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METHODS OF OSCILLATION THEORY OF HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate oscillatory properties of the second order half-linear equation

$$(r(t)\Phi(y'))' + c(t)\Phi(y) = 0, \quad \Phi(s) := |s|^{p-2}s. \tag{$*$}$$

Using the Riccati technique, the variational method and the reciprocity principle we establish new oscillation and nonoscillation criteria for (*). We also offer alternative methods of proofs of some recent oscillation results.

Keywords: half-linear equation, Riccati technique, variational principle, reciprocity principle, principal solution, oscillation and nonoscillation criteria

MSC 2000: 34C10

1. Introduction

The aim of this paper is to discuss the application of various methods in the oscillation theory of half-linear second order differential equations

$$(r(t)\Phi_p(y'))' + c(t)\Phi_p(y) = 0, \quad \Phi_p(s) = |s|^{p-2}s, \quad p > 1, \tag{1}$$

where the functions r, c are continuous and $r(t) > 0$.

It is known, see Elbert, Mirzov [8, 20], that the oscillation theory of (1) is very similar to that of the Sturm-Liouville linear equation

$$(r(t)y')' + c(t)y = 0 \tag{2}$$

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which is the special case $p = 2$ of (1). In particular, the Sturmian separation and comparison theory extends in a natural way to (1).

In the last decade, considerable effort has been made to generalize the linear oscillation and nonoscillation criteria to (1), see e.g. [3, 4, 7, 9, 14, 15, 16, 18] and the reference given therein. These investigations were mostly based on the so-called Riccati technique consisting in the fact that if y is a nonzero solution of (1) then $w(t) = \frac{r(t)\Phi(y')}{\Phi(y)}$ solves the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q = \frac{p}{p-1}. \tag{3}$$

Another method in the oscillation theory of (1), established only recently, see [13, 17, 19], consists in the relationship between the disconjugacy of (1) (i.e. the nonexistence of a nontrivial solution with two or more zeros in an interval under consideration) and the positivity of the functional

$$\mathcal{F}(y; a, b) = \int_a^b [r(t)|y'|^p - c(t)|y|^p] \,\mathrm{d}t. \tag{4}$$

More precisely, equation (1) is disconjugate in $[a, b]$ if and only if $\mathcal{F}(y; a, b) > 0$ for every nontrivial $y \in W^{1,p}(a, b)$ with $y(a) = 0 = y(b)$.

Finally, the third method we are going to discuss in the paper is the so-called reciprocity principle (this terminology comes from the linear case). If we denote $u := r(t)\Phi_p(y')$, where y is a solution of (1) and $c(t) > 0$ in (1), then by a direct computation one can verify that u solves the so-called *reciprocal* equation

$$\left(c^{1-q}(t)\Phi_q(u')\right)' + r^{1-q}(t)\Phi_q(u) = 0, \quad \Phi_q(s) := |s|^{q-2}s. \tag{5}$$

Conversely, if $y = c^{1-q}(t)\Phi_q(u')$, then this function satisfies the original equation (1). We will show by an elementary argument that (1) is oscillatory if and only if (5) has this property and then we will use this fact in order to offer alternative proofs of some known oscillation and nonoscillation criteria for (1).

2. Riccati technique

In this section we extend the linear Wintner nonoscillation criterion to (1). This criterion claims that if $\int^\infty r^{-1}(t)\,\mathrm{d}t = \infty$ and $\int^\infty c(t)\,\mathrm{d}t$ converges, then the linear Sturm-Liouville equation (2) is nonoscillatory provided

$$\limsup_{t\to\infty} \left(\int^t r^{-1}(s)\,\mathrm{d}s\right)\left(\int_t^\infty c(s)\,\mathrm{d}s\right) < \frac{1}{4}$$

and

$$\liminf_{t\to\infty}\Big(\int^t r^{-1}(s)\,\mathrm{d}s\Big)\Big(\int_t^\infty c(s)\,\mathrm{d}s\Big) > -\frac{3}{4}.$$

Similar sufficient conditions for nonoscillation of (2) can be formulated also in the case when $\int^\infty r^{-1}(t)\,\mathrm{d}t < \infty$.

Theorem 1. *Suppose that* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$ *and* $\int^\infty c(t)\,\mathrm{d}t = \lim\limits_{b\to\infty}\int^b c(t)\,\mathrm{d}t$ *converges. If*

$$\limsup_{t\to\infty}\Big(\int^t r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int_t^\infty c(s)\,\mathrm{d}s\Big) < \frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}, \tag{6}$$

$$\liminf_{t\to\infty}\Big(\int^t r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int_t^\infty c(s)\,\mathrm{d}s\Big) > -\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1} \tag{7}$$

then (1) *is nonoscillatory.*

Proof. We will find a solution of the Riccati type inequality

$$v' \leqslant -c(t) - (p-1)r^{1-q}(t)|v|^q \tag{8}$$

which is extensible up to ∞, i.e. it exists on some interval $[T,\infty)$. Then, if w is the solution of (3) given by the initial condition $w(T) = v(T)$, this solution satisfies the inequality $w(t) \geqslant v(t)$ for $t \geqslant T$. Hence it also exists on $[T,\infty)$ and this means that (1) is nonoscillatory.

To find the solution v of (8) we show that there exists an extensible up to ∞ solution of the differential inequality

$$\varrho' \leqslant (1-p)r^{1-q}(t)|\varrho + C(t)|^q, \quad C(t) := \int_t^\infty c(s)\,\mathrm{d}s \tag{9}$$

related to (8) by the substitution $\varrho = v - C$. This solution ϱ is

$$\varrho(t) = \beta\Big(\int^t r^{1-q}(s)\,\mathrm{d}s\Big)^{1-p}, \quad \beta := \Big(\frac{p-1}{p}\Big)^p.$$

Indeed, $\varrho' = (1-p)\beta r^{1-q}(t)\big(\int^t r^{1-q}(s)\,\mathrm{d}s\big)^{-p}$ and the right-hand side of (9) is

$$\begin{aligned}(1-p)r^{1-q}(t)|\varrho + C(t)|^q &= (1-p)r^{1-q}(t)\Big|\beta\Big(\int^t r^{1-q}\Big)^{1-p} + C(t)\Big|^q\\ &= (1-p)r^{1-q}(t)\Big|\beta + \Big(\int^t r^{1-q}\Big)^{p-1}C(t)\Big|^q\Big(\int^t r^{1-q}\Big)^{(1-p)q}\\ &= (1-p)r^{1-q}(t)\Big|\beta + \Big(\int^t r^{1-q}\Big)^{p-1}C(t)\Big|^q\Big(\int^t r^{1-q}\Big)^{-p}.\end{aligned}$$

Consequently, (9) is equivalent to the inequality

$$\beta \geqslant \left| \beta + \left(\int^t r^{1-q} \right)^{p-1} C(t) \right|^q. \tag{10}$$

However, since (6) and (7) hold, there exists $\varepsilon > 0$ such that

$$-\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1} + \varepsilon < \left(\int^t r^{1-q} \right)^{p-1} C(t) < \frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1} - \varepsilon$$

for large t and by a direct computation it is not difficult to verify that (10) really holds. □

The following theorem completes the previous statement and deals with the "complementary" case $\int^\infty r^{1-q}(t)\,\mathrm{d}t < \infty$.

Theorem 2. *Suppose that $\int^\infty r^{1-q}(t)\,\mathrm{d}t < \infty$. If*

$$\limsup_{t\to\infty} \left(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s \right)^{p-1} \left(\int^t c(s)\,\mathrm{d}s \right) < \frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}$$

and

$$\liminf_{t\to\infty} \left(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s \right)^{p-1} \left(\int^t c(s)\,\mathrm{d}s \right) > -\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1},$$

then (1) *is nonoscillatory.*

Proof. One can show in the same way as in the previous proof that the function

$$\varrho(t) = -\beta\left(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s \right)^{1-p}, \quad \beta = \Big(\frac{p-1}{p}\Big)^p$$

satisfies the inequality

$$\varrho' \leqslant (1-p) r^{1-q}(t) \big|\varrho - \tilde{C}(t)\big|^q, \quad \tilde{C}(t) = \int^t c(s)\,\mathrm{d}s,$$

which implies that $v = \varrho - \tilde{C}$ satisfies the Riccati inequality (8). □

3. Variational principle

The relationship between the disconjugacy of (1) in $[a,b]$ and the positivity of the functional $\mathcal{F}(y;a,b)$ over $W_0^{1,p}(a,b)$ given in the first section shows that to prove oscillation of (1), it suffices to construct, for any $T\in\mathbb{R}$ sufficiently large, a nontrivial function $y\in W^{1,p}(T,\infty)$ with compact support in (T,∞), such that $\mathcal{F}(y;T,\infty)\leqslant 0$. On the other hand, (1) is nonoscillatory provided we show that there exists $T\in\mathbb{R}$ such that $\mathcal{F}(y;T,\infty)>0$ for *every* $y\in W^{1,p}(T,\infty)$ with supp $\subset[T,\infty)$.

The variational approach was used e.g. in [4], where we proved, among other, that (1) is nonoscillatory provided $\int^\infty r^{1-q}(t)\,\mathrm{d}t=\infty$ and

$$\limsup_{t\to\infty}\left(\int^t r^{1-q}(s)\,\mathrm{d}s\right)^{p-1}\left(\int_t^\infty c_+(s)\,\mathrm{d}s\right)<\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \tag{11}$$

where $c_+(t)=\max\{0,c(t)\}$. Clearly, (11) is a particular case of the criterion (6), (7) since the nonoscillation of

$$(r(t)\Phi(y)')'+c_+(t)\Phi(y)=0$$

implies by the Sturmian comparison theorem the nonoscillation of (1). The proof of this statement is based on the Wirtinger-type inequality

$$\int_T^\infty |M'(t)||y|^p\,\mathrm{d}t\leqslant p^p\int_T^\infty\frac{M^p(t)}{|M'(t)|^{p-1}}|y'|^p\,\mathrm{d}t, \tag{12}$$

where M is a differentiable function with $M'\neq 0$ on $[T,\infty)$, which holds for every $y\in W^{1,p}(T,\infty)$, $\operatorname{supp} y\subset(T,\infty)$. Note also that the nonoscillation criterion (11) is not actually new and was proved for the first time in [15] by using the Riccati technique.

As a first statement of this section we prove via the variational technique the following variant of Theorem 3.5 of [14] which is in [14] proved using the Riccati technique.

Theorem 3. *Suppose that $\int^\infty r^{1-q}(t)\,\mathrm{d}t<\infty$, denote $R(t)=\int_t^\infty r^{1-q}(s)\,\mathrm{d}s$ and suppose that*

$$\int^\infty c(t)R^p(t)\,\mathrm{d}t$$

converges. If

$$\liminf_{t\to\infty}\frac{1}{R(t)}\int_t^\infty c(s)R^p(s)\,\mathrm{d}s>1 \tag{13}$$

then (1) *is oscillatory. Moreover, if* $c(t) \geqslant 0$ *for large* t *then the statement remains valid if* $\liminf$ *in* (13) *is replaced by* $\limsup$.

P r o o f. Let $T \in \mathbb{R}$ be arbitrary and $T < t_0 < t_1 < t_2 < t_3$ (these points will be specified later). Here and also in the remaining part of the paper we do not sometimes write explicitly the integration variable in an integral if no ambiguity may arise, i.e. we write e.g. $\int_{t_0}^t r^{1-q}$ instead of $\int_{t_0}^t r^{1-q}(s)\,\mathrm{d}s$. Define a function $y \in W^{1,p}(T,\infty)$ as follows:

$$y(t) = \begin{cases} 0, & T \leqslant t \leqslant t_0, \\ f(t), & t_0 \leqslant t \leqslant t_1, \\ h(t), & t_1 \leqslant t \leqslant t_2, \\ g(t), & t_2 \leqslant t \leqslant t_3, \\ 0, & t_3 \leqslant t < \infty, \end{cases}$$

where $h(t) = \int_t^\infty r^{1-q}(s)\,\mathrm{d}s = R(t)$,

$$f(t) = \left(\int_{t_0}^t r^{1-q}(s)\,\mathrm{d}s\right)\frac{R(t_1)}{\int_{t_0}^{t_1} r^{1-q}}, \quad g(t) = \left(\int_t^{t_3} r^{1-q}(s)\,\mathrm{d}s\right)\frac{R(t_2)}{\int_{t_2}^{t_3} r^{1-q}}.$$

Then using the fact that f, g, h are solutions of the equation $(r\Phi(y'))' = 0$ satisfying $f(t_0) = 0$, $f(t_1) = h(t_1)$, $g(t_2) = h(t_2)$, $g(t_3) = 0$, we have

$$\begin{aligned} \int_{t_0}^{t_3} r(t)|y'|^p\,\mathrm{d}t &= r(t)f(t)\Phi(f'(t))\big|_{t_0}^{t_1} + r(t)h(t)\Phi(h'(t))\big|_{t_1}^{t_2} + r(t)g(t)\Phi(g'(t))\big|_{t_2}^{t_3} \\ &= R(t_1)\left[\frac{R^{p-1}(t_1)}{\left(\int_{t_0}^{t_1} r^{1-q}\right)^{p-1}} + 1\right] + R(t_2)\left[-1 + \frac{R^{p-1}(t_2)}{\left(\int_{t_2}^{t_3} r^{1-q}\right)^{p-1}}\right]. \end{aligned}$$

Since the functions f/h, g/h are monotone in (t_0,t_1) and (t_2,t_3), respectively (this can be verified directly or using the same argument as in [5]), by the second mean value theorem of the integral calculus there exist $\xi_1 \in (t_0,t_1)$, $\xi_2 \in (t_2,t_3)$ such that

$$\int_{t_0}^{t_1} c(t)f^p(t)\,\mathrm{d}t = \int_{t_0}^{t_1} c(t)h^p(t)\Big(\frac{f(t)}{h(t)}\Big)^p\,\mathrm{d}t = \int_{\xi_1}^{t_1} c(t)h^p(t)\,\mathrm{d}t$$

and

$$\int_{t_2}^{t_3} c(t)g^p(t)\,\mathrm{d}t = \int_{t_2}^{\xi_2} c(t)h^p(t)\,\mathrm{d}t.$$

Consequently, $\int_{t_0}^{t_3} c(t)y^p(t)\,\mathrm{d}t = \int_{\xi_1}^{\xi_2} c(t)h^p(t)\,\mathrm{d}t$. Combining the above given computations, we get

$$\begin{aligned}\mathcal{F}(y;t_0,t_3) &= R(t_1)\biggl\{\frac{R^{p-1}(t_1)}{\bigl(\int_{t_0}^{t_1} r^{1-q}\bigr)^{p-1}} + 1 + \frac{R(t_2)}{R(t_1)}\biggl[-1 + \frac{R^{p-1}(t_2)}{\bigl(\int_{t_2}^{t_3} r^{1-q}\bigr)^{p-1}}\biggr] \\ &\qquad - \frac{1}{R(t_1)}\int_{\xi_1}^{\xi_2} ch^p\,\mathrm{d}t\biggr\} \\ &\leqslant R(t_1)\biggl\{\frac{R^{p-1}(t_1)}{\bigl(\int_{t_0}^{t_1} r^{1-q}\bigr)^{p-1}} + 1 + \frac{R^{p-1}(t_2)}{\bigl(\int_{t_2}^{t_3} r^{1-q}\bigr)^{p-1}} + \frac{1}{R(\xi_1)}\int_{\xi_1}^{\xi_2} c(t)h^p(t)\,\mathrm{d}t\biggr\}\end{aligned}$$

since $\int_{\xi_1}^{\xi_2} c(t)h^p(t)\,\mathrm{d}t > 0$ according to (8) if ξ_1, ξ_2 are sufficiently large.

Now, let $\varepsilon > 0$ be such that

$$\liminf_{t\to\infty} \frac{1}{R(t)}\int_t^\infty c(s)h^p(s)\,\mathrm{d}s > 1 + 4\varepsilon.$$

This implies that t_0 can be chosen in such a way that

$$R^{-1}(\xi)\int_\xi^\infty c(t)h^p(t)\,\mathrm{d}t > 1 + 3\varepsilon \tag{14}$$

whenever $\xi > t_0$. The number $t_1 > t_0$ is taken such that

$$\biggl(\int_{t_1}^\infty r^{1-q}\biggr)^{p-1}\biggl(\int_{t_0}^{t_1} r^{1-q}\biggr)^{1-p} < \varepsilon.$$

The relation (14) implies that there exists $t_2 > t_1$ such that

$$R^{-1}(\xi)\int_\xi^t c(s)h^p(s)\,\mathrm{d}s > 1 + 2\varepsilon$$

whenever $t > t_2$. Finally, we fix $t_3 > t_2$ in such a way that

$$R^{p-1}(t_2)\biggl(\int_{t_2}^{t_3} r^{1-q}\biggr)^{1-p} < 1 + \varepsilon.$$

Summarizing all the estimates, we have

$$\mathcal{F}(y;t_0,t_3) \leqslant R(t_1)\{\varepsilon + 1 + \varepsilon - (1 + 2\varepsilon)\} \leqslant 0,$$

which means that (1) is oscillatory. If $c(t) \geqslant 0$ for large t, then

$$\begin{aligned}\int_{t_0}^{t_3} c(t)y^p(t)\,\mathrm{d}t &= \int_{t_0}^{t_1} c(t)f^p(t)\,\mathrm{d}t + \int_{t_1}^{t_2} c(t)h^p(t)\,\mathrm{d}t + \int_{t_2}^{t_3} c(t)g^p(t)\,\mathrm{d}t \\ &\geqslant \int_{t_1}^{t_2} c(t)h^p(t)\,\mathrm{d}t\end{aligned}$$

and it is easy to see that ξ_1, ξ_2 do not come into play in this case and hence the same proof as above can be realized if lim inf in (13) is replaced by lim sup. □

Using a slight modification of the method applied in the previous proof we have the following statement.

Theorem 4. *If* $\int^\infty r^{1-q}(t)\,\mathrm{d}t < \infty$, *then* (1) *is oscillatory provided*

$$\liminf_{t\to\infty}\Big(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int^t c(s)\,\mathrm{d}s\Big) > 1. \tag{15}$$

Moreover, if $c(t) \geqslant 0$ *then* lim inf *in* (15) *may be replaced by* lim sup.

P r o o f. Define the test function y by

$$y(t) = \begin{cases} 0, & T \leqslant t \leqslant t_0, \\ \int_{t_0}^t r^{1-q}\big(\int_{t_0}^{t_1} r^{1-q}\big)^{-1}, & t_0 \leqslant t \leqslant t_1, \\ 1, & t_1 \leqslant t \leqslant t_2, \\ \int_t^{t_3} r^{1-q}\big(\int_{t_2}^{t_3} r^{1-q}\big)^{-1}, & t_2 \leqslant t \leqslant t_3, \\ 0, & t_3 \leqslant t < \infty. \end{cases}$$

Then in the case $c(t) \geqslant 0$ for large t we have similarly to the previous proof

$$\begin{aligned}\mathcal{F}(y;t_0,t_3) &\leqslant \Big(\int_{t_0}^{t_1} r^{1-q}\Big)^{1-p} + \Big(\int_{t_2}^{t_3} r^{1-q}\Big)^{1-p} + \int_{t_1}^{t_2} c(t)\,\mathrm{d}t \\ = \Big(\int_{t_2}^{t_3} r^{1-q}\Big)^{1-p}&\Big\{\Big(\frac{\int_{t_1}^{t_2} r^{1-q}}{\int_{t_2}^{t_3} r^{1-q}}\Big)^{p-1} + 1 + \Big(\int_{t_2}^{t_3} r^{1-q}\Big)^{p-1}\Big[\int_T^{t_2} c(t)\,\mathrm{d}t - \int_T^{t_1} c(t)\,\mathrm{d}t\Big]\Big\}\end{aligned}$$

and taking $t_0 < t_1 < t_2 < t_3$ appropriately, we have $\mathcal{F}(y;t_0,t_3) < 0$ provided (15) holds with lim sup instead of lim inf. If the assumption $c(t) \geqslant 0$ for large t is not satisfied, we use the second mean value theorem of integral calculus in computing $\int_{t_0}^{t_1} c(t)f^p(t)\,\mathrm{d}t$ and $\int_{t_2}^{t_3} c(t)g^p(t)\,\mathrm{d}t$ and then we apply the same idea as in the proof of Theorem 3. □

Remark 1. (i) Comparing the Riccati and the variational method, a typical feature is that the application of the variational principle gives "worse" oscillation constant in the oscillation and nonoscillation criteria, but generally under less restrictive assumptions on the coefficient c in (1). For example, Kusano et al. [15] proved that if $\int^\infty r^{1-q}(s)\,\mathrm{d}s < \infty$ and $c(t) \geqslant 0$ eventually, then (1) is oscillatory provided

$$\liminf_{t\to\infty} \frac{1}{R(t)} \int_t^\infty c(s)R^p(s)\,\mathrm{d}s > \frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1} =: K_p.$$

In Theorem 3 we have a bigger constant than K_p (this constant equals 1), but under *no sign restriction on the function c*.

(ii) In our recent paper [7] we have proved that if $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$, $c(t) \geqslant 0$ for large t and there exists a function $\tilde c$ such that the equation

$$(r(t)\Phi(y'))' + \tilde c(t)\Phi(y) = 0 \tag{16}$$

possesses an eventually positive solution h satisfying $h'(t) > 0$ for large t,

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) = L < \infty$$

exists and

$$\int^\infty r(t)(h'(t))^p\,\mathrm{d}t = \infty,$$

then (1) is oscillatory provided

$$\liminf_{t\to\infty}\Big(\int^t \frac{\mathrm{d}s}{r(s)h^2(s)(h'(s))^{p-2}}\Big)\int_t^\infty (c(s)-\tilde c(s))h^p(s)\,\mathrm{d}s > \frac{1}{2q}. \tag{17}$$

In particular, if

$$\tilde c(t) = \frac{\gamma}{t^p}, \quad \gamma = \Big(\frac{p-1}{p}\Big)^{p-1}, \quad h(t) = t^{\frac{p-1}{p}}$$

then equation (1) with $r \equiv 1$ is oscillatory if

$$\liminf_{t\to\infty} \lg t \int_t^\infty \Big(c(s) - \frac{\gamma}{s^p}\Big)s^{p-1}\,\mathrm{d}s > \tilde K_p := \frac{1}{2}\Big(\frac{p-1}{p}\Big)^{p-1}. \tag{18}$$

This criterion was proved using the Riccati technique. In [4] it was proved, using the variational principle, that (1) with $r \equiv 1$ (and without any sign restriction on the function c) is oscillatory if (18) holds with the bigger constant $4\tilde K_p$. This is a partial confirmation of the statement from the beginning of this remark that the Riccati technique provides a better oscillation constant then the variational method.

Using the variational method we can also prove the following modification of the oscillation criterion (17). In this criterion, (1) is viewed as a perturbation of the nonoscillatory equation (16). First let us recall the concept of the principal solution of a nonoscillatory half-linear equation (1) introduced in [21] and rediscovered in the recent paper [5] (an alternative approach based on the generalized Prüfer transformation can be found in [11]). If (1) is nonoscillatory, then among all solutions of the associated Riccati equation (3) one can find a solution which is less than any other extensible up to infinity solution and this solution is called (by the analogue with linear case) a *distinguished* solution of (3). Now, having defined the distinguished solution $\overline{w}$ of (3), the *principal solution* of (1) is defined as the solution which determines this "Riccati" distinguished solution, i.e. as a solution of the first order equation

$$y' = r^{q-1}(t)|\overline{w}|^{q-1} \operatorname{sgn} \overline{w}\, y.$$

In the next theorem we use the following notation. Let h be the principal solution of (16) and let f, g be the solutions of this equation satisfying the boundary conditions $f(t_0) = 0$, $f(t_1) = h(t_1)$, $g(t_2) = h(t_2)$, $g(t_3) = 0$ (we suppose that $t_0 < t_1 < t_2 < t_3$ are sufficiently large so that the solutions f, g exist). Denote by $w(t, t_0)$, $w_h(t)$ and $w(t, t_3)$ the solutions of the Riccati equation associated with (16) corresponding to f, h, g, respectively, i.e.

$$w(\cdot, t_0) = \frac{r\Phi(f')}{\Phi(f)}, \quad w_h(\cdot) = \frac{r\Phi(h')}{\Phi(h)}, \quad w(\cdot, t_3) = \frac{r\Phi(g')}{\Phi(g)}. \tag{19}$$

Theorem 5. *Suppose that* (16) *is nonoscillatory and* $w(\cdot, t_0)$, w_h *are defined as above. If*

$$\liminf_{t_0\to\infty} \left\{ \liminf_{t\to\infty} \frac{1}{h^p(t)\,[w(t,t_0) - w_h(t)]} \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,\mathrm{d}s \right\} > 1 \tag{20}$$

then (1) *is oscillatory. Moreover, if* $c(t) \geqslant \tilde{c}(t)$ *for large* t, *then* lim inf *in braces of* (20) *can be replaced by* lim sup.

P r o o f. We proceed similarly as in the previous two theorems of this section. We sketch the proof in the case $c(t) \geqslant \tilde{c}(t)$ for large t. If this assumption is not satisfied, we use again the second mean value theorem of integral calculus.

Let $T \in \mathbb{R}$ be arbitrary and let $T < t_0 < t_1 < t_2 < t_3$ (these points will be again specified later). Define the test function y by

$$y(t) = \begin{cases} 0, & T \leqslant t \leqslant t_0, \\ f(t), & t_0 \leqslant t \leqslant t_1, \\ h(t), & t_1 \leqslant t \leqslant t_2, \\ g(t), & t_2 \leqslant t \leqslant t_3, \\ 0, & t_3 \leqslant t < \infty, \end{cases}$$

where the functions f, g, h are defined above. Then using the same computation as in the proofs of Theorems 3 and 4 we have

$$\begin{aligned} \mathcal{F}(y; t_0, t_3) &= \int_{t_0}^{t_3} [r(t)|y'|^p - c(t)|y|^p] \, \mathrm{d}t \\ &= h^p(t_1) \, [w(t_1, t_0) - w_h(t_1)] + h^p(t_2) \, [w_h(t_2) - w(t_2, t_3)] \\ &\quad + \int_{t_1}^{t_2} (c(s) - \tilde{c}(s)) \, h^p(s) \, \mathrm{d}s. \end{aligned}$$

Denote further

$$\tilde{G}(t, t_0) := h^p(t)[w(t, t_0) - w_h(t)], \quad H(t, t_3) = h^p(t)[w_h(t) - w(t, t_3)].$$

Then

$$\mathcal{F}(y; t_0, t_3) = \tilde{G}(t_1, t_0) \left\{ 1 + \frac{H(t_2, t_3)}{\tilde{G}(t_1, t_0)} + \frac{1}{\tilde{G}(t_1, t_0)} \int_{t_1}^{t_2} (c(s) - \tilde{c}(s)) \, h^p(s) \, \mathrm{d}s \right\}$$

and (20) together with the fact that h is the principal solution of (16) (compare [5, Theorem 1]) imply that $t_0 < t_1 < t_2 < t_3$ can be chosen in such a way that $\mathcal{F}(y; t_0, t_3) < 0$, which means that (1) is oscillatory. $\square$

Remark 2. (i) Let $p = 2$, i.e. (16) reduces to the usual Sturm-Liouville linear equation

$$(r(t)y')' + \tilde{c}(t)y = 0. \tag{21}$$

Then, if this equation is nonoscillatory, $t_0, \bar{t}_0$ are sufficiently large and $\tilde{G}$ is the same as in the previous proof (with $p = 2$), we have

$$\lim_{t \to \infty} \frac{\tilde{G}(t, t_0)}{\tilde{G}(t, \bar{t}_0)} = 1, \quad \tilde{G}(t, t_0) \sim \frac{y(t)}{h(t)} \text{ as } t \to \infty,$$

where y is any nonprincipal solution of (21). Consequently, in the linear case the "$\liminf\limits_{t_0 \to \infty}$" operation can be omitted in (20) and the linear version of Theorem 5 reads as follows:

Corollary 1. *Equation* (2) *is oscillatory provided*

$$\liminf_{t\to\infty} \frac{y(t)}{h(t)} \int_t^\infty (c(s)-\tilde{c}(s))h^2(s)\,\mathrm{d}s > 1,$$

where h, y are the principal and nonprincipal solutions of (21), *respectively.*

(ii) In the linear case the above ratio $\frac{y(t)}{h(t)}$ can be expressed in the form

$$\frac{y(t)}{h(t)} = \int^t \frac{1}{r(s)h^2(s)}\,\mathrm{d}s.$$

If we compare this expression with (17), we see that the Riccati technique and the variational method give the same factor

$$G(t) = \int^t \frac{1}{r(s)h^2(s)}\,\mathrm{d}s$$

in oscillation (and also nonoscillation) criteria. It is an open problem whether in the half-linear case we have the same situation. The only known (nontrivial) result along this line is the case when $r(t) \equiv 1$ and $\tilde{c}(t) = \frac{\gamma_0}{t^p}$ with $\gamma_0 = \bigl(\frac{p-1}{p}\bigr)^{p-1}$, i.e. (16) is the generalized Euler equation with the critical coefficient γ_0. Then both the Riccati technique and the variational method give the same factor $G(t) = \lg t$, see [4, 7].

4. Reciprocity principle

In this final short section we discuss the application of the reciprocity principle mentioned in the introductory section in the oscillation theory of half-linear equations. This section actually contains no new results (comparing with the parts devoted to Riccati technique and variational principle), but shows how the reciprocity principle can be used to "transfer" the oscillation/nonoscillation criteria for (1) assuming the divergence of the integral $\int^\infty r^{1-q}(t)\,\mathrm{d}t$ to the case when this integral converges. Throughout this section we suppose that $c(t) > 0$ for large t.

The statement that (1) is nonoscillatory if and only if the reciprocal equation

$$\bigl(c^{1-q}(t)\Phi_q(u')\bigr)' + r^{1-q}(t)\Phi_q(u) = 0 \tag{22}$$

is nonoscillatory is a simple consequence of the Rolle theorem of differential calculus. Indeed, if y is an oscillatory solution of (1) then its derivative and hence also $u = r\Phi_p(y')$ oscillates. Conversely, if u oscillates then $c^{1-q}(t)\Phi_q(u') = -y$ oscillates as well. This relationship between the oscillation of (1) and (22) can be viewed also from the following "Riccati point of view". Equation (1) is nonoscillatory if and only if there exists a solution w of Riccati-type equation (3) which is defined on some interval $[T,\infty)$. The fact that $r(t)>0$, $c(t)>0$ for large t implies that w is eventually monotone, i.e. eventually of one sign. Now, by a direct computation one can verify that the function $v=-\frac{1}{\Phi_q(w)}$ satisfies the equation

$$v'+r^{1-q}(t)+(q-1)c(t)|v|^p=0,$$

which is just the Riccati-type equation associated with (22), i.e. (1) is really nonoscillatory if and only if (22) has the same property.

Recall that if $\int^\infty r^{1-q}(t)\,\mathrm{d}t=\infty$ and $\int^\infty c(t)\,\mathrm{d}t<\infty$ then (1) is oscillatory if

$$\liminf_{t\to\infty}\Big(\int^t r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int_t^\infty c(s)\,\mathrm{d}s\Big)>K_p:=\frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1} \tag{23}$$

and it is nonoscillatory if

$$\limsup_{t\to\infty}\Big(\int^t r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int_t^\infty c(s)\,\mathrm{d}s\Big)<K_p, \tag{24}$$

see [14]. Now we will show how this statement can be reformulated via the reciprocity principle for the case when $\int^\infty r^{1-q}(t)\,\mathrm{d}t<\infty$.

Theorem 6. *Suppose that $\int^\infty r^{1-q}(t)\,\mathrm{d}t<\infty$. If*

$$\liminf_{t\to\infty}\Big(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int^t c(s)\,\mathrm{d}s\Big)>K_p \tag{25}$$

then (1) *is oscillatory and if*

$$\limsup_{t\to\infty}\Big(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s\Big)^{p-1}\Big(\int^t c(s)\,\mathrm{d}s\Big)<K_p \tag{26}$$

then this equation is nonoscillatory.

Proof. Let us apply (23) to the reciprocal equation (22). First observe that (25) implies that $\int^\infty c=\infty$. Taking into account that p, q are mutually conjugate numbers, i.e. $(p-1)(q-1)=1$, we have $\int^\infty c^{(1-q)(1-p)}=\int^\infty c=\infty$, hence by (23) equation (22) is oscillatory if

$$\liminf_{t\to\infty}\Big(\int^t c(s)\,\mathrm{d}s\Big)^{q-1}\Big(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s\Big)>K_q:=\frac{1}{q}\Big(\frac{q-1}{q}\Big)^{q-1}. \tag{27}$$

Now, taking the $(p-1)$-th power of both sides of the last inequality, we see that (27) is equivalent to

$$\liminf_{t\to\infty}\left(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s\right)^{p-1}\left(\int^t c(s)\,\mathrm{d}s\right) > K_q^{p-1} = K_p$$

which we needed to prove.

Concerning the proof of the "nonoscillatory" part of the theorem, first consider the case $\int^\infty c(s)\,\mathrm{d}s < \infty$. If this happens, then we use the transformation of independent variable

$$s = \int^t r^{1-q}(\tau)\,\mathrm{d}\tau, \quad x(s) = y(t) \tag{28}$$

which transforms (1) into the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi_p\left(\frac{\mathrm{d}}{\mathrm{d}s}x\right)\right) + r^{q-1}(t(s))c(t(s))\Phi_p(x) = 0, \tag{29}$$

where $t = t(s)$ is the inverse function of $s = s(t)$ given by (28). The convergence of $\int^\infty r^{1-q}(t)\,\mathrm{d}t$ implies that the new variable s runs through a bounded interval where (29) has no singularity, hence any solution of this equation has only a finite number of zeros in this interval, which means that (1) is nonoscillatory. If $\int^\infty c(t)\,\mathrm{d}t = \infty$ we proceed in the same way as in the first part of the proof and use (24) instead of (23). $\square$

At the end of this section let us discuss one open problem concerning the reciprocity principle and the principal solutions of half-linear equations. In the linear case $p = 2$ it is known that if $\int^\infty r^{-1}(t)\,\mathrm{d}t = \infty$ and equation (2) is nonoscillatory then the fact that y is the principal solution of this equation implies that $u = ry'$ is the principal solution of the reciprocal equation

$$\left(\frac{1}{c(t)}u'\right)' + \frac{1}{r(t)}u = 0;$$

for a more comprehensive treatment of this problem see [1, 2, 6].

We conjecture here that a similar statement holds also for half-linear equations, namely, if $\int^\infty r^{1-q}(s)\,\mathrm{d}s = \infty$ and y is the principal solution of (1) then $u = r\Phi_p(y')$ is the principal solution of the reciprocal equation (22).

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