ℓ-subgroups of $[g]$ which do not contain the element g, and let $C_2(g)$ be the set of all maximal elements of $C_1(G)$.

A lattice ordered group G is said to be normal valued if, whenever $0 \neq g \in G$ and $G' \in C_2(g)$, then G' is a normal subgroup of $[g]$.

Let $\mathscr{N}$ be as above. From the fact that $\mathscr{N}$ is a variety (cf., e.g., [1]) we infer that $\mathscr{N}$ belongs to $\mathscr{C}$. Hence the class $\mathscr{C}_{nv}$ of all convexities X with $X \subseteq \mathscr{N}$ is the interval $[X_0, \mathscr{N}]$ of $\mathscr{C}$. Therefore 1.7 yields

1.8. Corollary. *$\mathscr{C}_{nv}$ is a distributive lattice.*

The following result is well-known.

1.9. Lemma. *Let $G \in \mathscr{N}$, $0 \neq g \in G$, $G' \in C_2(G)$. Then there exists an isomorphism φ of the lattice ordered group $[g]/G'$ into R.*

We start by investigating the convexity $C(Z)$.

Let $\mathbb{N}$ be the set of all positive integers and for each $n \in \mathbb{N}$ let $G_n = Z$. Denote $G^1 = \prod_{n\in\mathbb{N}} G_n$. For $g \in G^1$ we denote by g_n the n-th component of g. If there exists a positive integer m such that $|g(n)| \leqslant m$ for each $n \in \mathbb{N}$, then g will be said to be bounded. (In an analogous sense we apply the notion of boundedness also when dealing with any direct product of ℓ-subgroups of R.) The set of all bounded elements of G^1 will be denoted by G^2.

The set G^2 is an ℓ-ideal of G^1, thus we can construct the factor lattice ordered group $\overline{G^1} = G^1/G^2$ and we have $\overline{G^1} \in C(Z)$. For $g \in G^1$ we put $\overline{g} = g + G^2$.

2.1. Lemma. *The lattice ordered group $\overline{G^1}$ is divisible.*

P r o o f. It suffices to verify that for each positive integer m and for each strictly positive element $\overline{g} = g + G^2$ of $\overline{G^1}$ there exists $g' \in G^1$ such that $m\overline{g'} = \overline{g}$.

If $\overline{g} > \overline{0}$, then without loss of generality we can suppose that $g_n \geqslant 0$ for each $m \in \mathbb{N}$, and that the sequence $(g_n)_{n\in\mathbb{N}}$ is not bounded.

Let $m, n \in \mathbb{N}$. There is a real x_n such that $mx_n = g_n$. Next, there is a real z_n such that $0 \leqslant z_n < 1$ and $x_n + z_n \in Z$. Put $x_n + z_n = y_n$.

There are $g', z' \in G'$ such that

$$g'_n = y_n, \quad z'_n = 1 \quad \text{for each } n \in \mathbb{N}.$$

Hence

$$g_n = mx_n \leqslant my_n = mx_n + mz_n < g_n + m.$$

Thus $g \leqslant mg' < g + z'$ and therefore

$$\overline{g} \leqslant m\overline{g'} \leqslant \overline{g} + m\overline{z'}.$$

Clearly $z' \in G^2$, whence $\overline{z'} = \overline{0}$. We conclude that $\overline{g} = m\overline{g'}$. □

2.2. Lemma. *There exists $\{0\} \neq G^3 \in C(Z)$ such that*

(i) *G^3 is an ℓ-subgroup of R;*

(ii) *G^3 is divisible.*

P r o o f. Let $\overline{G^1}$ be as in 2.1. Then $\overline{G^1}$ belongs to $C(Z)$. There exists $\overline{g} \in \overline{G^1}$ with $\overline{g} \neq \overline{0}$. Let $G' \in C_2(\overline{g})$ and denote

$$G^3_0 = [\overline{g}]/G'$$

(with respect to the lattice ordered group $\overline{G^1}$). Then G_0^3 is a nonzero lattice ordered group. We have $[\overline{g}] \in C(Z)$ and hence $G_0^3 \in C(Z)$. In view of 2.1, $[\overline{g}]$ is divisible and thus G_0^3 is divisible as well. Now it suffices to apply 1.9. □

Let us consider the following condition for a convexity X:

($*$) There exists G^3 in X satisfying the conditions (i) and (ii) from 2.2.

Suppose that ($*$) is valid. Put $H_n = G^3$ for each $n \in \mathbb{N}$ and

$$G^4 = \prod_{n\in\mathbb{N}} H_n.$$

For $g \in G^4$ let g_n be the component of g in H_n. Next, let K be the set of all bounded elements of G^4. We investigate the factor lattice ordered group G^4/K; we denote $\overline{g} = g + K$. Since G^3 is divisible, so are G^4 and G^4/K. Hence for each $q \in Q$ we can construct $qg \in G^4$ and $q\overline{g} \in G^4/K$.

Let $0 < r \in R$. Put

$$Q_1 = \{g \in Q : 0 < q < r\}, \quad Q_2 = \{q \in Q : r < q\}.$$

There exists a sequence $\{q_{(n)}\}_{n\in\mathbb{N}}$ of elements of Q_1 such that $q_{(n)} < q_{(n+1)}$ for each $n \in \mathbb{N}$, and $\sup\{q_{(n)}\}_{n\in\mathbb{N}} = r$.

Under the above notation we have

2.3. Lemma. *Let $0 < g \in G^4$ with $\overline{g} > \overline{0}$ and let $0 < r \in R$. There exists $g' \in G^4$ such that $q_1\overline{g} \leqslant \overline{g'} \leqslant q_2\overline{g}$ for each $q_1 \in Q_1$ and each $q_2 \in Q_2$.*

Proof. There is $g' \in G^4$ such that

$$g'_n = q_{(n)}g_n$$

for each $n \in \mathbb{N}$. Thus $g'_n < q_2 g_n$ for each $n \in \mathbb{N}$ and each $q_2 \in Q_2$. Hence

$$\overline{g'} \leqslant q_2\overline{g} \quad \text{for each } q_2 \in Q_2. \tag{1}$$

Let $q_1 \in Q_1$. There exists $m \in \mathbb{N}$ such that $q_1 < q_{(m)}$. Further, there are elements $g_{(1,m)}$ and $g_{(2,m)}$ in G^4 such that

$(g_{(1,m)})_n = g_n$ if $n < m$, and $(g_{(1,m)})_n = 0$ otherwise;

$(g_{(2,m)})_n = g_n$ if $n \geqslant m$, and $(g_{(2,m)})_n = 0$ otherwise.

Similarly we construct $g'_{(1,m)}$ and $g'_{(2,m)}$ (with g replaced by g'). Thus

$$g = g_{(1,m)} + g_{(2,m)}, \tag{2}$$

$$g' = g'_{(1,m)} + g'_{(2,m)}. \tag{3}$$

Clearly $g_{(1,m)}, g'_{(1,m)} \in K$, whence

$$\overline{g} = \overline{g_{(2,m)}}, \quad \overline{g'} = \overline{g'_{(2,m)}}. \tag{4}$$

We have also

$$q_1 g_{(2,m)} < g'_{(2,m)},$$

whence

$$q_1 \overline{g_{(2,m)}} \leqslant \overline{g'_{(2,m)}}. \tag{5}$$

The relations (2)–(5) yield that $q_1 \overline{g} \leqslant \overline{g'}$ for each $q_1 \in Q_1$. Hence, by virtue of (1), the proof is complete. □

2.4. Lemma. *Let X be a convexity satisfying the condition $(*)$. Then $R \in X$.*

P r o o f. Let G^4, g and r be as in 2.3. Then $G^4 \in X$. Also, G^4/K belongs to X. Let $G' \in C_2(\overline{g})$. We construct the lattice ordered group $[\overline{g}]/G'$ (with respect to G^4/K). For $\overline{x} \in [\overline{g}]$ we denote $\overline{\overline{x}} = \overline{x} + G'$.

According to 1.9 there exists an isomorphism φ of $[\overline{g}]/G'$ into R. Denote $\varphi(\overline{\overline{g}}) = r_0$. Then $r_0 > 0$. From this we infer that there exists an isomorphism φ_1 of $[\overline{g}]/G'$ into R such that $\varphi_1(\overline{\overline{g}}) = 1$.

Let $\overline{g'}$ be as in 2.3. Put $\varphi_1(\overline{\overline{g'}}) = r'$. In view of 2.3 we have

$$q_1 \varphi_1(\overline{\overline{g}}) \leqslant \varphi_1(\overline{\overline{g'}}) \leqslant q_2 \varphi_1(\overline{\overline{g}}),$$

hence $q_1 \leqslant r' \leqslant q_2$ whenever $q_1 < r < q_2$. Thus $r = r'$ and therefore $r \in \varphi_1([g]/G')$. Then, clearly, φ_1 is an epimorphism. Since $[\overline{g}]/G'$ belongs to X we get that R belongs to X as well. □

2.5. Lemma. *$R \in C(Z)$.*

P r o o f. This is a consequence of 2.1–2.4. □

Now let G be an ℓ-subgroup of R such that $G \neq \{0\}$ and G fails to be isomorphic to Z. By an elementary argument we obtain

2.6. Lemma. *Let $0 < x \in R$. Then there exists $g_0 \in G$ such that $0 < g_0 < x$.*

For each $n \in \mathbb{N}$ let $G_n = G$. Put $G_0^1 = \prod\limits_{n \in \mathbb{N}} G_n$. Next, let G^2 be the ℓ-subgroup of G_0^1 consisting of all bounded elements of G_0^1. We denote $\overline{G_0^1} = G_0^1/G^2$, $\overline{g} = g + G^2$, where $g \in G_0^1$.

2.7. Lemma. *The lattice ordered group $\overline{G_0^1}$ is divisible.*

P r o o f. We proceed in the same way as in the proof of 2.1 with the distinction that instead of

$$y_n = x_n + z_n \in Z$$

we now consider the relation

$$y_n = x_n + z_n \in G$$

and apply 2.6. □

2.8. Lemma. *$R \in C(G)$.*

P r o o f. This is a consequence of 2.7, 2.3 and 2.4. □

2.9. Theorem. *Let X be a convexity of normal valued lattice ordered groups and let $X \neq X_0$. Then $C(R) \subseteq X$.*

P r o o f. There is $G_1 \in X$ with $G_1 \neq \{0\}$. Hence in view of 1.9 there exists $G \in X$ such that $G \neq \{0\}$ and G is an ℓ-subgroup of R. According to 2.5 and 2.8, R belongs to X. Thus $C(R) \subseteq X$. □

In other words, we have proved that the interval $[X_0, \mathscr{N}]$ of $\mathscr{C}$ has a unique atom.

2.10. Corollary. *$C(R) \subset C(Z)$.*

P r o o f. In view of 1.2, $C(R)$ is divisible. Hence $Z \notin C(R)$ and thus, according to 2.9, $C(R) \subset C(Z)$. □

We denote by $X_{v\ell}$ the class of all lattice ordered groups G which satisfy the following condition:

(vℓ) We can define a multiplication of elements of G by reals such that G turns out to be a vector lattice.

It is obvious that

(i) $X_{v\ell}$ is closed with respect to H, C and P; hence $X_{v\ell}$ is a convexity;

(ii) if G_1 is an ℓ-subgroup of R with $\{0\} \neq G_1 \in X_{v\ell}$, then $G_1 = R$.

The following result generalizes 2.10.

2.11. Proposition. *Let G_1 be an ℓ-subgroup of R such that $\{0\} \neq G_1 \neq R$. Then $C(R) \subset C(G_1)$.*

P r o o f. In view of 2.9 we have $C(R) \subseteq C(G_1)$. By way of contradiction, suppose that $C(R) = C(G_1)$. Hence $G_1 \in C(R)$. But $X \in X_{v\ell}$ and thus according to (i), G_1 belongs to $X_{v\ell}$ as well. This contradicts (ii). □

In particular, $C(R) \subset C(Q)$. An open question: what are the relations between $C(Q)$ and $C(Z)$?

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