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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 4, 673–680

Persistent URL: <http://dml.cz/dmlcz/127603>

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ON THE INDIVIDUAL ERGODIC THEOREM
IN D -POSETS OF FUZZY SETS

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(Received November 26, 1996)

Abstract. Calculus for observables in a space of functions from an abstract set to the unit interval is developed and then the individual ergodic theorem is proved.

Keywords: individual ergodic theorem, d -posets of fuzzy sets, state and observable

MSC 2000: 28E10, 58F11

1. INTRODUCTION

We will consider a measurable space (Ω, \mathcal{S}) , where $\Omega \in \mathcal{S}$ and \mathcal{F} is the family of all measurable functions $f: \Omega \rightarrow \langle 0, 1 \rangle$. In our quantum mechanics model ([5], [3]) there are two basic notions: state and observable.

A state is a mapping $m: \mathcal{F} \rightarrow \langle 0, 1 \rangle$ satisfying the following conditions:

- (i) $m(1_\Omega) = 1$.
- (ii) If $f, g, h \in \mathcal{F}$, $f = g + h$, then $m(f) = m(g) + m(h)$.
- (iii) If $f_n \in \mathcal{F}$ ($n = 1, 2, \dots$), $f_n \nearrow f$, $f \in \mathcal{F}$, then $m(f_n) \nearrow m(f)$.

Of course, by a theorem of Butnariu and Klement ([1]) there is a probability measure $\mu: \mathcal{S} \rightarrow \langle 0, 1 \rangle$ such that

$$m(f) = \int_{\Omega} f \, d\mu$$

for all $f \in \mathcal{F}$, hence our model coincides with the classical one. On the other hand, the notion of an observable gives a new point of view, new possibilities and new problems.

Supported by grant VEGA 2/6087/99.

An observable is a mapping $x: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}$ (where $\mathcal{B}(\mathbb{R})$ is the σ -algebra of all Borel subsets of \mathbb{R}) satisfying the following conditions:

- (i) $x(\mathbb{R}) = 1_\Omega$.
- (ii) If $A, B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.
- (iii) If $A_n \in \mathcal{B}(\mathbb{R})$, $A_n \subset A_{n+1}$ ($n = 1, 2, \dots$), then

$$x\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} x(A_n).$$

As an example of an observable a random variable on a probability space $(\Omega, \mathcal{S}, \mu)$ can be considered. If $\xi: \Omega \rightarrow \mathbb{R}$ is a random variable, then $x: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}$, defined by the formula $x(A) = \chi_{\xi^{-1}(A)}$, is an observable.

Let $m: \mathcal{F} \rightarrow \langle 0, 1 \rangle$ be a state, $x: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}$ an observable. Then the composite mapping $m_x = m \circ x: \mathcal{B}(\mathbb{R}) \rightarrow \langle 0, 1 \rangle$ is a probability measure. This notion corresponds to the notion of the probability distribution $\mu_\xi: \mathcal{B}(\mathbb{R}) \rightarrow \langle 0, 1 \rangle$ of a random variable $\xi: (\Omega, \mathcal{S}, \mu) \rightarrow \mathbb{R}$. Indeed, μ_ξ is defined by the formula

$$\mu_\xi(A) = \mu(\xi^{-1}(A)).$$

On the other hand (in this classical case)

$$\begin{aligned} m_x(A) &= m(x(A)) = m(\chi_{\xi^{-1}(A)}) \\ &= \int_{\Omega} \chi_{\xi^{-1}(A)} d\mu = \mu(\xi^{-1}(A)). \end{aligned}$$

By help of the probability distribution m_x the mean value $E(x)$ can be defined. Namely, in the case of a random variable $\xi: \Omega \rightarrow \mathbb{R}$ the mean value $E(\xi)$ is defined as the integral $\int_{\Omega} \xi d\mu$. But by the integral transformation formula we have

$$E(\xi) = \int_{\Omega} \xi d\mu = \int_{\mathbb{R}} t d\mu_\xi(t).$$

Therefore we define

$$E(x) = \int_{\mathbb{R}} t dm_x(t),$$

if the integral exists. In this case we say that the observable x is integrable.

There are some results concerning the probability theory for observables and states in the fuzzy quantum model (e.g., the strong law of large numbers in [4]). In this paper the individual ergodic theorem will be formulated and proved.

2. FORMULATION

First we recall the classical ergodic theorem ([9], Th. 1.5). Let (X, σ, P) be a probability space, $T: X \rightarrow X$ a measure preserving transformation (i.e., $A \in \sigma \Rightarrow T^{-1}(A) \in \sigma$ and $P(T^{-1}(A)) = P(A)$), let $\xi: X \rightarrow \mathbb{R}$ be an integrable observable. Then there is an integrable observable ξ^* such that the following conditions are satisfied:

- (i) $E(\xi) = E(\xi^*)$.
- (ii) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi \circ T^i = \xi^*$ P -almost everywhere.

We have defined the mean value of an observable. We must define a state preserving transformation, operations with observables and the m -almost everywhere convergence of a sequence of observables.

A mapping $\tau: \mathcal{F} \rightarrow \mathcal{F}$ is called an m -preserving transformation if the following conditions are satisfied:

- (i) $\tau(1_\Omega) = 1_\Omega$.
- (ii) If $f, g, h \in \mathcal{F}$, $f = g + h$, then $\tau(f) = \tau(g) + \tau(h)$.
- (iii) If $f_n \in \mathcal{F}$ ($n = 1, 2, \dots$), $f \in \mathcal{F}$, $f_n \nearrow f$, then $\tau(f_n) \nearrow \tau(f)$.
- (iv) $\tau(f) \cdot \tau(g) = \tau(f \cdot g)$ and $m(\tau(f)) = m(f)$ for all $f, g \in \mathcal{F}$.

By a theorem of [7] for every observables x_1, \dots, x_n there exists a mapping $h_n: \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{F}$ satisfying the following conditions:

- (i) $h_n(\mathbb{R}^n) = 1_\Omega$.
- (ii) If $A, B \in \mathcal{B}(\mathbb{R}^n)$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.
- (iii) If $A_i \in \mathcal{B}(\mathbb{R}^n)$, $A_i \subset A_{i+1}$ ($i = 1, 2, \dots$), then

$$h_n\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} h_n(A_i).$$

- (iv) $h_n(A_1 \times \dots \times A_n) = x_1(A_1) \cdot \dots \cdot x_n(A_n)$ for every

$$A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}).$$

The function h_n is called the joint observable of observables x_1, \dots, x_n . By the help of the joint observable h_n some operations can be defined. If $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is a Borel measurable function, then we define a mapping $g(x_1, \dots, x_n): \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}$ by the formula

$$g(x_1, \dots, x_n)(A) = h_n(g^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}).$$

The motivation is the following. If $(\xi_1, \dots, \xi_n) = U$ is a random vector then $g(\xi_1, \dots, \xi_n) = g \circ U$ is a random variable and

$$(g \circ U)^{-1}(A) = U^{-1}(g^{-1}(A)).$$

In the general situation $U^{-1}: \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{S}$ induces the joint distribution $h_n(A) = \chi_{U^{-1}(A)}$.

Finally, we shall define the m -almost everywhere convergence of a sequence of observables. If $(y_n)_n$ is a sequence of observables, we say that $\limsup_{n \rightarrow \infty} y_n$ exists, if there exists an observable \bar{y} such that

$$\begin{aligned} m(\bar{y}((-\infty, t))) &= \lim_{p \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{i \rightarrow \infty} m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \\ &= m\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \end{aligned}$$

for every $t \in \mathbb{R}$.

We say that $\liminf_{n \rightarrow \infty} y_n$ exists, if there exists an observable \underline{y} such that

$$\begin{aligned} m(\underline{y}((-\infty, t))) &= \lim_{p \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{i \rightarrow \infty} m\left(\bigvee_{n=k}^{k+i} y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \\ &= m\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \end{aligned}$$

for every $t \in \mathbb{R}$.

We say that a sequence $(y_n)_n$ of observables converges m -almost everywhere to an observable y , if $\limsup_{n \rightarrow \infty} y_n = \bar{y}$ and $\liminf_{n \rightarrow \infty} y_n = \underline{y}$ exist and

$$m(\underline{y}(-\infty, t)) = m(\bar{y}(-\infty, t)) = m(y(-\infty, t))$$

for every $t \in \mathbb{R}$.

The main result of the paper is contained in the following theorem.

Theorem 1. *Let x be an integrable observable, τ an m -preserving transformation. Then there is an integrable observable x^* satisfying the following conditions:*

- (i) $E(x^*) = E(x)$.
- (ii) $\frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x \rightarrow x^*$ m -a.e.

3. PROOF

The main idea of the proof consists in the construction of a probability space and an application of the classical individual ergodic theorem.

Let $x_n (n = 1, 2, \dots)$ be the observable defined by the formula $x_n = \tau^{n-1} \circ x$. Let \mathbb{N} be the set of all positive integers, $\emptyset \neq J \subset \mathbb{N}$, J finite, $J = \{j_1, \dots, j_k\}$. Then we define a probability measure $P_J: \mathcal{B}(\mathbb{R}^{|J|}) \rightarrow \langle 0, 1 \rangle$ determined by the formula

$$P_J(A_1 \times \dots \times A_k) = m(x_{j_1}(A_1) \cdot \dots \cdot x_{j_k}(A_k)),$$

$A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. It is not difficult to prove that the family

$$\{P_J; J \subset \mathbb{N}, J \neq \emptyset, J \text{ finite}\}$$

is a consistent system of probability measures. That is, if $J_1 \subset J_2$, $J_1 \neq \emptyset$, J_2 is finite and $\pi_{J_2, J_1}: \mathbb{R}^{|J_2|} \rightarrow \mathbb{R}^{|J_1|}$ is the projection, then

$$P_{J_1}(A) = P_{J_2}(\pi_{J_2, J_1}^{-1}(A))$$

for every $A \in \mathcal{B}(\mathbb{R}^{|J_1|})$. Therefore the Kolmogorov theorem is applicable. Denote by \mathcal{V} the family of all cylinders $B \subset \mathbb{R}^{\mathbb{N}}$, i.e. the sets of the form

$$\pi_J^{-1}(A) = \{(x_n)_n; (x_{j_1}, \dots, x_{j_k}) \in A\},$$

where $A \in \mathcal{B}(\mathbb{R}^{|J|})$, $J \neq \emptyset$, $J \subset \mathbb{N}$, J finite and $\pi_J: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{|J|}$ is the projection. If $\sigma(\mathcal{V})$ is the σ -algebra generated by \mathcal{V} , then there exists exactly one probability measure P such that

$$P(\pi_J^{-1}(A)) = P_J(A)$$

for every $\pi_J^{-1}(A) \in \mathcal{V}$.

Proposition 1. *Let $T: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$ be the transformation defined by the formula $T((t_n)_n) = (s_n)_n$, where $s_n = t_{n+1}$ ($n = 1, 2, \dots$). Then T preserves the probability measure P , i.e. $P(A) = P(T^{-1}(A))$ for every $A \in \sigma(\mathcal{V})$.*

Proof. It is sufficient to prove the equality $P(A) = P(T^{-1}(A))$ for sets of the form $A = \pi_J^{-1}(B)$, where B is the product of $k = |J|$ sets of $\mathcal{B}(\mathbb{R})$. Let

$J = \{i_1, \dots, i_k\}$, $B = B_1 \times \dots \times B_k$, $J_1 = \{i_1 + 1, \dots, i_k + 1\}$. Then

$$\begin{aligned}
 P(T^{-1}(A)) &= P(T^{-1}(\pi_J^{-1}(B_1 \times \dots \times B_k))) \\
 &= P(T^{-1}(\{(s_n)_n; s_{i_1} \in B_1, \dots, s_{i_k} \in B_k\})) \\
 &= P(\{(t_n)_n; t_{i_1+1} \in B_1, \dots, t_{i_k+1} \in B_k\}) \\
 &= P_{J_1}(\{(t_{i_1+1}, \dots, t_{i_k+1}); t_{i_1+1} \in B_1, \dots, t_{i_k+1} \in B_k\}) \\
 &= m(x_{i_1+1}(B_1) \cdot \dots \cdot x_{i_k+1}(B_k)) \\
 &= m(\tau^{i_1}(x(B_1)) \cdot \dots \cdot \tau^{i_k}(x(B_k))) \\
 &= m(\tau^{i_1-1}(x(B_1)) \cdot \dots \cdot \tau^{i_k-1}(x(B_k))) \\
 &= m(x_{i_1}(B_1) \cdot \dots \cdot x_{i_k}(B_k)) \\
 &= P_J(B_1 \times \dots \times B_k) = P_J(B) \\
 &= P(\pi_J^{-1}(B)) = P(A).
 \end{aligned}$$

□

Proposition 2. Let $\xi_n: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ be the projection defined by the formula $\xi_n((t_i)_i) = t_n$. Then for every $n \in \mathbb{N}$, ξ_n is a random variable, $P(\xi_n^{-1}(A)) = m(x_n(A))$, $A \in \mathcal{B}(\mathbb{R})$ and $P\left(\left\{(u_i)_i; \frac{1}{n} \sum_{i=1}^n \xi_i(u) < t\right\}\right) = m\left(\left(\frac{1}{n} \sum_{i=1}^n x_i\right)((-\infty, t)\right)$, $t \in \mathbb{R}$.

Proof. If $A \in \mathcal{B}(\mathbb{R})$, then $\xi_n^{-1}(A) = \{(t_i)_i; t_n \in A\} = \pi_{\{n\}}^{-1}(A) \in \mathcal{V} \subset \sigma(\mathcal{V})$, hence $\xi_n: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ is a random variable. Moreover,

$$\begin{aligned}
 P(\xi_n^{-1}(A)) &= P(\{(t_i)_i; t_n \in A\}) = P(\pi_{\{n\}}^{-1}(A)) \\
 &= P_{\{n\}}(A) = m(x_n(A)).
 \end{aligned}$$

Finally, let $J_n = \{1, \dots, n\}$. Then

$$\begin{aligned}
 P\left(\left\{(u_i)_i; \frac{1}{n} \sum_{i=1}^n \xi_i(u) < t\right\}\right) &= P_{J_n}\left(\left\{(u_1, \dots, u_n); \frac{1}{n} \sum_{i=1}^n u_i < t\right\}\right) \\
 &= m \circ h_n\left(\left\{(u_1, \dots, u_n); \frac{1}{n} \sum_{i=1}^n u_i < t\right\}\right).
 \end{aligned}$$

If we put $g(u_1, \dots, u_n) = \frac{1}{n} \sum_{i=1}^n u_i$, then by definition

$$\frac{1}{n} \sum_{i=1}^n x_i = g(x_1, \dots, x_n) = h_n \circ g^{-1},$$

hence

$$\begin{aligned} P\left(\left\{(u_i)_i; \frac{1}{n} \sum_{i=1}^n \xi_i(u) < t\right\}\right) &= P(g^{-1}((-\infty, t))) \\ &= m(h_n(g^{-1}((-\infty, t)))) = m\left(\left(\frac{1}{n} \sum_{i=1}^n x_i\right)((-\infty, t))\right). \end{aligned}$$

□

Proposition 3. *There exists an observable x^* such that $\frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x \rightarrow x^*$ m -a.e.*

P r o o f. We have proved that $\xi_1: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ is a random variable (with respect to $\sigma(\mathcal{Y})$) and ξ_1 and x have the same probability distribution defined by the formula

$$P_{\xi_1}(A) = P(\xi_1^{-1}(A)) = m(x_1(A)) = m(x(A)) = m_x(A).$$

Since x is integrable (i.e. $\int_{\mathbb{R}} t dm_x(t)$ exists), ξ_1 is integrable, too. Therefore, by the individual ergodic theorem, there exists an integrable, invariant random variable ξ^* such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \xi_1 \circ T^i \rightarrow \xi^* \quad P\text{-a.e.}$$

Of course, $\xi_1 \circ T^i = \xi_{i+1}$, hence

$$(*) \quad \frac{1}{n} = \sum_{j=1}^n \xi_j \rightarrow \xi^* \quad P\text{-a.e.}$$

Theorem 3 of [8] states that P -a.e. convergence of the sequence $(g_n(\xi_1, \dots, \xi_n))_n$ implies m -a.e. convergence of the sequence $(g_n(x_1, \dots, x_n))_n$ to an observable x^* and

$$\begin{aligned} P\left(\left\{u \in \mathbb{R}^{\mathbb{N}}; \limsup_{n \rightarrow \infty} g_n(\xi_1(u), \dots, \xi_n(u)) < t\right\}\right) \\ &= m\left(\limsup_{n \rightarrow \infty} g_n(x_1, \dots, x_n)(-\infty, t)\right) \\ &= m\left(\liminf_{n \rightarrow \infty} g_n(x_1, \dots, x_n)(-\infty, t)\right) = m(x^*((-\infty, t))) \end{aligned}$$

for every $t \in \mathbb{R}$. Put

$$g_n(u_1, \dots, u_n) = \frac{1}{n} \sum_{i=1}^n u_i.$$

Then by (*) and Proposition 2 we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x = \frac{1}{n} \sum_{i=1}^n x_i \rightarrow x^* \quad m\text{-a.e.}$$

□

Proposition 4. *Let x^* be the observable introduced in Proposition 3. Then x^* is integrable and $E(x^*) = E(x)$.*

Proof. Since ξ_1 is an integrable random variable, by the individual ergodic theorem ξ^* is integrable, too and

$$E(\xi_1) = E(\xi^*),$$

$$P(\{u; \xi^*(u) < t\}) = P\left(\left\{u; \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_1 \circ T^i(u) < t\right\}\right)$$

for every $t \in \mathbb{R}$. By Theorem 3 of [8]

$$\begin{aligned} m_{x^*}((-\infty, t)) &= m(x^*((-\infty, t))) \\ &= P(\{u; \xi^*(u) < t\}) = P_{\xi^*}((-\infty, t)). \end{aligned}$$

Therefore $m_{x^*} = P_{\xi^*}$. Since also $m_x = P_{\xi_1}$, we have

$$\begin{aligned} E(x) &= \int_{\mathbb{R}} t \, dm_x(t) = \int_T t \, dP_{\xi_1} = E(\xi_1) = E(\xi^*) \\ &= \int_{\mathbb{R}} t \, dP_{\xi^*}(t) = \int_{\mathbb{R}} t \, dm_{x^*}(t) = E(x^*). \end{aligned}$$

□

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