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DR-IRREDUCIBILITY OF CONNECTED MONOUNARY ALGEBRAS

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This paper is a continuation of [6], where irreducibility in the sense of Duffus and Rival (DR-irreducibility) of monounary algebras was defined. The definition is analogous to that introduced by Duffus and Rival [1] for the case of posets. In [6] we found all connected monounary algebras A possessing a cycle and such that A is DR-irreducible.

The main result of the present paper is Thm. 4.1 which describes all connected monounary algebras A without a cycle and such that A is DR-irreducible.

Other types of irreducibility of monounary algebras defined by means of the notion of a retract were studied in [2]–[5].

0. Preliminaries

Let $A = (A, f)$ be a monounary algebra. A nonempty subset M of A is said to be a retract of A if there is a mapping h of A onto M such that h is an endomorphism of A and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M. Further, we denote by $R(A)$ the system of all monounary algebras B such that B is isomorphic to (M, f) for some retract M of A.

A monounary algebra A is said to be irreducible in the sense of Duffus and Rival (DR-irreducible), if, whenever $A \in R\Big(\prod\limits_{i\in I} B_i\Big)$ and $B_i \in R(A)$ for each $i \in I$, then there is $j \in I$ such that $A \in R(B_j)$.

We will use the notion of the degree of an element $x \in B$, where (B, f) is a monounary algebra; for this notion cf. e.g. [8], [7] and [2]. The degree of x is an ordinal or the symbol ∞ and is denoted by $s_f(x)$.

The following theorem proved in [2] is essentially applied in several proofs below:

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(Thm) Let (A,f) be a monounary algebra and let (M,f) be a subalgebra of (A,f). Then M is a retract of (A,f) if and only if the following conditions are satisfied:

(a) If $y \in f^{-1}(M)$, then there is $z \in M$ such that $f(y)=f(z)$ and $s_f(y) \leqslant s_f(z)$.

(b) For any connected component K of (A,f) with $K \cap M = \emptyset$, the following conditions are satisfied.

(b1) If K contains a cycle with d elements, then there is a connected component K' of (A,f) with $K' \cap M \neq \emptyset$ and there is $n \in \mathbb{N}$ such that $n|d$ and K' has a cycle with n elements.

(b2) If K contains no cycle and x_0 is a fixed element of K, then there is $y_0 \in M$ such that $s_f(f^k(x_0)) \leqslant s_f(f^k(y_0))$ for each $k \in \mathbb{N} \cup \{0\}$.

1. Some DR-irreducible algebras

1.1. Notation. Let $\mathbb{N} = (\mathbb{N}, f)$ be a monounary algebra such that $f(n) = n+1$ for each $n \in \mathbb{N}$ and let $\mathbb{Z} = (\mathbb{Z}, f)$ be a monounary algebra such that $f(n) = n+1$ for each $n \in \mathbb{Z}$.

1.2. Lemma. *The algebras* $\mathbb{N}$ *and* $\mathbb{Z}$ *are* DR-*irreducible.*

Proof. The assertion follows from the fact that $\mathbb{N}$ and $\mathbb{Z}$ have no nontrivial retracts. □

1.3. Notation. For $n \in \mathbb{N}$ let $n' = (n,1)$. Further, denote $\mathbb{N}' = \{n' : n \in \mathbb{N}\}$, $E = \mathbb{Z} \cup \mathbb{N}'$. For $k \in \mathbb{Z}$ put $f(k) = k+1$ and for $n \in \mathbb{N}$ let

$$f(n') = \begin{cases} (n-1)' & \text{if } n > 1, \\ 0 & \text{if } n = 1. \end{cases}$$

Then $E = (E,f)$ is a connected monounary algebra and $s_f(x) = \infty$ for each $x \in E$.

1.4. Notation. For $k \in \mathbb{N}$ put $k' = (k,1)$ and $k'' = (k,2)$. Let $n \in \mathbb{N}$. Denote $E'_n = \{1', 2', \ldots, n'\}$, $E''_n = \{1'', 2'', \ldots, n''\}$, $E_n = E'_n \cup E''_n \cup \mathbb{N}$. Further, define a unary operation f on E_n as follows: $f(1') = f(1'') = 1$, $f(2') = 1', \ldots, f(n') = (n-1)'$, $f(2'') = 1'', \ldots, f(n'') = (n-1)''$ and $f(j) = j+1$ for each $j \in \mathbb{N}$.

1.5. Lemma. (a) *The algebra* E *is* DR-*irreducible.*

(b) *If* $n \in \mathbb{N}$, *then the algebra* E_n *is* DR-*irreducible.*

Proof. Let $A = E$ or $A = E_n$ for some $n \in \mathbb{N}$ and suppose that A is DR-reducible. Then there exist monounary algebras $B_i \in R(A)$ for $i \in I$ such that

(1) $A \in R\Big(\prod_{i \in I} B_i\Big)$,

(2) $A \notin R(B_i)$ for each $i \in I$.

The relation (2) implies that if $i \in I$, then $A \not\cong B_i$, and since $B_i \in R(A)$, we get

if $x \in B_i$, then card $f^{-1}(x) \leqslant 1$.

This implies

if $b \in \prod_{i \in I} B_i$, then card $f^{-1}(b) \leqslant 1$.

Hence A is not isomorphic to any subalgebra of $\prod_{i \in I} B_i$, which is a contradiction to (1). □

1.6. Notation. Let $k \in \mathbb{N}$, $m_1, \ldots, m_k, p_1, \ldots, p_k \in \mathbb{N}$ and $m_1 < p_1 < m_2 < p_2 < \ldots < m_k < p_k$. If $i \in \{1, \ldots, k\}$, let

$$Y_i = \{(i, j) \colon j \in \{0, \ldots, m_i - 1\}\}.$$

The symbol $Y(m_1, p_1; m_2, p_2; \ldots; m_k, p_k)$ will denote the monounary algebra defined on the set

$$\mathbb{N} \cup \bigcup_{i \in \{1, \ldots, k\}} Y_i$$

such that if $n \in \mathbb{N}$, $i \in \{1, \ldots, j\}$, then

$$f(n) = n + 1,$$
$$f((i,j)) = \begin{cases} (i, j+1) & \text{if } j \in \{0, \ldots, m_i - 2\}, \\ (i+1, p_i) & \text{if } i \neq k, j = m_i - 1, \\ p_k & \text{if } i = k, j = m_i - 1. \end{cases}$$

(For the case $Y(2, 4; 6, 8)$ cf. Fig. 1.)

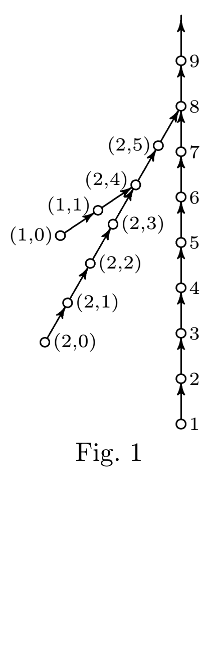


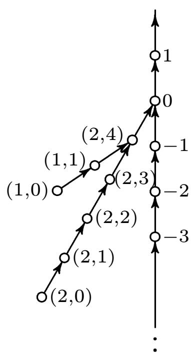
Fig. 1

1.7. Notation. Let $k \in \mathbb{N}$, $m_1, \ldots, m_k, p_1, \ldots, p_{k-1} \in \mathbb{N}$ and $m_1 < p_1 < m_2 < p_2 < \ldots < p_{k-1} < m_k$. If $i \in \{1, \ldots, k\}$, let Y_i be as in 1.6. The symbol $Y(m_1, p_1; m_2, p_2; \ldots; m_k)$ will denote the monounary algebra defined on the set

$$\mathbb{Z} \cup \bigcup_{i \in \{1, \ldots, k\}} Y_i$$

such that if $n \in \mathbb{Z}$, $i \in \{1, \ldots, k\}$, then

$$f(n) = n + 1,$$
$$f((i,j)) = \begin{cases} (i, j+1) & \text{if } j \in \{0, \ldots, m_i - 2\}, \\ (i+1, p_i) & \text{if } i \neq k, j = m_i - 1, \\ 0 & \text{if } i = k, j = m_i - 1. \end{cases}$$



$Y(2,4,5)$

Fig. 2

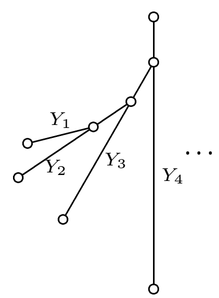


Fig. 3

1.8. Notation. Let $m_1 < p_1 < m_2 < p_2 < \ldots < m_i < p_i < \ldots$ be positive integers. For $i \in \mathbb{N}$ let Y_i be as in 1.6. The symbol $Y(m_1, p_1; m_2, p_2; \ldots)$ will denote the monounary algebra defined on the set

$$\bigcup_{i \in \mathbb{N}} Y_i$$

such that

$$f((i,j)) = \begin{cases} (i, j+1) & \text{if } j \in \{0, \ldots, m_i - 2\} \\ (i+1, p_i) & \text{if } j = m_i - 1. \end{cases}$$

1.9. Definition. We will say that A is of type $(\alpha 1)$ $((\alpha 2)$, $(\alpha 3)$, respectively), if A is isomorphic to some algebra defined in 1.6 (1.7, 1.8). If A is of a type of $(\alpha 1)$, $(\alpha 2)$, $(\alpha 3)$, then A is said to be of type (α).

1.10. Lemma. *Let A be one of the algebras defined in* 1.6–1.8. *If M is a retract of A and $(1,0) \in M$, then $M = A$.*

P r o o f. Let the assumption hold and suppose that M is a retract of A such that $(1,0) \in M$. Further, let φ be the corresponding retraction endomorphism. Then

(1) $\varphi((1,0)) = (1,0)$.

Since φ is a homomorphism, the relation (1) implies

$\varphi(x) = x$ for each $x \in A$.

Therefore $M = A$. □

1.11. Lemma. *If A is of type (α), then A is* DR*-irreducible.*

P r o o f. Let A be of type (α) and suppose that A is DR-reducible. Without loss of generality, A is one of the algebras defined in 1.6–1.8. Then there exist monounary algebras $B_i \in R(A)$ for $i \in I$ such that

(1) $A \in R\Big(\prod\limits_{i \in I} B_i\Big)$,

(2) $A \notin R(B_i)$ for each $i \in I$.

Hence there is a retract T of $\prod\limits_{i \in I} B_i$ such that

(3) $T \cong A$.

Let t be the element of T corresponding to the element $(1,0)$ (in the isomorphism (3)). In A the relation

(4) $f^{-(m_1+1)}(f^{m_1}((1,0))) \neq \emptyset$

is valid, thus (3) yields

(4′) $f^{-(m_1+1)}(f^{m_1}(t)) \neq \emptyset$.

We have $f^{-1}((1,0)) = \emptyset$, hence there is $i \in I$ with $f^{-1}(t(i)) = \emptyset$. Without loss of generality we can suppose that B_i is a subalgebra of A. The relation $f^{-1}(t(i)) = \emptyset$ implies

$$t(i) = (l,0)$$

for some $l \in \mathbb{N}$. If $l = 1$, then 1.10 yields that $B_i = A$, a contradiction to (2). Thus $l > 1$. In A, hence also in B_i, we have

$$f^{-(m_1+1)}(f^{m_1}((l,0))) = \emptyset,$$

i.e.,

$$f^{-(m_1+1)}(f^{m_1}(t(i))) = \emptyset,$$

which implies

$f^{-(m_1+1)}(f^{m_1}(t)) = \emptyset$,

a contradiction to (4′). □

2. Infinite degrees

In this section we suppose that A is a connected monounary algebra such that $A \ncong \mathbb{Z}$, $A \ncong E$, A possesses no cycle and $s_f(x) = \infty$ for each $x \in A$.

2.1. Construction. Let $\lambda = \operatorname{card} A$. Further, let I_j for $j \in \mathbb{Z}$ be disjoint sets of indices such that $\operatorname{card} I_j = \lambda$ for each $j \in \mathbb{Z}$ and $I = \bigcup_{j \in \mathbb{Z}} I_j$. For $i \in I$ put $B_i = E$,

$$B = \prod_{i \in I} B_i.$$

Denote by K the connected component of B such that K contains the element $q \in B$ with $q(i) = k$ for each $i \in I_k$, $k \in \mathbb{Z}$.

2.2. Lemma. (a) *$s_f(x) = \infty$ for each $x \in B$.*
(b) $\operatorname{card} f^{-1}(x) \geqslant \lambda$ *for each $x \in K$.*

P r o o f. (a) If $x \in B$, $i \in I$, then $s_f(x(i)) = \infty$ by 2.1. Then $s_f(x) = \infty$ as well.

(b) Let $x \in K$. Then x and q belong to the same connected component, thus there are $m, n \in \mathbb{N}$ such that $f^m(x) = f^n(q)$. Let $i \in I_{m-n}$. We obtain

$$f^m(x(i)) = f^n(q(i)) = f^n(m-n) = m - n + n = m,$$

i.e.,

$$x(i) \in f^{-m}(m) = \{0\},$$

thus

(1) $f^{-1}(x(i)) = \{-1, 1'\}$ for each $i \in I_{m-n}$.

Further, we have

(2) $f^{-1}(x(j)) \neq \emptyset$ for each $j \in I$.

The relation $\operatorname{card} I_{m-n} = \lambda$ together with (1) and (2) then yields

$$\operatorname{card} f^{-1}(x) \geqslant 2^\lambda,$$

therefore (b) is valid. □

2.3. Lemma. *A is* DR-*reducible.*

P r o o f. Let B and K be as in 2.1. According to 2.2(b), there is a subalgebra T of K with

(1) $A \cong T$.

Then $s_f(x) = \infty$ for each $x \in T$. According to (Thm), this and the fact that no connected component of B contains a cycle imply that T is a retract of B, thus

(2) $A \in R(B)$.

Further, $A \ncong E$ and $A \ncong \mathbb{Z}$, thus A is not isomorphic to any retract of B_i (for $i \in I$), hence

(3) $A \notin R(B_i)$ for each $i \in I$.

Obviously,

(4) $B_i \in R(A)$ for each $i \in I$.

Hence (1)–(4) yield that A is DR-reducible. □

3. Auxiliary results

Suppose that A is a connected monounary algebra possessing no cycle, $A \ncong \mathbb{N}$ and that there is $c \in A$ with $s_f(c) \neq \infty$.

Then the set

$$S_0 = \{x \in A \colon\ f^{-1}(x) = \emptyset\}$$

is nonempty. For $x \in S_0$ there exists the least positive integer $n_1(x)$ such that

$$\operatorname{card} f^{-1}(f^{n_1(x)}(x)) > 1 \text{ and } \operatorname{card} f^{-n_1(x)}(f^{n_1(x)}(x)) > 1.$$

For $x \in S_0$ we denote

$$P(x) = \bigcup_{m \in \mathbb{N} \cup \{0\}} f^{-m}(f^{n_1(x)-1}(x)).$$

Obviously, if $y \in P(x)$, then $f^{-n_1(x)}(y) = \emptyset$.

Let $n \in \mathbb{N}$. Put

$$\begin{aligned} J^{(n)} &= \{x \in S_0 \colon\ n_1(x) = n\}, \\ V^{(n)} &= \{f^n(x) \colon\ x \in J^{(n)}\}. \end{aligned}$$

For each $v \in V^{(n)}$ with the property

$$f^{-n}(v) \subseteq J^{(n)}$$

we choose a fixed element of the set $f^{-n}(v)$ and denote it by v'. Then we define

$$\begin{aligned} I^{(n)} = \{x \in J^{(n)} \colon\ & f^{-n}(f^n(x)) \nsubseteq J^{(n)}\} \cup \\ \cup \{x \in J^{(n)} \colon\ & f^{-n}(f^n(x)) \subseteq J^{(n)}, x \neq (f^n(x))'\}. \end{aligned}$$

If $x \in I^{(n)}$, then there exists an endomorphism φ_x of A such that $\varphi_x(y) = y$ for each $y \in A - P(x)$ and if $y \in P(x)$, then $\varphi_x(y) \in A - \bigcup_{z \in I^{(n)}} P(z)$.

3.1. Lemma. *Suppose that there is $n \in \mathbb{N}$ such that card $I^{(n)} \geqslant 2$. Then A is* DR-*reducible.*

P r o o f. We shall now write I instead of $I^{(n)}$. Denote

$$\begin{aligned} A_0 &= A - \bigcup_{i \in I} P(i), \\ B_i &= A_0 \cup P(i) \text{ for each } i \in I, \\ B &= \prod_{i \in I} B_i. \end{aligned}$$

The definition of B_i implies

(1) $B_i \in R(A)$ for each $i \in I$.

Further, A is not isomorphic to any subalgebra of B_i for $i \in I$, thus

(2) $A \notin R(B_i)$ for each $i \in I$.

If $a \in A_0$, let $\overline{a} \in B$ be such that $\overline{a}(i) = a$ for each $i \in I$. Put

$$T_0 = \{\overline{a}\colon\ a \in A_0\},$$

and if $i \in I$, let

$$T_i = \{b \in B\colon\ (\exists y \in P(i))(b(i) = y, b(j) = \varphi_i(y) \text{ for each } j \in I - \{i\}\}.$$

Then

(3) $T = \bigcup_{i \in I \cup \{0\}} T_i \cong A.$

Take any fixed $k \in I$. We are going to prove that T is a retract of B. Let $b \in f^{-1}(T)$.

(a) Suppose that $f(b) = \overline{a}$, $a \in A_0$. Then $f(b(k)) = a$. We have either

(4.1) $b(k) \in A_0$

or

(4.2) $b(k) \in P(k)$.

Put

(5.1) $d = b(k)$ if (4.1) is valid,

(5.2) $d = \varphi_k(b(k))$ if (4.2) is valid and denote

(6) $z = \overline{d}$.

Then $z \in T_0 \subseteq T$ and for each $j \in I$ we have

(7.1) $f(z(j)) = f(b(k)) = a = f(b(j))$,

or

(7.2) $f(z(j)) = f(\varphi_k(b(k))) = \varphi_k(f(b(k))) = \varphi_k(a) = a = f(b(j))$,

hence

(8) $f(z) = f(b)$.

Further,

$$s_f(b) \leqslant s_f(b(k)) \leqslant s_f(\varphi_k(b(k)),$$
$$s_f(z) = \begin{cases} s_f(b(k)) & \text{if (5.1) holds,} \\ s_f(\varphi_k(b(k)) & \text{if (5.2) holds,} \end{cases}$$

which yields

(9) $s_f(b) \leqslant s_f(z)$.

(b) Suppose that (a) is not valid. Then there is $i \in I$ with $f(b) \in T_i$, i.e., there is $y \in P(i)$ such that

$$(f(b))(j) = \begin{cases} y & \text{if } j = i, \\ \varphi_i(y) & \text{if } j \in I - \{i\}. \end{cases}$$

Take $z \in T_i$ such that

$$z(j) = \begin{cases} b(i) & \text{if } j = i, \\ \varphi_i(b(i)) & \text{if } j \in I - \{i\}. \end{cases}$$

This implies

$$f(z(j)) = \begin{cases} f(b(i)) & \text{if } j = i, \\ f(\varphi_i(b(i))) = \varphi_i(f(b(i))) = \varphi_i(y) = f(b(j)) & \text{if } j \in I - \{i\}, \end{cases}$$

hence

$$f(z) = f(b).$$

Further,

$$s_f(b) \leqslant \min \{s_f(b(i)), s_f(\varphi_i(b(i)))\} = s_f(z).$$

We have proved

(10) for each $b \in f^{-1}(T)$ there exists $z \in T$ with $f(b) = f(z)$, $s_f(b) \leqslant s_f(z)$.

Let K be a connected component of B with $K \cap T = \emptyset$, $n \in K$. Then either

(11.1) $u(k) \in A_0$

or

(11.2) $u(k) \in P(k)$;

denote either

(12.1) $w = \overline{u(k)}$

or

(12.2) $w = \overline{\varphi_k(u(k))}$

if (11.1) or (11.2) is valid, respectively. Then $w \in T_0$. The mapping $\psi\colon u \to w$ is a homomorphism, since either (11.1) holds, thus $f(u(k)) \in A_0$ and

(13.1) $\psi(f(u)) = \overline{(f(u))(k)} = f(\overline{u(k)}) = f(\psi(u))$,

or (11.2) is valid and

(13.2) if $f(u(k)) \in A_0$, then

$$\psi(f(u)) = \overline{(f(u))(k)} = \overline{\varphi_k(f(u(k)))} = \overline{f(\varphi_k(u(k)))} = f(\overline{\varphi_k(u(k))}) = f(\psi(u)),$$

(13.3) if $f(u(k)) \in P(k)$, then

$$\psi(f(u)) = \overline{\varphi_k(f(u(k)))} = \overline{f(\varphi_k(u(k)))} = f(\overline{\varphi_k(u(k))}) = f(\psi(u)).$$

This and (10) imply (in view of (Thm)) that T is a retract of B. According to (1)–(3) we obtain that A is DR-reducible. $\square$

3.2. Lemma. *Suppose that there are $m, n \in \mathbb{N}$, $m < n$ and $x \in J^{(m)}$, $y \in J^{(n)}$ with $x \notin P(y)$. Then A is* DR*-reducible.*

P r o o f. In view of 2.1, we can assume that card $I^{(n)} \leqslant 1$, card $I^{(m)} \leqslant 1$. Denote

$$B_1 = A - P(y),$$
$$B_2 = A - P(x).$$

It is obvious that

(1) $B_1 \in R(A)$, $B_2 \in R(A)$.

Denote by $I^{(n)}(B_1)$ the set of elements of B_1 described analogously as $I^{(n)}$ for A. Then we get

$I^{(n)}(B_1) = \emptyset$.

Similarly,

$$I^{(m)}(B_2) = \emptyset.$$

Then A is not isomorphic to any subalgebra of B_1 and A is not isomorphic to any subalgebra of B_2, thus

(2) $A \notin R(B_1)$, $A \notin R(B_2)$.

Let $B = B_1 \times B_2$. Denote

$$T = \{(a,a)\colon\ a \in A - (P(x) \cup P(y))\} \cup \\ \cup \{(v, \varphi_x(v))\colon\ v \in P(x)\} \cup \{(\varphi_y(u), u)\colon\ u \in P(y)\}.$$

Then

(3) $A \cong T$.

Let us show that T is a retract of B. Let $b \in f^{-1}(T)$.

(a) If $f(b) = (a,a)$, $a \in A - (P(x) \cup P(y))$, then there is $d \in f^{-1}(a) - (P(x) \cup P(y))$; we put $z = (d,d)$. This yields

(4) $f(z) = f(b)$, $s_f(b) \leqslant s_f(z)$.

(b) If $f(b) = (v, \varphi_x(v))$, $v \in P(x)$, then put $z = (b(1), \varphi_x(b(1)))$; we obtain that (4) is valid, too.

(c) The case when $f(b) = (\varphi_y(u), u)$, $u \in P(y)$, is analogous; we put $z = (\varphi_y(b(2)), b(2))$.

Let K be a connected component of B with $K \cap T = \emptyset$, $t \in K$. If $t(1) \in A - P(x)$, then denote $w = (t(1), t(1))$. If $t(1) \in P(x)$, then put $w = (\varphi_x(t(1)), \varphi_x(t(1))$. It can be easily shown that the mapping $t \to w$ is a homomorphism of K into T. Hence (Thm) yields that T is a retract of B. According to (1)–(4) we conclude that A is DR-reducible. □

3.3. Lemma. *Let m be the smallest positive integer such that $J^{(m)} \neq \emptyset$. Further, let $A \ncong E_m$. If $I^{(m)} \neq J^{(m)}$, then A is* DR-*reducible.*

Proof. Suppose that A is DR-irreducible. By 3.1 there is $x \in A$ with

$$I^{(m)} = \{x\}.$$

Let $I^{(m)} \neq J^{(m)}$. Then there is $y \in A - \{x\}$ such that $J^{(m)} = \{x, y\}$. Since $A \ncong E_m$,

$$A \neq \{x, f(x), \ldots, f^{m-1}(x)\} \cup \{y, f(y), \ldots\}.$$

One of the following cases occurs:

a) $S_0 \neq \{x, y\}$. Then there is the least positive integer $n > m$ such that $I^{(n)} \neq \emptyset$. According to 3.1,

$$I^{(n)} = \{z\} \text{ for some } z \in A$$

and, in view of 3.2,

$$\{x, y\} \subseteq P(z).$$

There is $p \in \mathbb{N}$ such that

$$f^{m+p-1}(x) \notin \{f^j(z)\colon\ j \in \mathbb{N}\} \text{ and } f^{m+p}(x) \in \{f^j(z)\colon\ j \in \mathbb{N}\}.$$

Denote

(1) $u_0 = f^{m+p}(x)$.

Then there are $u_1, u_2, \ldots, u_{m+p} \in A - \{x, f(x), \ldots, f^{m+p-1}(x)\}$ with

(2) $f^{-1}(u_0) \underset{\neq}{\supset} \{u_1\}$, $f^{-1}(u_1) = \{u_2\}$, $f^{-1}(u_2) = \{u_3\}, \ldots, f^{-1}(u_{m+p+1}) = u_{m+p}$.

b) $S_0 = \{x, y\}$. Then there are $p \in \mathbb{N}$, $u_0 \in A$ and $u_1, u_2, \ldots, u_{m+p} \in A - \{x, f(x), \ldots, f^{m+p-1}(x)\}$ such that (1) and (2) are valid. Denote

$$B_1 = B_2 = A - \{y, f(y), \ldots, f^{m-1}(y)\}.$$

Obviously,

(3) $B_1 \in R(A)$, $B_2 \in R(A)$.

Further, let l be the least positive integer such that $J^{(l)}(B_1) \neq \emptyset$. Then l is greater than m, hence A is not isomorphic to any subalgebra of B_1 and

(4) $A \notin R(B_1)$, $A \notin R(B_2)$.

Let $\nu\colon\ A \to B_1 \times B_2$ be the mapping defined as follows: If $a = f^k(y)$, $k \in \{0, \ldots, m-1\}$, then put $\nu(a) = (f^k(x), u_{m-k})$. If $a \in B_1$, then put $\nu(a) = (a, f^p(a))$. Obviously, ν is injective. Denote

$T = \nu(A)$.

Let $a \in A$. If $\{a, f(a)\} \subseteq A - B_1$ or $\{a, f(a)\} \subseteq B_1$, then

$\nu(f(a)) = f(\nu(a))$.

Suppose that $a \in A - B_1$, $f(a) \in B_1$. Then $a = f^{m-1}(y)$ and we obtain

$$\begin{aligned} f(\nu(a)) &= f((f^{m-1}(x), u_1)) = (f^m(x), u_0) = (f^m(y), f^{m+p}(x)) = \\ &= (f^m(y), f^{m+p}(y)) = \nu(f^m(y)) = \nu(f(a)), \end{aligned}$$

hence

(5) ν is an isomorphism of A onto T.

We want to prove that T is a retract of $B_1 \times B_2$. If K is a connected component of $B_1 \times B_2$, $K \cap T = \emptyset$, then the mapping $\varphi\colon K \to T$ defined by the formula

$$\varphi(b) = \nu(b(1))$$

is a homomorphism. Suppose that $v \in f^{-1}(t)$, $t \in T$. First let $t = \nu(f^k(y))$, $k \in \{0, \ldots, m-1\}$. Then

$$t = (f^k(x), u_{m-k});$$

moreover, $k > 0$ and

$$f^{-1}(t) = \{(f^{k-1}(x), u_{m-k+1})\} \in T,$$

which yields that $v \in T$. Now let $t = (a, f^p(a))$, where $a \in B_1$. If $v(1) \in B_1$, then put $d = v(1)$. If $v_1 \in A - B_1$, then there is $d \in f^{-1}(a) \cap B_1$ such that $s_f(d) > s_f(v(1))$. Denote $r = \nu(d)$. We obtain that $r \in T$. Further,

$$\begin{aligned} s_f(r) &= \min\{s_f(r(1)), s_f(r(2))\} = \min\{s_f(d), s_f(f^p(d))\} = s_f(d),\\ s_f(v) &= s_f(v(1)) \leqslant s_f(d). \end{aligned}$$

Obviously, $f(r) = f(v)$, hence

(6) if $v \in f^{-1}(T)$, then there is $r \in T$ with $f(r) = f(v)$ and $s_f(r) \geqslant s_f(v)$.

In view of (Thm), T is a retract of $B_1 \times B_2$, therefore with respect to (3), (4) and (5), A is DR-reducible, which is a contradiction. □

4. Main result

The aim of this section is to prove

4.1. Theorem. *A connected monounary algebra A possessing no cycle is* DR-*irreducible if and only if either A is of type (α) or A is isomorphic to $\mathbb{N}$, $\mathbb{Z}$, E or E_n for some $n \in \mathbb{N}$.*

P r o o f. The sufficient condition for DR-irreducibility is valid in view of 1.2, 1.5 and 1.11.

Now suppose that A is DR-irreducible, A is not of type (α) and that A is not isomorphic to $\mathbb{N}, \mathbb{Z}, E$ or E_n for $n \in \mathbb{N}$. In view of Section 2, there is $x \in A$ with $s_f(x) \neq \infty$. Let us proceed like in Section 3. There exists the smallest positive integer m_1 such that $J^{(m_1)} \neq \emptyset$. By 3.3, $I^{(m_1)} = J^{(m_1)}$. Then 3.1 implies that there is $x_1 \in A$ such that

$$I^{(m_1)} = J^{(m_1)} = \{x_1\}.$$

If $J^{(k)} = \emptyset$ for each $k \in \mathbb{N}$, $k > m_1$, then (4.1) is valid; this case will be investigated later.

Suppose that there is the smallest positive integer $m_2 \in \mathbb{N}$, $m_2 > m_1$ such that $J^{(m_2)} \neq \emptyset$. As above, 3.3 and 3.1 yield that there is $x_2 \in A$ with

$$I^{(m_2)} = J^{(m_2)} = \{x_2\}.$$

Further,

$$x_1 \in P(x_2),$$

in virtue of 3.2.

If $J^{(k)} = \emptyset$ for each $k \in \mathbb{N}$, $k > m_2$, then (4.1) is valid. If not, then there is the smallest $m_3 \in \mathbb{N}$, $m_3 > m_2$ and there is $x_3 \in A$ with

$$I^{(m_3)} = J^{(m_3)} = \{x_3\}, x_2 \in P(x_3).$$

Now there are two possibilities:

I. After finitely many steps we finish this process and come to (4.1);

II. We get $x_1, x_2, \ldots \in A$, $m_1 < m_2 < \ldots$ such that if $k \in \mathbb{N}$, then $I^{(m_k)} = J^{(m_k)} = \{x_k\}$ and $x_k \in P(x_{k+1})$. Since A is not of type $(\alpha 3)$, this yields that there exists $z \in A$ with $s_f(z) = \infty$. The algebra A is connected, thus there are $j, l \in \mathbb{N}$ such that $f^j(x_1) = f^l(z)$. Further,

$$x_1 \in P(x_2) \subsetneqq P(x_3) \subsetneqq P(x_4) \ldots,$$

thus $f^j(x_1) \in P(x_i)$ for some $i \in \mathbb{N}$. Then $z \in P(x_i)$ for some $i \in \mathbb{N}$, and the relation $s_f(z) = \infty$ contradicts the relation $f^{-n_1(x_i)}(z) = \emptyset$.

Therefore we have

(4.1) there exist $k \in \mathbb{N}$, $m_1, \ldots, m_k \in \mathbb{N}$, $x_1, \ldots, x_k \in A$ such that $J^{(i)} = \emptyset$ for each $i > m_k$,

$$m_1 < m_2 < \ldots < m_k,$$
$$I^{(m_1)} = J^{(m_1)} = \{x_1\}, \ldots, I^{(m_k)} = J^{(m_k)} = \{x_k\},$$
$$x_1 \in P(x_2), \ldots, x_{k-1} \in P(x_k).$$

The algebra A is not of type $(\alpha 1)$, thus there is $z \in A$ with $s_f(z) = \infty$. Then

$$s_f(f^{m_k}(x_k)) = \infty$$

and there are distinct elements y_i for $i \in \mathbb{Z}$ such that $y_0 = f^{m_k}(x_k)$ and $f(y_i) = y_{i+1}$ for each $i \in \mathbb{Z}$. Further, A is not of type $(\alpha 2)$, hence there are $a, b \in A$, $a \neq b$ such that $f(a) = f(b)$ and $s_f(a) = s_f(b) = \infty$. Denote

$$B_1 = P(x_k) \cup \{y_i\colon\ i \in \mathbb{Z}\},$$
$$B_2 = A - P(x_k).$$

Obviously, B_1 and B_2 are subalgebras of A. Notice that $s_f(x) = \infty$ for each $x \in B_2$, thus A is not isomorphic to any subalgebra of B_2, hence $A \notin R(B_2)$. The existence of $a, b \in A$ implies that A is not isomorphic to any subalgebra of B_1, thus $A \notin R(B_1)$. Further, by the definition of a retract we get

$$B_1 \in R(A), B_2 \in R(A).$$

There exists a retract homomorphism $\psi\colon\ A \to \{y_i\colon\ i \in \mathbb{Z}\}$. Let us define a mapping $\nu\colon\ A \to B_1 \times B_2$ as follows:

$$\nu(t) = \begin{cases} (t, \psi(t)) \text{ if } t \in P(x_k), \\ (\psi(t), t) \text{ otherwise.} \end{cases}$$

Denote

$$T = \{\nu(t)\colon\ t \in A\}.$$

The mapping ν is injective, since if $t \in P(x_k)$, $r \in A - P(x_k)$, $\nu(t) = \nu(r)$, then $t = \psi(r)$, $r = \psi(t)$, thus $\{r, t\} \subseteq \{y_i\colon\ i \in \mathbb{Z}\}$, hence $\psi(r) = r$, $\psi(t) = t$ and $r = t$. Let us show that ν is a homomorphism. If $\{t, f(t)\} \subseteq P(x_k)$ or $\{t, f(t)\} \subseteq A - P(x_k)$, then obviously $\nu(f(t)) = f(\nu(t))$. Suppose that $t \in P(x_k)$, $f(t) \in A - P(x_k)$. Then $f(t) = y_0$, $\psi(y_0) = y_0$ and we have

$$\nu(f(t)) = \nu(y_0) = (\psi(y_0), y_0) = (y_0, y_0) = (y_0, \psi(y_0)) =$$
$$= (f(t), f(\psi(t))) = f(t, \psi(t)) = f(\nu(t)).$$

Hence T is a subalgebra of $B_1 \times B_2$ such that

$$T \cong A.$$

No connected component of $B_1 \times B_2$ contains a cycle and there is $q \in T$ with $s_f(q) = \infty$, thus (Thm) implies that for proving that T is a retract of $B_1 \times B_2$ it suffices to verify that for each $d \in f^{-1}(T)$ there is $v \in T$ with $f(d) = f(v)$ and $s_f(d) \leqslant s_f(v)$. Thus let $d \in f^{-1}(T)$. Then either

(1) $f(d) = (t, \psi(t))$, $t \in P(x_k)$,

or

(2) $f(d) = (\psi(t), t)$, $t \in A - P(x_k)$.

There is $i \in \mathbb{Z}$ with $\psi(t) = y_i$. If (1) is valid, then $d(1) \in f^{-1}(t)$, $t \in P(x_k)$, hence $d(1) \in P(x_k)$; take $v = (b(1), y_{i-1})$. This implies

$$\begin{aligned} f(v) &= (f(d(1)), f(y_{i-1})) = (t, y_i) = (t, \psi(t)) = f(d), \\ s_f(d) &= \min\{s_f(d(1)), s_f(d(2))\} = \min\{s_f(d(1)), \infty\} = \\ &= s_f(d(1), y_{i-1}) = s_f(v). \end{aligned}$$

Let (2) hold. Then $f(d(2)) = t$, $d(2) \in f^{-1}(t) \subseteq B_2$, $f(d(1)) = y_i$. Put $v = (\psi(d(2)), d(2))$. We get

$$f(v) = (f(\psi(d(2))), f(d(2))) = (\psi(f(d(2)), t) = (\psi(t), t) = f(d).$$

Since $b(2) \in B_2$, we get $s_f(d(2)) = \infty$, thus $s_f(\psi(d(2)) = \infty$ (ψ is a homomorphism), hence $s_f(v) = s_f\ (\psi(d(2)), d(2)) = \infty \geqslant s_f(d)$.

We have proved that A is DR-reducible, which is a contradiction, and this completes the proof. □

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