Czechoslovak Mathematical Journal

J. Muñoz; Josemar Rodríguez; F. J. Muriel
Weil bundles and jet spaces

Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 4, 721–748

Persistent URL: `http://dml.cz/dmlcz/127607`

Czechoslovak Mathematical Journal, 50 (125) (2000), 721–748

WEIL BUNDLES AND JET SPACES

J. Muñoz, J. Rodriguez, Salamanca,
and F. J. Muriel, Cáceres

(Received November 24, 1997)

Abstract. In this paper we give a new definition of the classical contact elements of a smooth manifold M as ideals of its ring of smooth functions: they are the kernels of Weil's near points. Ehresmann's jets of cross-sections of a fibre bundle are obtained as a particular case. The tangent space at a point of a manifold of contact elements of M is shown to be a quotient of a space of derivations from the same ring $C^\infty(M)$ into certain finite-dimensional local algebras. The prolongation of an ideal of functions from a Weil bundle to another one is the same ideal, when its functions take values into certain Weil algebras; following the same idea vector fields are prolonged, without any considerations about local one-parameter groups. As a consequence, we give an algebraic definition of Kuranishi's fundamental identification on Weil bundles, and study their affine structures, as a generalization of the classical results on spaces of jets of cross-sections.

Keywords: jet, near point, contact element, Weil bundle, fundamental identification

MSC 2000: 58A20

Introduction

Jets of smooth mappings between manifolds are usually defined as equivalence classes of mappings, following Ehresmann [2]. Although the classical formulation of jets has been a powerful tool in differential geometry for the last thirty years, it has several inconveniences, such as the need of using very frequently tedious calculations with local coordinates or changing the ring of functions each time a prolongation is defined; consequently some aspects of the theory and its applications seem not to be as clear as they must be. On the other hand, the concept of jets of cross-sections of a fibre bundle implies that the independent variables are fixed previously, hence it is more restrictive than the classical idea of the contact element which can be found in

Lie [9], who used to consider some of the coordinates as independent variables and the remainder as functions of the former, but in a dynamic way, without fixing them.

In [13], Weil outlines a new theory of jet spaces from a different point of view; his *near points* are the natural generalization of the rational points of a smooth manifold, understood as the real spectrum of its ring of smooth functions. The main idea is: if M is a smooth manifold and a point of M is an algebra homomorphism from $C^\infty(M)$ onto $\mathbb{R}$, when $\mathbb{R}$ is replaced by a local finite dimensional $\mathbb{R}$-algebra A, more general "points" are obtained (they were called *A-points* by Weil). For example, when A is the algebra of dual numbers, the A-points of M are the tangent vectors in M.

Weil claims in [13] that the points he defined include as a particular case the jets considered by Ehresmann, which are obtained when only algebras isomorphic to $\mathbb{R}^\ell_m = \mathbb{R}[[x_1 \dots x_m]] \big/ (x_1 \dots x_m)^{\ell+1}$ are considered. Such a restriction has the disadvantage that their subalgebras, quotients and tensor products are not in general of the same type (the last point is a serious obstacle in the theory of prolongations, as will be shown in this paper).

Weil's work seems to have been ignored until 1972, when Morimoto [10] summarized some of his ideas; nevertheless, from the method proposed by him to prolong vector fields to the spaces of near points we can see that he does not use the fundamental point of view in Weil's theory, which is not to replace the ring of functions, but only the algebra in which they take their values. Nowadays several books and papers can be found about the so-called Weil bundles and spaces of (n, r)-velocities, treated as an example of the theory of natural bundles (see [12], [7], [6] or [4]), even though in general they are still very close to Ehresmann's theory and therefore can not use all the power of the language of near points. We did not find in these references the relationship between near points and contact elements.

The aim of this paper is to continue the work started by Weil and continued by Morimoto, Kolář and other authors. It is divided into two sections; the first is devoted to the study of Weil bundles and the construction of the classical contact elements (which are called jets in this paper for reasons which will be explained later), and in the second we study the tangent spaces to the manifolds of near points and jets.

In the first section we develop the theory of near points following the ideas of [13], whose results are covered in the first and third paragraphs. In §1.4 we introduce the notion of prolongation of an ideal of $C^\infty(M)$ to the space of A-points of M without the assumption that it is the ideal of a submanifold of M; the main idea is that the prolongation of the ideal (f) is generated by the same f, considered as a function in M^A with values in A. This definition is intrinsic and avoids using local coordinates.

The remainder of this section is devoted to the study of what we call A-jets of M, which are defined as the kernels of the A-points of M; the classical contact elements are obtained when only the algebras $\mathbb{R}^{\ell}_{m}$ are considered. These spaces are shown to be locally jets of cross-sections of a fibre bundle. We also give a characterization of the points of $\mathcal{J}^{\ell}M$, ℓ-jets of cross-sections of a fibre bundle $\pi\colon\ M \to X$, as algebra homomorphisms from $C^{\infty}(M)$ onto the different $C^{\infty}(X)\big/\mathfrak{m}_{x}^{\ell+1}$, when $x \in X$; thus, they are $C^{\infty}(X)\big/\mathfrak{m}_{x}^{\ell+1}$-points of M. The two possibilities of understanding jets as ideals and as algebra homomorphisms make this theory more operative. In the last paragraph of this section we apply our techniques to the prolongation of ideals to jet spaces; we use the Taylor imbedding, which is defined in a straightforward way.

Section 2 is devoted to the study of the tangent spaces to the manifolds of near points and jets. The main idea is, the tangent vectors to the manifold of A-points of M are derivations from $C^{\infty}(M)$ into A; we again fix the ring of functions and change the algebra where they take values. The vertical tangent vectors to the bundle of jets of cross-sections of a fibre bundle $\pi\colon\ M \to X$ are $C^{\infty}(X)$-derivations from $C^{\infty}(M)$ into the different $C^{\infty}(X)\big/\mathfrak{m}_{x}^{\ell+1}$; this fact allows us to apply to these spaces most of the results obtained for the manifolds of near points. As an example of what we have said, if $\mathfrak{p}^{\ell}, \overline{\mathfrak{p}}^{\ell} \in \mathcal{J}^{\ell}M$ are over the same point of $\mathcal{J}^{\ell-1}M$, then its difference $\mathfrak{p}^{\ell} - \overline{\mathfrak{p}}^{\ell}$ is a $C^{\infty}(X)$-derivation from $C^{\infty}(M)$ with values in $\mathfrak{m}_{x}^{\ell}\big/\mathfrak{m}_{x}^{\ell+1}$, which makes it easier to understand the affine structure of jet bundles and Kuranishi's fundamental identification [8] (compare the definition given by Kuranishi, based on prolongations of one-parameter groups, with paragraph 2.5 bellow).

Although we restrict ourselves to the C^{∞} case, most of the results given in this paper are algebraic and still hold for analytic or regular algebraic varieties. Some of our techniques and results were used by R. Alonso in [1] to give a more natural definition of the Poincaré-Cartan form; in [11] we show how this point of view simplifies the study of Lie equations, and in further papers we will apply it to other topics such as differential invariants and formal integrability, for example.

1. Near points and jets of a smooth manifold

1.1. Spaces of near points of a smooth manifold.

By a *local algebra* (also called *Weil algebra* in [7]) we shall mean a finite dimensional local commutative $\mathbb{R}$-algebra A with unit.

If A is a local algebra and $\mathfrak{m}$ its maximal ideal, then there is a nonnegative integer ℓ such that $\mathfrak{m}^{\ell} \neq 0$ and $\mathfrak{m}^{\ell+1} = 0$; this integer is called the *height* of A, according to Weil [13]. The *width* of A is the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$.

Let A be a local algebra and ℓ its height; the subalgebras of A containing the unit and the quotient algebras of A are local algebras whose height is ℓ or less. Furthermore, if $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local algebras of height ℓ and r, respectively, then $A \otimes_{\mathbb{R}} B$ is a local algebra with height $\ell + r$ whose maximal ideal is $\mathfrak{m}_A \otimes B + A \otimes \mathfrak{m}_B$.

Let us denote $\mathbb{R}^\infty_m = \mathbb{R}[[X_1,\ldots,X_m]]$ and let $\mathfrak{m}(\mathbb{R}^\infty_m)$ be its maximal ideal; the quotient ring $\mathbb{R}^\ell_m = \mathbb{R}^\infty_m \big/ \mathfrak{m}(\mathbb{R}^\infty_m)^{\ell+1}$ is a local algebra of height ℓ. In general, if $m_1,\ldots,m_k,\ell_1,\ldots,\ell_k$ are positive integers, then the tensor product

$$\mathbb{R}^{\ell_1,\ldots,\ell_k}_{m_1,\ldots,m_k} = \mathbb{R}^{\ell_1}_{m_1} \otimes \ldots \otimes \mathbb{R}^{\ell_k}_{m_k}$$

is a local algebra of height $\ell_1 + \ldots + \ell_k$. Each local algebra A is a quotient of $\mathbb{R}^\infty_m$ by an ideal of finite codimension (for a proof see [7]).

Definition 1.1.1. Let M be a smooth manifold and A a local algebra. An *A-point* or *near point of type A* of M is an algebra homomorphism $p^A\colon C^\infty(M) \to A$. For each $f \in C^\infty(M)$ we write $f(p^A)$ for the image of f under p^A and call it the *value of f at p^A*. We will denote by M^A the set of A-points of M.

Examples. (1) The space of algebra homomorphisms $\mathrm{Hom}_{\mathbb{R}}(C^\infty(M),\mathbb{R})$ is well known to be M, hence the $\mathbb{R}$-points of M are the usual points of M. Thus, if A is a local algebra, the composition of each A-point $p^A \in M^A$ with the homomorphism $A \to A/\mathfrak{m} \approx \mathbb{R}$ is a point $p \in M$. Following Weil [13], we say that p^A is an A-point *near* p and that p is the *projection of p^A into* M. Note that the monomorphism $\mathbb{R} \to A$ allows the points of M to be viewed as A-points.

(2) It is easy to show that, if $\mathbb{D} = \mathbb{R}^1_1$, the algebra of dual numbers, then $M^{\mathbb{D}} = TM$, the tangent bundle to M.

(3) When $A = \mathbb{R}^\ell_m$, the space of $\mathbb{R}^\ell_m$-points of M will be denoted as M^ℓ_m; in general, if $A = \mathbb{R}^{\ell_1,\ldots,\ell_k}_{m_1,\ldots,m_k}$ we will write $M^{\ell_1,\ldots,\ell_k}_{m_1,\ldots,m_k} = M^A$.

If $p^\ell_m \in M^\ell_m$, let p be its projection into M. The kernel of p^ℓ_m contains the ideal $\mathfrak{m}_p^{\ell+1}$ of functions on $C^\infty(M)$ which vanish at p up to the order ℓ; if $\{y_1,\ldots,y_n\}$ is a coordinate system at p vanishing at p, then using Taylor's expansion we can write $C^\infty(M) = \mathbb{R}[y_1,\ldots,y_n] + \mathfrak{m}_p^{\ell+1}$, hence p^ℓ_m is completely determined by its action on $\mathbb{R}[y_1,\ldots,y_n]$. Therefore p^ℓ_m can be understood as the ℓ-jet at 0 of a smooth mapping from a neighbourhood of the origin in $\mathbb{R}^m$ into M. Thus $M^\ell_m = \mathcal{J}^\ell_0(\mathbb{R}^m, M)$.

Let M and A be as above; each function $f \in C^\infty(M)$ can be prolonged to a mapping $f^A\colon M^A \to A$ defined by $f^A(p^A) = p^A(f)$. We will simply write f instead of f^A when no confusion can arise.

Let $\{a_1,\ldots,a_d\}$ be a basis of A; $f(p^A)$ can be written in the form

$$f(p^A) = \sum_{k=1}^{d} f_k(p^A) a_k,$$

$f_1,\ldots,f_d$ being real-valued functions defined on M^A, called the *real components* of f in M^A with respect to the basis $\{a_1,\ldots,a_d\}$. Since A is a finite dimensional vector space, it can be endowed with a standard smooth structure, completely determined by the condition that linear forms on A be smooth. This fact allows us to define a smooth structure on M^A; to be more precise:

Theorem 1.1.2. *Let M be a smooth manifold and A a local algebra. The set M^A can be given a smooth structure canonically determined by the condition that each $f \in C^\infty(M)$ be smooth when considered as a mapping from M^A to A.*

Remark. Let $y_1, \ldots, y_n \in C^\infty(M)$ be a coordinate system on an open subset U of M; set $A = \mathbb{R}^\ell_m$ and take the basis $\{\frac{1}{\alpha!}x^\alpha \colon |\alpha| \leqslant \ell\}$ of A. If for each $p^\ell_m \in U^\ell_m$ we write

$$y_i(p^\ell_m) = \sum_{|\alpha|\leqslant\ell} \frac{1}{\alpha!} y_{i\alpha}(p^\ell_m) x^\alpha \qquad i = 1, \ldots, n,$$

the functions $y_{i\alpha}$ $(1 \leqslant i \leqslant n;\ |\alpha| \leqslant \ell)$ form a coordinate system in U^ℓ_m. The value at p^ℓ_m of each $f \in C^\infty(M)$ is obtained by considering the ℓ-Taylor expansion of $f(y_1, \ldots, y_n)$ at $p = p^0_m$ and replacing each y_i by $y_i(p^\ell_m)$, hence the real components f_α of f are polynomials in the $y_{i\beta}$ with coefficients in $C^\infty(U)$.

Definition 1.1.3. We will say that a near point $p^A \in M^A$ is *regular* if the algebra homomorphism $p^A \colon C^\infty(M) \to A$ is onto. If $\mathcal{B}$ is a subring of $C^\infty(M)$, p^A is said to be *$\mathcal{B}$-regular* or *regular over $\mathcal{B}$* if its restriction to $\mathcal{B}$ is onto.

The set of regular A-points of M will be denoted by $\check{M}^A$.

The following result is an immediate consequence of the inverse function theorem:

Proposition 1.1.4. *Let $p^\ell_m \in M^\ell_m$, where $\ell \geqslant 1$, and let $\varphi \colon \mathbb{R}^m \to M$ be a mapping such that $j^\ell_0\varphi = p^\ell_m$. The following statements are equivalent:*

1. *p^ℓ_m is regular.*
2. *$\varphi_* \colon T_0\mathbb{R}^m \to T_pM$ is injective.*
3. *φ defines a local diffeomorphism between a neighbourhood of the origin of $\mathbb{R}^m$ and a locally closed submanifold of M.*
4. *If $y_1, \ldots, y_n$ are local coordinates at $p = p^0_m$ in M, then the rank of the matrix*

$$\left(\frac{\partial\left(y_1(p^\ell_m), \ldots, y_n(p^\ell_m)\right)}{\partial\left(x_1, \ldots, x_m\right)}\right)_{x=0}$$

is m.

Corollary 1.1.5. *If $\ell \geqslant r \geqslant 0$, then the canonical projection $\pi^r_\ell \colon M^\ell_m \to M^r_m$ maps $\check{M}^\ell_m$ onto $\check{M}^r_m$. Furthermore, if $r \geqslant 1$ then $(\pi^r_\ell)^{-1}(\check{M}^r_m) = \check{M}^\ell_m$.*

Corollary 1.1.6. *$\check{M}^\ell_m$ is an open subset of M^ℓ_m. For each $\ell > 0$ and $m > n = \dim M$ we have $\check{M}^\ell_m = \emptyset$. If $m \leqslant n$, then $\check{M}^\ell_m$ is a dense subset of M^ℓ_m.*

If M and N are smooth manifolds, then $(M \times N)^A$ can be identified in a canonical way with $M^A \times N^A$.

If A is a local algebra, then the mapping $M \rightsquigarrow M^A$ is a covariant functor from the category of finite dimensional smooth manifolds into itself; in fact, each smooth mapping $\varphi\colon M \to N$ gives a mapping $\varphi^A\colon M^A \to N^A$ which associates with each $p^A \in M^A$ the algebra homomorphism

$$\begin{aligned} \varphi^A(p^A)\colon C^\infty(N) &\longrightarrow A \\ f &\longmapsto (\varphi^*(f))(p^A) = (p^A \circ \varphi^*)(f). \end{aligned}$$

It follows easily that if φ, ψ are smooth maps, then $(\psi \circ \varphi)^A = \psi^A \circ \varphi^A$. We will simply write φ instead of φ^A when no confusion can arise. As for $f \in C^\infty(N)$ and $p^A \in M^A$ we have

$$(\varphi^*(f))(p^A) = f(\varphi^A(p^A)),$$

if we fix a basis $a_1, \dots, a_d$ of A it follows that

$$(\varphi^*(f))_k = f_k \circ \varphi^A = (\varphi^A)^*(f_k)$$

for $k = 1, \dots, d$, hence $(\varphi^A)^*(f_k) \in C^\infty(M^A)$ for each $f \in C^\infty(N)$ and $1 \leqslant k \leqslant d$. Since the functions f_k determine a smooth structure in N^A, the mapping $\varphi^A\colon M^A \to N^A$ is smooth.

As a special case, each smooth automorphism of M gives a smooth automorphism of M^A, and the same is true for each one-parameter group of automorphisms of M.

On the other hand, if M is a smooth manifold, each homomorphism of local algebras $\sigma\colon A \to B$ gives a mapping $\sigma\colon M^A \to M^B$ such that if $\varphi\colon M \to N$ is smooth, then the diagram

$$\begin{tikzcd} M^A \arrow[r, "\varphi"] \arrow[d, "\sigma"'] & N^A \arrow[d, "\sigma"] \\ M^B \arrow[r, "\varphi"] & N^B \end{tikzcd}$$

is commutative.

1.2. The Taylor imbedding.

Now we deal with an important example of what was said at the end of the previous paragraph, given by what we call *Taylor's homomorphism* $T^{\ell,r}\colon \mathbb{R}^{\ell+r}_m \to \mathbb{R}^{\ell,r}_{m,m}$, induced by the homomorphism of local algebras

$$\begin{aligned} T\colon \mathbb{R}[[X_1, \dots, X_m]] &\longrightarrow \mathbb{R}[[Y_1, \dots, Y_m; Z_1, \dots, Z_m]] \\ X_i &\longmapsto Y_i + Z_i. \end{aligned}$$

The smooth mapping from $M_m^{\ell+r}$ to $M_{m,m}^{\ell,r}$ induced by $T^{\ell,r}$ is called *Taylor's injection*. As the image of $T^{\ell,r}$ is the set of elements

$$F(y,z)=\sum_{|\alpha|=0}^{\ell}\sum_{|\beta|=0}^{r}\frac{1}{\alpha!\beta!}F_{\alpha\beta}y^{\alpha}z^{\beta}$$

which satisfy the equations $F_{\alpha\beta}=F_{\alpha'\beta'}$ whenever $\alpha+\beta=\alpha'+\beta'$, the following theorem holds:

Theorem 1.2.1. *The Taylor injection represents $M_m^{\ell+r}$ as the closed submanifold of $M_{m,m}^{\ell,r}$ whose elements are the points $p_{m,m}^{\ell,r}\in M_{m,m}^{\ell,r}$ which satisfy all the equations $f_{\alpha\beta}(p_{m,m}^{\ell,r})=f_{\alpha'\beta'}(p_{m,m}^{\ell,r})$, where $f\in C^{\infty}(M)$ and $|\alpha|,|\alpha'|\leqslant\ell,|\beta|,|\beta'|\leqslant r,\alpha+\beta=\alpha'+\beta'$. If $y_1,\dots,y_n$ forn a coordinate system on an open subset U of M, then the subset $U_m^{\ell+r}=U_{m,m}^{\ell,r}\cap M_m^{\ell+r}$ of $U_{m,m}^{\ell,r}$ is defined by the equations*

$$y_{i\alpha\beta}=y_{i\alpha'\beta'},\ (i=1\dots,n;|\alpha|,|\alpha'|\leqslant\ell;|\beta|,|\beta'|\leqslant r;\alpha+\beta=\alpha'+\beta').$$

The following result, whose proof is straightforward, will be useful later:

Proposition 1.2.2. *Let $m,\ell,r,s\in N$; the diagram*

$$\begin{tikzcd} M_m^{\ell+r+s} \arrow[r] \arrow[d] & M_{m,m}^{\ell+r,s} \arrow[d] \\ M_{m,m}^{\ell,r+s} \arrow[r] & M_{m,m,m}^{\ell,r,s} \end{tikzcd}$$

is commutative.

1.3. Transitivity of prolongations.

In this paragraph we state an important theorem due to Weil [13]. We begin with some properties of the spaces of near points that can be proved without any difficulties.

If $\varphi\colon\ M\times M\to M$ is an internal operation in M, its prolongation φ^A to M^A is an internal operation in M^A and if φ is associative or commutative, then the same is true for φ^A. In particular, if M is a Lie group, then M^A is a Lie group.

If A is a local algebra, then $\mathbb{R}^A$ is an $\mathbb{R}$-algebra canonically isomorphic to A, and the prolongations of the sum and product in $\mathbb{R}$ are the same operations in A.

If E is a finite dimensional $\mathbb{R}$-vector space, then E^A is an A-module canonically isomorphic to $A\otimes_{\mathbb{R}}E$. If B is a commutative $\mathbb{R}$-algebra, then B^A is a commutative A-algebra canonically isomorphic to $A\otimes_{\mathbb{R}}B$, and the prolongations of the operations of B are the respective operations in $A\otimes_{\mathbb{R}}B$.

If we fix a basis $\{b_1, \ldots, b_s\}$ of B, the prolongation $f^B \colon M^B \to B$ of each $f \in C^\infty(M)$ can be written as $f^B = \sum_{k=1}^{s} f_k b_k$ with $f_k \in C^\infty(M^B)$. If f_k^A denotes the prolongation of f_k to $(M^B)^A$, then the prolongation of f^B is the mapping $(f^B)^A\colon (M^B)^A \to A \otimes B$ defined as $(f^B)^A = \sum_{k=1}^{s} f_k^A \otimes b_k$. From the definition of M^B it follows that, if $f, g \in C^\infty(M)$, then $(fg)^B = f^B g^B$; thus, if $g^B = \sum_{k=1}^{s} g_k b_k$, then

$$f^B g^B = \sum_{i,j=1}^{s} f_i g_j b_i \cdot b_j = \sum_{i,j,k=1}^{s} f_i g_j \lambda_{ij}^k b_k \qquad \text{where} \quad \lambda_{ij}^k \in \mathbb{R}$$

and

$$(f^B g^B)^A = \sum_{i,j,k=1}^{s} f_i^A g_j^A \lambda_{ij}^k \otimes b_k = \left(\sum_{i=1}^{s} f_i^A \otimes b_i\right)\left(\sum_{j=1}^{s} g_j^A \otimes b_j\right) = (f^B)^A (g^B)^A \ ,$$

that is to say, $((fg)^B)^A = (f^B)^A (g^B)^A$; therefore each point $P^A \in (M^B)^A$ can be considered in a natural way as the point $p^{A\otimes B} \in M^{A\otimes B}$ which attaches to each function $f \in C^\infty(M)$ the value

$$f(p^{A\otimes B}) = (f^B)^A (P^A).$$

Thus we have defined a map $(M^B)^A \to M^{A\otimes B}$; now we want to prove that it is a diffeomorphism. As the problem is local in M, we can suppose that $M = \mathbb{R}^n$, hence

$$(M^B)^A = (B \otimes M)^A = A \otimes (B \otimes M) = (A \otimes B) \otimes M.$$

The above discussion enables us to state the main theorem in this paragraph:

Theorem 1.3.1. *[Weil] Let M be a smooth manifold, and A, B local algebras. The manifolds $(M^B)^A$ and $M^{A\otimes B}$ are canonically diffeomorphic.*

Remark. From the identification of $(M^B)^A$ with $M^{A\otimes B}$ it follows that the consecutive prolongation of a function $f \in C^\infty(M)$ to M^B and then to $(M^B)^A$ agrees with the prolongation of f to $M^{A\otimes B}$.

1.4. Prolongation of ideals.

The following result is a straightforward consequence of the definitions:

Proposition 1.4.1. *Let X be a closed submanifold of M and $I(X)$ its ideal in $C^\infty(M)$. Given a local algebra A, the natural imbedding $X^A \subseteq M^A$ identifies X^A with the set of zeros of the ideal $I(X)$ in M^A, where each $f \in I(X)$ is understood as a function from M^A into A. The set X^A is a submanifold of M^A whose ideal is the set of functions $F \in C^\infty\big(M^A\big)$ which agree locally with functions of the ideal of $C^\infty\big(M^A\big)$ generated by the real components of the functions of $I(X)$.*

Definition 1.4.2. The *prolongation* of an ideal I of $C^\infty(M)$ to $C^\infty(M^A)$ is the ideal of this ring whose elements are the functions that agree locally with functions of the ideal generated by the real components of the functions of I, when $C^\infty(M)$ is considered as a ring of funtions from M^A into A. The *prolongation* of a submanifold X of M to M^A is, by definition, the submanifold X^A.

Remark. The above definition allows us to rephrase Proposition 1.4.1 as follows: The prolongation to M^A of a closed submanifold X of M is a closed submanifold of M^A whose ideal is the prolongation to $C^\infty(M^A)$ of the ideal of X in $C^\infty(M)$.

The following properties are easy consequences of the above definitions:

Proposition 1.4.3. *The iterated prolongation of an ideal I of $C^\infty(M)$ to $C^\infty(M^B)$ and then to $C^\infty((M^B)^A)$ agrees with its direct prolongation to $C^\infty(M^{A\otimes B})$.*

Proposition 1.4.4. *The operations of prolongation and specialization commute, in the following sense: If X is a closed submanifold of M, I an ideal of $C^\infty(M)$ and $\bar I$ the specialization of I to X, then the prolongation of $\bar I$ to $C^\infty(X^A)$ agrees with the specialization to X^A of the prolongation of I to M^A.*

Definition 1.4.5. The *prolongation* of an ideal I of the ring $C^\infty(M_m^\ell)$ to $C^\infty(M_m^{\ell+r})$ is the specialization to the submanifold $M_m^{\ell+r}$ of $(M_m^\ell)_m^r$ of the prolongation of I to $C^\infty((M_m^\ell)_m^r)$. The *prolongation* of a submanifold X of M_m^ℓ to $M_m^{\ell+r}$ is the intersection of X_m^r with $M_m^{\ell+r}$.

Remark. The prolongation to $M_m^{\ell+r}$ of a closed submanifold X of M_m^ℓ is the intersection of closed subsets of $M_{m,m}^{\ell,r}$, hence it is a closed subset, but it need not be a submanifold. This closed subset agrees with the set of zeros of the specialization to $M_m^{\ell+r}$ of the prolongation of the ideal I of X to $C^\infty(M_{m,m}^{\ell,r})$.

Proposition 1.4.6. *The prolongation of an ideal I of $C^\infty(M)$ to $C^\infty(M_{m,m}^{\ell,r})$ followed by its specialization to the submanifold $M_m^{\ell+r}$ of $M_{m,m}^{\ell,r}$ agrees with the direct prolongation of I to $C^\infty(M_m^{\ell+r})$.*

Proof. For each $f \in C^\infty(M)$ the restriction of $f_{\alpha,\beta}$ to $M_m^{\ell+r}$ is $f_{\alpha+\beta}$. □

Theorem 1.4.7. *The consecutive prolongation of an ideal I of $C^\infty(M_m^\ell)$ to $C^\infty(M_m^{\ell+r})$ and then to $C^\infty(M_m^{\ell+r+s})$ agrees with its direct prolongation to $C^\infty(M_m^{\ell+r+s})$.*

Proof. It is a consequence of Propositions 1.4.4 and 1.2.2. □

Let $y_1, \ldots, y_n$ be a coordinate system on an open subset U of M; then the functions $y_{i\alpha}$ $(1 \leqslant i \leqslant n; |\alpha| \leqslant \ell)$ form a coordinate system in the open subset U_m^ℓ of M_m^ℓ.

The prolongation of a function $f \in C^\infty(U_m^\ell)$ to a function in $U_{m,m}^{\ell,r}$ with values in $\mathbb{R}_m^r$ can be obtained by replacing each $y_{i\alpha}$ by the polynomial

$$Y_{i\alpha}(x_1, \dots, x_m) = \sum_{|\beta| \leqslant r} \frac{1}{\beta!} y_{i\alpha\beta} x^\beta$$

and then calculating the r-Taylor expansion of $f(\dots, Y_{i\alpha}(x_1, \dots, x_m), \dots)$, considered as a function of $\{y_{i\alpha\beta}, x_j\}$, with respect to the variables x_j at $x = 0$; the coefficient at x^β is $\frac{1}{\beta!} f_\beta$, where f_β is a function in the variables $y_{i\alpha\gamma}$ $(i = 1, \dots, n; |\alpha| \leqslant \ell; |\gamma| \leqslant r)$, the local coordinates in $U_{m,m}^{\ell,r}$. In order to get the specialization to $U_m^{\ell+r}$ we replace $y_{i\alpha\gamma}$ by $y_{i\alpha+\gamma}$; thus we obtain a set of generators of the prolongation to $U_m^{\ell+r}$ of the ideal (f). The prolongation of a finitely generated ideal is obtained by calculating the prolongations of a system of its generators. On the other hand, Theorem 1.4.7 allows us to prolongate ideals from $C^\infty(M_m^\ell)$ to $C^\infty(M_m^{\ell+r})$ as the result of a chain of consecutive prolongations from each $C^\infty(M_m^{\ell+j})$ to $C^\infty(M_m^{\ell+j+1})$ $(j = 0, \dots, r-1)$.

If we write $\varepsilon_k = (0, \dots, 1, \dots, 0)$ (1 on the k-th place), then the prolongation of an ideal (f) from $C^\infty(M_m^\ell)$ to $C^\infty(M_m^{\ell+1})$ is generated by $m+1$ functions

$$f_0 = f, \qquad f_k = f_{\varepsilon_k} = \sum_{i=1}^{n} \sum_{|\alpha| \leqslant \ell} \frac{\partial f}{\partial y_{i\alpha}} y_{i\alpha+\varepsilon_k}, \quad (1 \leqslant k \leqslant m).$$

1.5. Jet spaces.

Definition 1.5.1. Let M be a smooth manifold and A a local algebra. The *jet* of $p^A \in M^A$ is its kernel as a homomorphism from $C^\infty(M)$ into A.

The set of jets of A-points of M will be denoted by $J^A(M)$. For $A = \mathbb{R}_m^\ell$ we will write $J_m^\ell(M)$ instead of $J^A(M)$.

There exists a natural projection

$$\begin{aligned} \ker\colon\ M^A &\longrightarrow J^A(M) \\ p^A &\longmapsto \mathfrak{p}^A = \ker(p^A) \end{aligned}$$

which satisfies the following condition: each smooth mapping $\varphi\colon M \to N$ induces a mapping $\varphi_*\colon J^A(M) \to J^A(N)$, defined by $\varphi_*(\mathfrak{p}^A) = {\varphi^*}^{-1}(\mathfrak{p}^A)$, where $\varphi^*\colon C^\infty(N) \to C^\infty(M)$ is the ring homomorphism attached to φ, such that the diagram

$$\begin{tikzcd} M^A \arrow[r, "\varphi"] \arrow[d, "\ker"'] & N^A \arrow[d, "\ker"] \\ J^A(M) \arrow[r, "\varphi_*"] & J^A(N) \end{tikzcd}$$

is commutative. We will simply write φ instead of φ_* when no confusion can arise.

If $p^A \in M^A$ is regular, its jet $\mathfrak{p}^A$ is said to be regular. The set of jets of regular A-points of M will be denoted by $\mathcal{J}^A(M)$. We will use the notation $\mathcal{J}^\ell_m(M)$ for the set of regular jets of points of M^ℓ_m.

Examples. (1) Let $n = \dim M$, $p^\ell_n \in \check{M}^\ell_n$ and $p = p^0_n$. Since $\mathfrak{m}_p^{\ell+1} \subset \ker p^\ell_n$, p^ℓ_n gives rise to an onto $\mathbb{R}$-algebra homomorphism $C^\infty(M)\,/\mathfrak{m}_p^{\ell+1} \to \mathbb{R}^\ell_n$ which must be an isomorphism, because both the vector spaces have equal dimension. Thus, $\mathfrak{p}^\ell_n = \ker p^\ell_n = \mathfrak{m}_p^{\ell+1}$ and consequently $\mathcal{J}^\ell_n(M) \approx M$.

(2) *Jets of cross-sections of a fibre bundle.* Consider a fibre bundle $\pi\colon\, M \to X$ and let $m = \dim X$; π^* represents $C^\infty(X)$ as a subring of $C^\infty(M)$. It is easy to check that the image of the subset of $C^\infty(X)$-regular jets of $J^\ell_m(M)$ by the mapping $\pi\colon\, J^\ell_m(M) \to J^\ell_m(X)$ is $\mathcal{J}^\ell_m(X) \approx X$. It will cause no confusion if we denote by $\mathcal{J}^\ell M$ the set of $C^\infty(X)$-regular jets of $J^\ell_m(M)$. This set is the usual set of *jets of cross-sections of the fibre bundle* $\pi\colon\, M \to X$.

Let $\mathfrak{p}^\ell_m \in \mathcal{J}^\ell M$, let $p \in M$ be its projection onto M and $x = \pi(p)$. Since $\mathfrak{p}^\ell_m \cap C^\infty(X) = \mathfrak{m}_x^{\ell+1}$, each point $p^\ell_m \in M^\ell_m$ such that $\ker p^\ell_m = \mathfrak{p}^\ell_m$ gives rise to an isomorphism $C^\infty(X)\,/\mathfrak{m}_x^{\ell+1} \to \mathbb{R}^\ell_m$. For each $f \in C^\infty(M)$ the element $f(\mathfrak{p}^\ell_m)$ associated with $f(p^\ell_m)$ is the equivalence class of the only $(\bmod\, m_x^{\ell+1})$ function $g \in C^\infty(X)$ such that $f - g \in \mathfrak{p}^\ell_m$, hence it depends only on $\mathfrak{p}^\ell_m$, not on p^ℓ_m. Consequently, the points of $\mathcal{J}^\ell M$ can be understood as $C^\infty(X)$-algebra homomorphisms from $C^\infty(M)$ onto the different $C^\infty(X)\,/\mathfrak{m}_x^{\ell+1}$ $(x \in X)$, that is, each $\mathfrak{p}^\ell_m \in \mathcal{J}^\ell M$ is a $C^\infty(X)\,/\mathfrak{m}_x^{\ell+1}$-point of M.

If $s\colon\, X \to M$ is a cross-section of π, then the mapping $s_*\colon\, \mathcal{J}^\ell_m(X) \to \mathcal{J}^\ell_m(M)$ is denoted by $j^\ell s$ and called the *ℓ-jet prolongation* of s. It is obvious that the image of $j^\ell s$ is a subset of $\mathcal{J}^\ell M$; for each $x \in X$ we will write $j^\ell_x s = (j^\ell s)(x)$, and this point will be called the *ℓ-jet of s at x*.

1.6. Smooth structure on $\mathcal{J}^\ell_m(M)$.

The set $\mathrm{Aut}(\mathbb{R}^\ell_m)$ of $\mathbb{R}$-algebra automorphisms of $\mathbb{R}^\ell_m$ is a closed subgroup of the linear group $GL(N, \mathbb{R})$, where $N = \binom{m+\ell}{\ell}$, hence it is a Lie group. Its Lie algebra is the set of derivations of $\mathbb{R}^\ell_m$, $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^\ell_m)$ (for a proof, see [5]).

The group $\mathrm{Aut}(\mathbb{R}^\ell_m)$ operates in M^ℓ_m through the mapping

$$\begin{aligned} \mathrm{Aut}(\mathbb{R}^\ell_m) \times M^\ell_m &\longrightarrow M^\ell_m \\ (\sigma, p^\ell_m) &\longmapsto \sigma(p^\ell_m) = \sigma \circ p^\ell_m . \end{aligned}$$

This mapping is easily seen to be smooth and transforms regular points into regular points. Furthermore, the following assertion holds:

Lemma 1.6.1. *The orbits of* $\mathrm{Aut}(\mathbb{R}^\ell_m)$ *in* $\check{M}^\ell_m$ *are submanifolds diffeomorphic to this group.*

P r o o f. Let $p^\ell_m \in \check{M}^\ell_m$ and $y_1, \dots, y_m \in C^\infty(M)$ be such that

$$y_i(p^\ell_m) = y_i(p) + x_i \qquad (1 \leqslant i \leqslant m).$$

Let $y_{m+1}, \dots, y_n \in C^\infty(M)$ be such that $y_1, \dots, y_m, y_{m+1}, \dots, y_n$ is a coordinate system in an open neighborhood U of p in M.

For each $\sigma \in \mathrm{Aut}(\mathbb{R}^\ell_m)$ set $\xi_i\,(x_1, \dots, x_m) = \sigma(x_i)$ $(1 \leqslant i \leqslant m)$; then

$$y_i(\sigma(p^\ell_m)) = y_i(p) + \xi_i(x) \qquad (1 \leqslant i \leqslant m),$$

hence the coordinates $y_{i\alpha}$ $(1 \leqslant i \leqslant m;\ |\alpha| \leqslant \ell)$ of $\sigma(p^\ell_m)$ in M^ℓ_m are exactly those of σ in $\mathrm{Aut}(\mathbb{R}^\ell_m)$ and, when p^ℓ_m is fixed and σ runs through $\mathrm{Aut}(\mathbb{R}^\ell_m)$, the other coordinates of $\sigma(p^\ell_m)$ are smooth functions of the former ones. □

Proposition 1.6.2. *Two points of $\check{M}^\ell_m$ belong to the same orbit of $\mathrm{Aut}(\mathbb{R}^\ell_m)$ if and only if they have the same jet.*

P r o o f. It follows from the fact that each point p^ℓ_m such that $\ker p^\ell_m = \mathfrak{p}^\ell_m$ gives rise to an isomorphism $C^\infty(M)/\mathfrak{p}^\ell_m \to \mathbb{R}^\ell_m$. □

Corollary 1.6.3. *$\mathcal{J}^\ell_m(M)$ can be identified in a canonical way with the quotient set of $\check{M}^\ell_m$ by the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$; this group operates freely on each fibre of the canonical projection $\check{M}^\ell_m \to \mathcal{J}^\ell_m(M)$.*

Remarks. (1) The action of the group $\mathrm{Aut}(\mathbb{R}^\ell_m)$ is not transitive in the fibres of the canonical projection $M^\ell_m \to J^\ell_m(M)$. In order to get an example it suffices to find two isomorphic subalgebras of $\mathbb{R}^\ell_m$ such that the isomorphism cannot be prolonged to an automorphism of $\mathbb{R}^\ell_m$.

For example, consider the subrings

$$A = \{a + b(x_1^2 + x_2^2 + x_3^2) \colon\ a, b \in \mathbb{R}\},$$
$$B = \{a + bx_1x_2 \colon\ a, b \in \mathbb{R}\}$$

of $\mathbb{R}^2_3$. They are isomorphic in an obvious way, but there exists no automorphism of $\mathbb{R}^2_3$ which sends A to B, because the polynomial $x_1^2 + x_2^2 + x_3^2$ is irreducible, while x_1x_2 is not.

(2) It is easily shown that the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$ on the fibres of the projection $M^\ell_m \to J^\ell_m(M)$ is not free, in general.

The remainder of this section is devoted to endowing $\mathcal{J}^\ell_m(M)$ with a smooth structure in such a way that the canonical projection $\check{M}^\ell_m \to \mathcal{J}^\ell_m(M)$ is a principal fibre bundle (see [4] for a different proof). We divide the proccess in three steps.

(1) For each subring $\mathcal{B}$ of $C^\infty(M)$ the set of $\mathcal{B}$-regular points of M^ℓ_m is an open subset (perhaps empty) of $\check{M}^\ell_m$, invariant under the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$.

Let $p^\ell_m, q^\ell_m \in \check{M}^\ell_m$; since M is a Hausdorff space, if $p^0_m \neq q^0_m$ then it is obvious that if $\mathcal{J}^\ell_m(M)$ is endowed with the quotient topology then there are disjoint open neighborhoods of $\mathfrak{p}^\ell_m$ and $\mathfrak{q}^\ell_m$. If $p^0_m = q^0_m = p$ and U is a coordinated neighborhood of p, then there exist functions $z_1, \dots, z_m \in C^\infty(U)$ such that p^ℓ_m and q^ℓ_m are regular over the same subring $\mathbb{R}[z_1, \dots, z_m]$ of $C^\infty(U)$. Thus, to prove that $\mathcal{J}^\ell_m(M)$ is a Hausdorff space we only need to show that for each open subset U of M with a coordinate system $y_1, \dots, y_n$ the image of the subset of points of U^ℓ_m regular over the subring $\mathbb{R}[y_{i_1}, \dots, y_{i_m}]$ of $C^\infty(U)$ $(i_1 < \dots < i_m)$ is a Hausdorff subspace of $\mathcal{J}^\ell_m(U)$.

(2) Let $y_1, \dots, y_n$ be a coordinate system in an open subset U of M and let $\underline{U}^\ell_m$ be the open set of the points of $\check{U}^\ell_m$ regular over $\mathbb{R}[y_1, \dots, y_m]$. Our next goal is to show that $\underline{\mathcal{J}}^\ell_m(U)$ is a Hausdorff subspace of $\mathcal{J}^\ell_m(U)$, finding convenient local coordinates.

Set $p^\ell_m \in \underline{U}^\ell_m$ and let $p = p^0_m$; then p^ℓ_m gives rise to an $\mathbb{R}$-linear isomorphism

$$p^\ell_m \colon \mathbb{R}[y_1, \dots, y_m] \Big/ \big(y_1 - y_1(p), \dots, y_m - y_m(p)\big)^{\ell+1} \to \mathbb{R}^\ell_m.$$

Therefore, for each $f \in C^\infty(U)$ there exists a unique polynomial $\overline{P}_f \in \mathbb{R}[y_1, \dots, y_m]$ of degree ℓ such that

$$f - \overline{P}_f \in \mathfrak{p}^\ell_m = \ker p^\ell_m.$$

Lemma 1.6.4. *For each $f \in C^\infty(U)$ there exist $\binom{m+\ell}{m}$ functions $F_\alpha \colon \underline{U}^\ell_m \to \mathbb{R}$ $(0 \leqslant |\alpha| \leqslant \ell)$ such that for each $p^\ell_m \in \underline{U}^\ell_m$,*

$$f(p^\ell_m) = \sum_{|\alpha| \leqslant \ell} \frac{1}{\alpha!} F_\alpha(p^\ell_m)(y(p^\ell_m) - y(p))^\alpha \tag{1.1}$$

where for $\alpha = (\alpha_1, \dots, \alpha_m)$ we write $y^\alpha = y_1^{\alpha_1} \dots y_m^{\alpha_m}$. The functions F_α are smooth in $\underline{U}^\ell_m$ and they can be written as rational functions of $y_{i\beta}, f_\gamma$ $(1 \leqslant i \leqslant m, |\beta|, |\gamma| \leqslant \ell)$ whose denominators do not vanish in $\underline{U}^\ell_m$; they are linear homogeneous in the f_γ. The ideal of $C^\infty(\underline{U}^\ell_m)$ generated by the F_α $(|\alpha| \leqslant \ell)$ agrees with the one generated by the f_γ $(|\gamma| \leqslant \ell)$.

Proof. The first assertion is obvious. On the other hand, equation (1.1) can be rewritten as

$$f = \sum_{|\gamma| \leqslant \ell} \frac{1}{\gamma!} f_\gamma x^\gamma = \sum_{|\alpha| \leqslant \ell} \frac{1}{\alpha!} F_\alpha (y - y_0)^\alpha, \tag{1.2}$$

understood as an equality between smooth functions from $\underline{U}^\ell_m$ into $\mathbb{R}^\ell_m$, and the statement follows from applying Cramer's rule to the attached system of linear equations in the unknown functions $F_\alpha(p^\ell_m)$. □

We will denote by P_f or $P(F_\alpha, y_i)$ the polynomial in $y_1, \dots, y_m$ whose coefficients are the functions F_α which appear on the right hand side of (1.1). From their definition it follows that the $F_\alpha(p^\ell_m)$ do not depend on p^ℓ_m, but only on $\mathfrak{p}^\ell_m$, hence the functions F_α are left invariant by $\mathrm{Aut}(\mathbb{R}^\ell_m)$ and consequently they are functions in $\underline{\mathcal{J}}^\ell_m(U)$. When $f = y_{m+j}$ $(j = 1 \dots, n-m)$ we will write $Y_{m+j,\alpha} = F_\alpha$; the functions $y_{i\alpha}, Y_{m+j,\beta}$ $(1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n-m; |\alpha|, |\beta| \leqslant \ell)$ are a new coordinate system in $\underline{U}^\ell_m$. If $p^\ell_m \in \underline{U}^\ell_m$, then $\mathfrak{p}^\ell_m = \ker p^\ell_m$ is the sum of $\mathfrak{m}_p^{\ell+1}$ and the ideal of $C^\infty(U)$ generated by the $n-m$ polynomials $y_{m+j} - P(Y_{m+j,\alpha}(p^\ell_m), y_i)$. Since $Y_{m+j,0} = y_{m+j,0}$, $\mathfrak{p}^\ell_m$ is completely determined by $y_i(p), Y_{m+j,\alpha}(p^\ell_m)$ $(1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n-m; |\alpha| \leqslant \ell)$; therefore the functions $y_{i0}, Y_{m+j,\alpha}$ separate points in $\underline{\mathcal{J}}^\ell_m(U)$, hence it is a Hausdorff space.

(3) The group $\mathrm{Aut}(\mathbb{R}^\ell_m)$ operates freely in $\check{M}^\ell_m$ and the quotient space $\mathcal{J}^\ell_m(M)$ is Hausdorff, the graph of the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$ in $\check{M}^\ell_m$ is closed in $\check{M}^\ell_m \times \check{M}^\ell_m$ and homeomorphic to $\mathrm{Aut}(\mathbb{R}^\ell_m) \times \check{M}^\ell_m$, as follows from the calculus made in the proof of Lemma 1.6.1, hence we have completed the proof of

Theorem 1.6.5. *$\mathcal{J}^\ell_m(M)$ can be given in a canonical way a structure of smooth manifold such that the natural projection $\check{M}^\ell_m \to \mathcal{J}^\ell_m(M)$ is a principal fibred bundle with structure group* $\mathrm{Aut}(\mathbb{R}^\ell_m)$.

Remarks. (1) The map $\eta\colon \underline{\mathcal{J}}^\ell_m(U) \to \underline{U}^\ell_m$ which assigns to each point $\mathfrak{p}^\ell_m \in \underline{\mathcal{J}}^\ell_m(U)$ the point $p^\ell_m \in \underline{U}^\ell_m$ defined by

$$\begin{aligned} y_i(p^\ell_m) &= y_i(p) + x_i, \\ y_{m+j}(p^\ell_m) &= \sum_{|\alpha| \leqslant \ell} \frac{1}{\alpha!} Y_{m+j,\alpha}(\mathfrak{p}^\ell_m) x^\alpha \end{aligned}$$

$(1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n-m; |\alpha| \leqslant \ell)$ is a local cross-section of the fibred bundle $\check{U}^\ell_m \to \mathcal{J}^\ell_m(U)$. Choosing U smaller if neccesary we can suppose that the coordinate system $y_1, \dots, y_m, y_{m+1}, \dots, y_n$ represents it as an open subset $U' \times U''$, where U' is an open subset of $\mathbb{R}^m$ and U'' is an open subset of $\mathbb{R}^{n-m}$. Then $U^\ell_m \approx U'^\ell_m \times U''^\ell_m$ (each point p^ℓ_m of U^ℓ_m is identified with the couple $\left(p'^\ell_m, p''^\ell_m\right)$, restrictions of $p^\ell_m\colon C^\infty(U) \to \mathbb{R}^\ell_m$ to $C^\infty(U')$ and $C^\infty(U'')$, respectively). The image of a jet $\mathfrak{p}^\ell_m \in \underline{\mathcal{J}}^\ell_m(U)$ by the composition of the section $\eta\colon \underline{\mathcal{J}}^\ell_m(U) \to \underline{U}^\ell_m$ and the projection $\underline{U}^\ell_m \to U' \times U''^\ell_m$ is the couple $\left(p', p''^\ell_m\right)$, whose coordinates are

$$\begin{aligned} y_i(p') &= y_{i0}(\mathfrak{p}^\ell_m) && (1 \leqslant i \leqslant m), \\ y_{m+j,\alpha}(p''^\ell_m) &= Y_{m+j,\alpha}(\mathfrak{p}^\ell_m) && (1 \leqslant j \leqslant n-m; |\alpha| \leqslant \ell). \end{aligned}$$

Therefore the above mapping $\underline{\mathcal{J}}^\ell_m(U) \to U' \times U''^\ell_m$ is a diffeomorphism.

The set of ℓ–jets of cross-sections of the trivial fibred bundle $U' \times U'' \to U'$ is diffeomorphic to $U' \times U''^{\ell}_{m}$: in fact, as was shown in §1.5 each $\mathfrak{p}^\ell \in \mathcal{J}^\ell(U' \times U'')$ can be understood as a $C^\infty(U')$-algebra homomorphism from $C^\infty(U)$ into $C^\infty(U') \big/ \mathfrak{m}_{p'}^{\ell+1} \approx \mathbb{R}^\ell_m$; the last isomorphism is not canonical, but it is completely determined by attaching to the class $y_i(\mathfrak{p}^\ell_m)$ the element $y_{i0}(p') + x_i$ of $\mathbb{R}^\ell_m$ $(1 \leqslant i \leqslant m)$. Thus we get an isomorphism $\mathcal{J}^\ell(U' \times U'') \approx U' \times U''^{\ell}_{m}$ which applies each jet $\mathfrak{p}^\ell$ to the point of $U' \times U''^{\ell}_{m}$ defined by the restrictions of $\mathfrak{p}^\ell$ to $C^\infty(U')$ and $C^\infty(U'')$; we have thus proved that the space $\mathcal{J}^\ell_m(M)$ is covered by open sets which are sets of ℓ–jets of local cross-sections of fibred bundles.

(2) From the proof of Theorem 1.6.5 it follows that, if $\pi\colon M \to X$ is a fibred bundle, then $\mathcal{J}^\ell(M)$ is a smooth manifold (it is an open subset of $\mathcal{J}^\ell_m(M)$, where $m = \dim X$).

(3) If $n = \dim M$, the mapping $\check{M}^\ell_n \to \mathcal{J}^\ell_n(M) \approx M$ is a principal fibred bundle whose structure group is $\mathrm{Aut}(\mathbb{R}^\ell_n)$; it is known as the *$\ell$-th order frame bundle*. The points $p^\ell_n \in \check{M}^\ell_n$ are homomorphisms from $C^\infty(M)$ onto $\mathbb{R}^\ell_n$ and hence they give isomorphisms $p^\ell_n\colon C^\infty(M)/\mathfrak{m}_p^{\ell+1} \to \mathbb{R}^\ell_n$. This particular case of Theorem 1.6.5 can be found, without a proof, in [13, p. 117].

1.7. The Taylor imbedding in jet spaces.

Consider the canonical projection $\check{M}^\ell_m \to \mathcal{J}^\ell_m(M)$; for each $r \geqslant 0$ there exists a mapping $J^r_m(\check{M}^\ell_m) \to J^r_m(\mathcal{J}^\ell_m(M))$ such that the diagram

$$\begin{tikzcd}
(\check{M}^\ell_m)^r_m \arrow[r] \arrow[d] & (\mathcal{J}^\ell_m(M))^r_m \arrow[d] \\
J^r_m(\check{M}^\ell_m) \arrow[r] & J^r_m(\mathcal{J}^\ell_m(M))
\end{tikzcd}$$

is commutative. Thus we obtain a mapping from $(\check{M}^\ell_m)^r_m$ into $J^r_m(\mathcal{J}^\ell_m(M))$ which we call the *canonical mapping*; the image of each point $P^r_m \in (\check{M}^\ell_m)^r_m$ is the ideal of $C^\infty\big(\mathcal{J}^\ell_m(M)\big)$ intersection of this ring with the jet of $P^r_m\colon C^\infty\big(\check{M}^\ell_m\big) \to \mathbb{R}^r_m$.

Theorem 1.7.1. *The Taylor injection $\check{M}^{\ell+r}_m \to (\check{M}^\ell_m)^r_m$ gives rise to a mapping $\varphi\colon \mathcal{J}^{\ell+r}_m(M) \to J^r_m(\mathcal{J}^\ell_m(M))$ such that the diagram*

$$\begin{tikzcd}
\check{M}^{\ell+r}_m \arrow[r, "\text{Taylor injection}"] \arrow[d, "\text{jet}"'] & (\check{M}^\ell_m)^r_m \arrow[d, "\text{canonical mapping}"] \\
\mathcal{J}^{\ell+r}_m(M) \arrow[r, "\varphi"] & J^r_m(\mathcal{J}^\ell_m(M))
\end{tikzcd}$$

is commutative; φ is injective, its image is a subset of $\mathcal{J}^r_m\big(\mathcal{J}^\ell_m(M)\big)$ and it defines a diffeomorphism between $\mathcal{J}^{\ell+r}_m(M)$ and a closed submanifold of $\mathcal{J}^r_m\big(\mathcal{J}^\ell_m(M)\big)$.

P r o o f. Let $p_m^{\ell+r}$ be a point of $\check{M}_m^{\ell+r}$; the image $p_{m,m}^{\ell,r}$ of $p_m^{\ell+r}$ in $\check{M}_{m,m}^{\ell,r}$ by the Taylor imbedding is identified with a point $P_m^r \in (\check{M}_m^\ell)_m^r$ by the canonical isomorphism; let $\overline{P}_m^r$ be the restriction of P_m^r to $C^\infty\big(\mathcal{J}_m^\ell(M)\big)$. The restriction of $\overline{P}_m^r$ to $C^\infty(M)$ is p_m^r, the projection of $p_m^{\ell+r}$ to $\check{M}_m^r$; as $p_m^{\ell+r}$ is regular, so is $\overline{P}_m^r$. Therefore $\ker \overline{P}_m^r$ is the sum of the ideal $\mathfrak{m}_{\overline{P}_m^0}^{r+1}$ of $C^\infty\big(\mathcal{J}_m^\ell(M)\big)$ and an ideal I generated by a family of $\dim \mathcal{J}_m^\ell(M) - m = (n-m)\binom{m+\ell}{m}$ functionally independent functions.

The point $p_m^{\ell+r}$ being regular, there exists a local coordinate system $y_1, \dots, y_n$ on an open neighborhood of $p = p_m^0$ in M such that $y_{m+1}, \dots, y_n \in \ker p_m^{\ell+r}$; then the $(n-m)\binom{m+\ell}{m}$ functions $Y_{m+j,\alpha}$ belong to $\ker \overline{P}_m^r$ and, as they are functionally independent, they generate I. Thus we have proved that $\ker \overline{P}_m^r$ is the sum of the ideal $\mathfrak{m}_{\overline{P}_m^0}^{r+1}$ with the ideal I generated by the functions F_α when f runs through the ideal $\ker p_m^{\ell+r}$, which shows the existence of the mapping φ and, as $\overline{P}_m^r$ is regular, that the image of φ is contained in $\mathcal{J}_m^r\big(\mathcal{J}_m^\ell(M)\big)$.

It remains to show that φ is an imbedding. As the problem is local in M, we can replace M by an open subset U with local coordinates $y_1, \dots, y_n$ which map U into a product of open subsets $U' \times U'' \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$. Through the isomorphisms

$$\mathcal{J}^{\ell+r}(U) \approx U' \times {U''}_m^{\ell+r},$$
$$\mathcal{J}^r\big(\mathcal{J}^\ell(U)\big) \approx \mathcal{J}^r\big(U' \times {U''}_m^\ell\big) \approx U' \times \big({U''}_m^\ell\big)_m^r$$

the mapping $\varphi\colon\ \mathcal{J}^{\ell+r}(U) \to \mathcal{J}^r\big(\mathcal{J}^\ell(U)\big)$ is attached to the mapping

$$U' \times {U''}_m^{\ell+r} \xrightarrow{\text{id.}\times\text{Taylor}} U' \times \big({U''}_m^\ell\big)_m^r,$$

which completes the proof. $\square$

From the proof of the above theorem it follows that, if $\pi\colon\ M \to X$ is a fibred bundle, then there exists an imbedding $\mathcal{J}^{\ell+r}(M) \to \mathcal{J}^r\big(\mathcal{J}^\ell(M)\big)$ which applies $j_x^{\ell+r}s$ to $j_x^r(j^\ell s)$.

1.8. Prolongation of ideals from $C^\infty\big(\mathcal{J}_m^\ell(M)\big)$ to $C^\infty\big(\mathcal{J}_m^{\ell+r}(M)\big)$.

Definition 1.8.1. The *prolongation of an ideal I of $C^\infty(M)$ to $C^\infty\big(\mathcal{J}_m^\ell(M)\big)$* is the intersection with this subring of $C^\infty\big(\check{M}_m^\ell\big)$ of the prolongation of I to $C^\infty\big(\check{M}_m^\ell\big)$.

The following result is a consequence of Lemma 1.6.4 and Theorem 1.6.5:

Proposition 1.8.2. *The prolongation of an ideal I from $C^\infty(M)$ to $C^\infty\big(\check{M}_m^\ell\big)$ is locally generated by its prolongation to $C^\infty\big(\mathcal{J}_m^\ell(M)\big)$, that is to say, by functions invariant by* $\mathrm{Aut}(\mathbb{R}_m^\ell)$.

This proposition and Proposition 1.4.1 yield

Proposition 1.8.3. *Let X be a closed submanifold of M and I the ideal of X in $C^\infty(M)$. Then $\mathcal{J}^{\ell}_m(X)$ is a closed submanifold of $\mathcal{J}^{\ell}_m(M)$ (it may be empty) whose ideal is the prolongation of I to $C^\infty\big(\mathcal{J}^{\ell}_m(M)\big)$.*

Definition 1.8.4. The *prolongation of an ideal I of $C^\infty\big(\mathcal{J}^{\ell}_m(M)\big)$ to $C^\infty\big(\mathcal{J}^{\ell+r}_m(M)\big)$* is the specialization to the closed submanifold $\mathcal{J}^{\ell+r}_m(M)$ of $\mathcal{J}^{r}_m\big(\mathcal{J}^{\ell}_m(M)\big)$ of the prolongation of I to $C^\infty\big(\mathcal{J}^{r}_m\big(\mathcal{J}^{\ell}_m(M)\big)\big)$.

In order to obtain a result similar to Theorem 1.4.7 consider the inclusions of rings

$$\overset{(1)}{C^\infty\big(\mathcal{J}^{\ell}_m(M)\big)} \subseteq \overset{(2)}{C^\infty\big(\check{M}^{\ell}_m\big)},$$

$$\overset{(3)}{C^\infty\big(\mathcal{J}^{r}_m\big(\mathcal{J}^{\ell}_m(M)\big)\big)} \subseteq \overset{(4)}{C^\infty\Big(\big(\check{\mathcal{J}^{\ell}_m(M)}\big)^r_m\Big)} \subseteq \overset{(5)}{C^\infty\big((\mathring{\check{M}}{}^{\ell}_m)^r_m\big)},$$

where $(\mathring{\check{M}}{}^{\ell}_m)^r_m$ is the inverse image of $(\check{\mathcal{J}^{\ell}_m(M)})^r_m$ in $(\check{M}^{\ell}_m)^r_m$.

According to Definition 1.8.1, to obtain the prolongation of an ideal I of (1) to (3) we must prolongate it from (1) to (4) and then cut this prolongation with (3).

If $\pi\colon X \to Y$ is a surjective submersion, then the existence of local sections of π immediately implies that, if $\mathcal{F}$ is a sheaf of ideals of Y, then $\pi^*(\mathcal{F}) \cap C^\infty_Y = \mathcal{F}$. Applying this result to the surjective submersion

$$\pi\colon\ (\mathring{\check{M}}{}^{\ell}_m)^r_m \to \big(\check{\mathcal{J}^{\ell}_m(M)}\big)^r_m$$

we see that, when the prolongation of I to (4) is prolongated to (5) and cut with (4), the result is the prolongation of I to (4); therefore, in order to obtain the prolongation of I from (1) to (3) it may be prolongated from (1) to (5) and then cut with (3).

By Proposition 1.8.3 the prolongation of I from (1) to (3) generates locally the prolongation of I from (1) to (4); as the latter generates the prolongation of I from (1) to (5), the prolongation of I from (1) to (3) generates locally the prolongation of I from (1) to (5).

Let us consider a system of generators of the prolongation of I from (1) to (5) contained in its prolongation from (1) to (3). Their specializations to the submanifold $\check{M}^{\ell+r}_m$ of $(\mathring{\check{M}}{}^{\ell}_m)^r_m$ is a system of generators of the prolongation of $IC^\infty\big(\check{M}^{\ell}_m\big)$ to $C^\infty\big(\check{M}^{\ell+r}_m\big)$. From the commutativity of the diagram of Theorem 1.7.1 it follows that the intersection with $C^\infty\big(\mathcal{J}^{\ell+r}_m(M)\big)$ of the prolongation of I from $C^\infty\big(\mathcal{J}^{\ell}_m(M)\big)$ to $C^\infty\big(\check{M}^{\ell+r}_m\big)$ agrees with the prolongation of I to this ring in the sense of Definition 1.8.4.

The above discussion and Theorem 1.4.7 allows us to state the following

Theorem 1.8.5. *The prolongation of an ideal I of $C^\infty(\mathcal{J}^\ell_m(M))$ to $C^\infty(\mathcal{J}^{\ell+r}_m(M))$ is the intersection with this ring of the prolongation of $IC^\infty(\check{M}^\ell_m)$ to $C^\infty(\check{M}^{\ell+r}_m)$, and it generates locally the prolongation of I to $C^\infty(\check{M}^{\ell+r}_m)$. The prolongation of I to $C^\infty(\mathcal{J}^{\ell+r}_m(M))$ can be obtained by means of successive prolongations from $C^\infty(\mathcal{J}^{\ell+i}_m(M))$ to $C^\infty(\mathcal{J}^{\ell+i+1}_m(M))$ $(0 \leqslant i \leqslant r-1)$.*

Finally we give the expression in local coordinates of the prolongation of an ideal. The above theorem reduces the prolongation of an ideal I from one jet space to another to the prolongation from $C^\infty(M)$ to $C^\infty(\mathcal{J}^1_m(M))$; such a prolongation can be obtained by means of a system of generators of I, hence we restrict ourselves to the case $I = (f)$.

We can suppose that M is an open subset of $\mathbb{R}^n$ with coordinates $y_1, \dots, y_n$; then $y_{i0}; Y_{m+j,k}$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-m, 0 \leqslant k \leqslant m)$ form a coordinate system in the open subset $\underline{\mathcal{J}}^1_m(M)$ of regular jets over $\mathbb{R}[y_1, \dots, y_m]$, where $Y_{m+j,k} = Y_{m+j,\varepsilon_k}$.

There is an equality between functions from the open subset $\underline{M}^1_m$ of M^1_m with values in $\mathbb{R}^1_m$:

$$f_0 + f_1 x_1 + \ldots + f_m x_m = f = F_0 + F_1(y_1 - y_{10}) + \ldots + F_m(y_m - y_{m0}). \tag{1.3}$$

Replacing each y_i by $y_{i0} + \sum\limits_{k=1}^{m} y_{ik} x_k$ we obtain a system of linear equations for F_k:

$$\begin{aligned} f_0 &= F_0, \\ f_1 &= y_{11} F_1 + \ldots + y_{m1} F_m, \\ &\ldots\ldots\ldots, \\ f_m &= y_{1m} F_1 + \ldots + y_{mm} F_m. \end{aligned} \tag{1.4}$$

If we solve this system and have in mind the equations obtained for the functions f_k of (1.3) at the end of Section 1.3 we see that the prolongation of (f) to $C^\infty(\underline{\mathcal{J}}^1_m(M))$ is generated by the functions

$$\begin{aligned} F_0 &= f(y_{i0}; y_{m+j,0}), \\ F_i &= \partial^\#_i f = \frac{\partial f}{\partial y_i}(y_{i0}; y_{m+j,0}) + \sum_{j=1}^{n-m} \frac{\partial f}{\partial y_{m+j}}(y_{i0}; y_{m+j,0}) Y_{m+j,i} \qquad (1 \leqslant i \leqslant m). \end{aligned}$$

Remark. In this paragraph the prolongation of an ideal I of $C^\infty(\mathcal{J}^\ell_m(M))$ was defined without the hypothesis that I is the ideal of a submanifold of $\mathcal{J}^\ell_m(M)$. The above calculus shows that our definition coincides whith the one given by Kuranishi [8, p. 15], who defined the prolongation of an ideal I locally, as the result of applying the operators $\partial^\#_i$ to its generators.

2.1. Tangent module and tangent space at a near point of M.

According to Theorem 1.3.1, for each local algebra A the manifolds $(M^A)^1_1$ and $M^{A\otimes\mathbb{R}^1_1}$ are canonically diffeomorphic. We know that the first is isomorphic to the tangent bundle $T(M^A)$; on the other hand, each point $p^{A\otimes\mathbb{R}^1_1} \in M^{A\otimes\mathbb{R}^1_1}$ is an $\mathbb{R}$-algebra homomorphism from $C^\infty(M)$ into $A\otimes\mathbb{R}^1_1$ whose composition with the projection $\mathbb{R}^1_1 \to \mathbb{R}$ is a point $p^A \in M^A$. If each $a \in A$ is identified with $a\otimes 1 \in A\otimes\mathbb{R}^1_1$, the homomorphism $p^A\colon C^\infty(M) \to A$ can be viewed as valued in $A\otimes\mathbb{R}^1_1$, and $p^{A\otimes\mathbb{R}^1_1} - p^A$ is a derivation of $C^\infty(M)$ with values in A, that is to say, $p^{A\otimes\mathbb{R}^1_1}$ is the sum of the point $p^A \in M^A$ and a derivation of $C^\infty(M)$ with values in A, endowed with the $C^\infty(M)$-module structure induced by p^A. Thus we have proved our next fundamental result:

Theorem 2.1.1. *For each point $p^A \in M^A$ there exists a canonical linear isomorphism between $T_{p^A}M^A$, the tangent space to M^A at p^A, and $\operatorname{Der}_{\mathbb{R}}\bigl(C^\infty(M), A\bigr)$, where A is considered as a $C^\infty(M)$-module through the homomorphism p^A.*

By virtue of this theorem the tangent space $T_{p^A}M^A$ can be understood as a space of derivations from $C^\infty(M)$ into A; in this case it will be denoted by $\mathcal{T}_{p^A}M^A$ and called the *tangent module to M^A at p^A*. It is a free A-module of rank $n = \dim M$.

Fix a basis $a_1, \dots, a_d$ in A; if we regard a function $f \in C^\infty(M)$ as a function from M^A into A, we write it as $f = \sum\limits_{k=1}^{d} f_k a_k$, where $f_k \in C^\infty(M^A)$. Each tangent vector $\overline{D}_{p^A} \in T_{p^A}M^A$ defines a point $(p^A)^{\mathbb{R}^1_1} \in (M^A)^1_1$ such that, for $k = 1\dots, d$,

$$f_k((p^A)^{\mathbb{R}^1_1}) = f_k(p^A)\otimes 1 + (\overline{D}_{p^A} f_k)\otimes x,$$

where x is a generator of the maximal ideal of $\mathbb{R}^1_1$. Hence when f is considered as a function from $(M^A)^{\mathbb{R}^1_1}$ into $A\otimes\mathbb{R}^1_1$ its value at $(p^A)^{\mathbb{R}^1_1}$ is

$$f((p^A)^{\mathbb{R}^1_1}) = f(p^A)\otimes 1 + \left(\sum_{k=1}^{d} \left(\overline{D}_{p^A} f_k\right) a_k\right)\otimes x;$$

thus we have proved that, for each $\overline{D}_{p^A} \in T_{p^A}M^A$, the derivation $D_{p^A} \in \mathcal{T}_{p^A}M^A$ attached to it according to the above theorem maps each $f \in C^\infty(M)$ into the element

$$D_{p^A} f = \sum_{k=1}^{d} \bigl(\overline{D}_{p^A} f_k\bigr) a_k.$$

We will use the same notation D_{p^A} for the vector $\overline{D}_{p^A}$ of $T_{p^A}M^A$ attached to this derivation.

Let $\varphi\colon M \to N$ be a smooth mapping and let $p^A \in M^A$; the linear mapping $\varphi_*\colon \mathcal{T}_{p^A} M^A \to \mathcal{T}_{\varphi(p^A)} N^A$ attached to the differential of the mapping $\varphi^A\colon M^A \to N^A$ at p^A according to Theorem 2.1.1 is defined by:

$$\varphi_*(D_{p^A}) = D_{p^A} \circ \varphi^*,$$

where $\varphi^*\colon C^\infty(N) \to C^\infty(M)$ is the ring homomorphism induced by φ.

On the other hand, if $\sigma\colon A \to B$ is a homomorphism between local algebras and $\sigma\colon M^A \to M^B$ is the smooth mapping induced by σ, then

$$\sigma_*(D_{p^A}) = \sigma \circ D_{p^A}.$$

Let D be a vector field on M; for each $p^A \in M^A$ the mapping $D_{p^A}\colon C^\infty(M) \to A$ defined as $D_{p^A}(f) = (Df)(p^A)$ is an element of $\mathcal{T}_{p^A} M^A$ which we call the *value of D at p^A*; we will use the same name value of D at p^A when we mean the tangent vector attached to D_{p^A} by Theorem 2.1.1. Thus we obtain a vector field on M^A which will be called the *prolongation of D to M^A*. Using the above notation for the real components of a function $f \in C^\infty(M)$, it is easy to show that $(Df)_k(p^A) = D_{p^A} f_k$, that is to say, $Df_k = (Df)_k$. This implies immediately that the prolongation of vector fields preserves the Lie brackets.

Remark. According to our definition the prolongation to M^A of a vector field D on M is only the same field, considered at each point $p^A \in M^A$ as a derivation from $C^\infty(M)$ into A. This idea gives in a canonical way the prolongation of a tangent vector field to the manifolds of near points and jets, without any consideration about the one-parameter groups generated by the vector field.

Proposition 2.1.2. *A point $p^A \in M^A$ is regular if and only if each tangent vector to M^A at p^A is the value at p^A of a vector field on M.*

Proof. Let $p^A \in M^A$ and let p be its projection into M. Let $y_1, \dots, y_n$ be a local coordinate system on a neighbourhood U of p in M. If D is a tangent vector field on M, it can be written in U as $D = \sum_{i=1}^n f_i \frac{\partial}{\partial y_i}$, where f_i are smooth functions in M. Then

$$D_{p^A} = \sum_{i=1}^n f_i(p^A) \left(\frac{\partial}{\partial y_i}\right)_{p^A}.$$

Thus, when D runs through the set of tangent vector fields on M, D_{p^A} runs through the set of derivations $\sum_{i=1}^n \xi_i \left(\frac{\partial}{\partial y_i}\right)_{p^A}$, where $\xi_i \in \operatorname{Im} p^A$. As $\left(\frac{\partial}{\partial y_i}\right)_{p^A}$ $(1 \leqslant i \leqslant n)$ span $\mathcal{T}_{p^A} M^A$, we conclude. $\square$

2.2. The kernel of the projection $T_{p^\ell_m} M^\ell_m \to T_{p^{\ell-1}_m} M^{\ell-1}_m$.

Let $p^\ell_m \in M^\ell_m$; by Theorem 2.1.1, the differential at p^ℓ_m of the canonical projection $\pi^{\ell-1}_\ell \colon M^\ell_m \to M^{\ell-1}_m$ can be viewed as the natural mapping $\mathcal{T}_{p^\ell_m} M^\ell_m \to \mathcal{T}_{p^{\ell-1}_m} M^{\ell-1}_m$ which applies each derivation $D_{p^\ell_m} \colon C^\infty(M) \to \mathbb{R}^\ell_m$ to the derivation $D_{p^{\ell-1}_m} \colon C^\infty(M) \to \mathbb{R}^{\ell-1}_m$ obtained as the composition of $D_{p^\ell_m}$ and the canonical projection $\pi^{\ell-1}_\ell \colon \mathbb{R}^\ell_m \to \mathbb{R}^{\ell-1}_m$. This easily yields

Proposition 2.2.1. *For each $p^\ell_m \in M^\ell_m$ the kernel $Q_{p^\ell_m} M^\ell_m$ of the canonical projection $T_{p^\ell_m} M^\ell_m \to T_{p^{\ell-1}_m} M^{\ell-1}_m$ is identified by the isomorphism given by Theorem* 2.1.1 *with the set of $\mathbb{R}$-derivations from $C^\infty(M)$ into $\mathbb{R}^\ell_m$ valued in $\mathfrak{m}\big(\mathbb{R}^\ell_m\big)^\ell$.*

Proposition 2.2.2. *Let $D_{p^\ell_m} \in Q_{p^\ell_m} M^\ell_m$ be considered as a derivation from $C^\infty(M)$ into $\mathbb{R}^\ell_m$. For each derivation $Z \colon \mathbb{R}^\ell_m \to \mathbb{R}^{\ell-1}_m$ the mapping $Z \circ D_{p^\ell_m}$ belongs to $Q_{p^{\ell-1}_m} M^{\ell-1}_m$. Furthermore, if $\overline{Z} \colon \mathbb{R}^\ell_m \to \mathbb{R}$ is the composition of Z with the canonical projection $\mathbb{R}^{\ell-1}_m \to \mathbb{R}$, then $Z \circ D_{p^\ell_m}$ depends only on $D_{p^\ell_m}$ and $\overline{Z}$.*

Let $\ell \geqslant 1$ and let us consider $\mathbb{R}$ endowed with the $\mathbb{R}^\ell_m$-module structure given by the natural projection; then $\mathrm{Der}_{\mathbb{R}}\big(\mathbb{R}^\ell_m, \mathbb{R}\big)$ can be identified with $T_0(\mathbb{R}^m)$. Thus, according to Proposition 2.2.2 each $D_{p^\ell_m} \in Q_{p^\ell_m} M^\ell_m$ can be understood as a linear mapping from $T_0(\mathbb{R}^m)$ into $Q_{p^{\ell-1}_m} M^{\ell-1}_m$ which applies each tangent vector $\overline{Z} \in T_0(\mathbb{R}^m)$ to the derivation $Z \circ D_{p^\ell_m}$ of $Q_{p^{\ell-1}_m} M^{\ell-1}_m$. Iterating the proccess we can define a multilinear mapping

$$\begin{aligned} T_0(\mathbb{R}^m) \times \ldots \times T_0(\mathbb{R}^m) &\longrightarrow Q_{p^{\ell-r}_m} M^{\ell-r}_m \\ \big(\overline{Z}_1, \ldots, \overline{Z}_r\big) &\longmapsto Z_r \circ \ldots \circ Z_1 \circ D_{p^\ell_m}. \end{aligned} \tag{2.1}$$

If $D_{p^\ell_m} \neq 0$, there must exist a derivation Z_1 from $\mathbb{R}^\ell_m$ to $\mathbb{R}^{\ell-1}_m$ such that $Z_1 \circ D_{p^\ell_m} \neq 0$; if we repeat the argument we see that, for each $r \leqslant \ell$, the mapping which assigns to each $D_{p^\ell_m} \in Q_{p^\ell_m} M^\ell_m$ the multilinear mapping (2.1) is injective. On the other hand, we can suppose that the representants of Z_i are determined by vector fields on $\mathbb{R}^m$ which commute, hence it is also symmetric.

We can summarize the above discussion as follows:

Theorem 2.2.3. *Set $p^\ell_m \in M^\ell_m$; for each $r \leqslant \ell$ there exists a cannonical injection*

$$Q_{p^\ell_m} M^\ell_m \subseteq S^r\left[T^*_0(\mathbb{R}^m)\right] \otimes_{\mathbb{R}} Q_{p^{\ell-r}_m} M^{\ell-r}_m \tag{2.2}$$

which attaches to each tangent vector $D_{p^\ell_m} \in Q_{p^\ell_m} M^\ell_m$ the multilinear mapping (2.1)*. In particular, for $r = \ell$ this mapping is an isomorphism, because it is injective and the two vector spaces have the same dimension, that is to say:*

$$Q_{p^\ell_m} M^\ell_m \approx S^\ell\left[T^*_0(\mathbb{R}^m)\right] \otimes_{\mathbb{R}} T_p M, \quad \textit{provided} \quad p = p^0_m. \tag{2.3}$$

Remark. As a consequence ot the translation of this theorem to jet spaces we will define in an easy way what Kuranishi [8] calls the *fundamental identification.*

The coordinate expression of the mapping (2.2) may be obtained as the result of an easy calculus:

Proposition 2.2.4. *Set $D_{p^\ell_m} \in Q_{p^\ell_m} M^\ell_m \subseteq T_{p^\ell_m} M^\ell_m$ and denote*

$$F_i\left(x_1,\dots,x_m\right) = \sum_{|\alpha|\leqslant\ell} \frac{1}{\alpha!}\left(D_{p^\ell_m} y_{i\alpha}\right) x^\alpha.$$

Then the F_i are homogeneous polynomials of degree ℓ and the image of $D_{p^\ell_m}$ by the mapping (2.2) *is the linear mapping (from $S^r T_0(\mathbb{R}^m)$ into $Q_{p^{\ell-r}_m} M^{\ell-r}_m$)*

$$\sum_{i=1}^n \left(d^{(r)} F_i\right) \otimes \left(\frac{\partial}{\partial y_i}\right)_p.$$

2.3. The projection $M^\ell_m \to M^{\ell-1}_m$ as an affine fibred bundle.

Let p^ℓ_m be a point of M^ℓ_m; for each $r \leqslant \ell$ we write $p^{\ell-r}_m = \pi^{\ell-r}_\ell\left(p^\ell_m\right)$. As was shown in the last paragraph, the elements of $Q_{p^\ell_m} M^\ell_m$ can be viewed as derivations from $C^\infty(M)$ into $\mathfrak{m}(\mathbb{R}^\ell_m)^\ell$; therefore, if $D \in Q_{p^\ell_m} M^\ell_m$, then D is a derivation not only at p^ℓ_m, but at each point of M^ℓ_m belonging to the fibre of $p^{\ell-r}_m$ in the projection $\pi^{\ell-r}_\ell$. Consequently, the vector space $Q_{p^\ell_m} M^\ell_m$ does not depend on p^ℓ_m, but only on $p^{\ell-r}_m$, hence the natural projection $Q\left(M^\ell_m\right) \to M^{\ell-r}_m$ is a vector fibred bundle. In particular, for $\ell = r$ the projection $Q\left(M^\ell_m\right) \to M$ is a vector fibred bundle.

We will denote by $Q_{p^{\ell-r}_m} M^\ell_m$ the fibre of $p^{\ell-r}_m$ under the mapping $Q\left(M^\ell_m\right) \to M^{\ell-r}_m$.

Let $p^\ell_m, q^\ell_m \in M^\ell_m$ and suppose that $p^{\ell-1}_m = q^{\ell-1}_m$; if they are viewed as algebra homomorphisms from $C^\infty(M)$ into $\mathbb{R}^\ell_m$ then $p^\ell_m - q^\ell_m$ is a derivation from $C^\infty(M)$ into $\mathbb{R}^\ell_m$, endowed with the $C^\infty(M)$-module structure given by any of them, with values in $\mathfrak{m}\left(\mathbb{R}^\ell_m\right)^\ell$; in other words, it is an element of $Q_{p^{\ell-1}_m} M^\ell_m$.

Conversely, given $p^\ell_m \in M^\ell_m$ and $D \in Q_{p^{\ell-1}_m} M^\ell_m$, if p^ℓ_m is considered as an $\mathbb{R}$-algebra homomorphism from $C^\infty(M)$ into $\mathbb{R}^\ell_m$ and D as a derivation at p^ℓ_m from $C^\infty(M)$ into $\mathbb{R}^\ell_m$ valued in $\mathfrak{m}\left(\mathbb{R}^\ell_m\right)^\ell$, then the mapping $q^\ell_m = p^\ell_m + D\colon C^\infty(M) \to \mathbb{R}^\ell_m$ is an $\mathbb{R}$-algebra homomorphism, that is, a point of M^ℓ_m, and $q^{\ell-1}_m = p^{\ell-1}_m$.

The following theorem is a consequence of the definitions and the above discussion:

Theorem 2.3.1. *The mapping*

$$\begin{aligned} Q\left(M^\ell_m\right) \times_{M^{\ell-1}_m} M^\ell_m &\longrightarrow M^\ell_m \\ \left(D, p^\ell_m\right) &\longmapsto p^\ell_m + D \end{aligned} \tag{2.4}$$

defines an operation of $Q\big(M_m^\ell\big)$ on M_m^ℓ which preserves the fibres of the projection $\pi_\ell^{\ell-1}\colon\ M_m^\ell \to M_m^{\ell-1}$; the latter is an affine fibred bundle modelled over the vector fibred bundle $Q\big(M_m^\ell\big) \to M_m^{\ell-1}$. This operation is functorial, in the following sense: if $\varphi\colon\ M \to N$ is a smooth mapping, then the diagram

$$\begin{array}{ccc} Q\big(M_m^\ell\big) \times_{M_m^{\ell-1}} M_m^\ell & \longrightarrow & M_m^\ell \\ {\scriptstyle \varphi_* \times \varphi}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\ Q\left(N_m^\ell\right) \times_{N_m^{\ell-1}} N_m^\ell & \longrightarrow & N_m^\ell \end{array} \tag{2.5}$$

is commutative.

Let $\varphi\colon\ M \to N$ be a smooth mapping; it follows from the commutativeness of diagram (2.5) that, for each couple of natural numbers m, ℓ, the mapping $\varphi_m^\ell\colon\ M_m^\ell \to N_m^\ell$ is a morphism of affine fibred bundles over $\varphi_m^{\ell-1}$ whose associated morphism of vector fibred bundles is the restriction of φ_* to $Q\big(M_m^\ell\big)$. In particular, the following statement holds:

Proposition 2.3.2. *The Taylor embedding $M_m^{\ell+r} \to M_{m,m}^{\ell,r}$ is a morphism of affine fibred bundles over $M_m^{\ell+r-1} \to M_{m,m}^{\ell,r-1}$ whose associated morphism of vector fibred bundles $Q\big(M_m^{\ell+r}\big) \to Q\big(M_{m,m}^{\ell,r}\big)$ agrees, up to a constant factor, with the fundamental identification.*

2.4. The tangent space to $\mathcal{J}_m^\ell(M)$ at a point $\mathfrak{p}_m^\ell$.

Given $\mathfrak{p}_m^\ell \in \mathcal{J}_m^\ell(M)$, let $p_m^\ell \in \check{M}_m^\ell$ be such that $\mathfrak{p}_m^\ell = \ker p_m^\ell$. As the differential at p_m^ℓ of the surjective submersion $\check{M}_m^\ell \to \mathcal{J}_m^\ell(M)$ is surjective and $\mathbb{R}_m^\ell \approx C^\infty(M)\,/\mathfrak{p}_m^\ell$, Theorem 2.1.1 implies that $T_{\mathfrak{p}_m^\ell}\mathcal{J}_m^\ell(M)$ is isomorphic to the quotient vector space of $T_{p_m^\ell}M_m^\ell \approx \mathrm{Der}_\mathbb{R}\left(C^\infty(M), C^\infty(M)\,/\mathfrak{p}_m^\ell\right)$ by the subspace of the vectors tangent to the fibre of the projection $\check{M}_m^\ell \to \mathcal{J}_m^\ell(M)$ at $\mathfrak{p}_m^\ell$. If $\mathfrak{p}_m^\ell$ is considered as an ideal of $C^\infty(M)$, then each function $f \in \mathfrak{p}_m^\ell$ vanishes in such fibre; therefore each derivation from $C^\infty(M)$ into $C^\infty(M)\,/\mathfrak{p}_m^\ell \approx \mathbb{R}_m^\ell$ that, considered as belonging to $T_{p_m^\ell}M_m^\ell$ by Theorem 2.1.1, is tangent to the fibre which contains p_m^ℓ, annihilates the ideal $\mathfrak{p}_m^\ell$, hence it gives a derivation from $C^\infty(M)\,/\mathfrak{p}_m^\ell$ into $C^\infty(M)\,/\mathfrak{p}_m^\ell$.

Hence there exists a surjective homomorphism of vector spaces

$$\mathrm{Der}_\mathbb{R}\left(C^\infty(M), C^\infty(M)\,/\mathfrak{p}_m^\ell\right)/\mathrm{Der}_\mathbb{R}\left(C^\infty(M)\,/\mathfrak{p}_m^\ell\,, C^\infty(M)\,/\mathfrak{p}_m^\ell\right) \to T_{\mathfrak{p}_m^\ell}\mathcal{J}_m^\ell(M)$$

which must be an isomorphism, because the two spaces have the same dimension.

We summarize the discussion as follows:

Theorem 2.4.1. *Let $\mathfrak{p}_m^\ell \in \mathcal{J}_m^\ell(M)$. Then the tangent space $T_{\mathfrak{p}_m^\ell}\mathcal{J}_m^\ell(M)$ is isomorphic to the vector space*

$$\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(M)\,/\mathfrak{p}_m^\ell\right)/\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M)\,/\mathfrak{p}_m^\ell\,, C^\infty(M)\,/\mathfrak{p}_m^\ell\right).$$

The remainder of this paragraph is devoted to computing the local coordinate expression of the prolongation to the jet spaces of a vector field on M.

Let D be a tangent vector field in M; the *prolongation of D to M_m^ℓ* is defined as the vector field whose value at each point $p_m^\ell \in \check{M}_m^\ell$ is the vector attached to $D_{p_m^\ell}$ by Theorem 2.1.1. The restriction of this field to $\check{M}_m^\ell$ is projectable to a vector field on $\mathcal{J}_m^\ell(M)$ which we call the *prolongation of D to $\mathcal{J}_m^\ell(M)$*.

We will use the notation of §1.6. Let $y_1\ldots,y_m,y_{m+1},\ldots,y_n$ be a coordinate system on an open subset U of M; then $\{y_{i\alpha},y_{m+j,\alpha}\}$ is a coordinate system on $\underline{U}_m^\ell$ and $\{y_{i0},Y_{m+j,\alpha}\}$ is a coordinate system in $\underline{\mathcal{J}}_m^\ell(U)\approx U'\times {U''}_m^\ell$. Let $\mathfrak{p}_m^\ell\in\underline{\mathcal{J}}_m^\ell(U)$ and $p_m^\ell=\eta(\mathfrak{p}_m^\ell)$; then $y_i(p_m^\ell)=y_i(p)+x_i$ $(1\leqslant i\leqslant m)$. If the local expression of D in the coordinates $\{y_1,\ldots,y_n\}$ is

$$D=\sum_{i=1}^m f_i\frac{\partial}{\partial y_i}+\sum_{j=1}^{n-m} f_{m+j}\frac{\partial}{\partial y_{m+j}},$$

then

$$D_{p_m^\ell}y_i=f_i(p_m^\ell)=\sum_{|\gamma|\leqslant\ell}\frac{1}{\gamma!}F_{i,\gamma}(\mathfrak{p}_m^\ell)x^\gamma\qquad i=1,\ldots,m$$

where $F_{i0}=f_{i0}$ is f_i, considered in $C^\infty\big(\underline{\mathcal{J}}_m^\ell(U)\big)$. Further,

(2.6)

$$\begin{aligned}D_{p_m^\ell}y_{m+j}&=D_{p_m^\ell}\Bigg[\sum_{|\alpha|\leqslant\ell}\frac{1}{\alpha!}Y_{m+j,\alpha}(y-y_0)^\alpha\Bigg]\\&=\sum_{|\alpha|\leqslant\ell}\frac{1}{\alpha!}\Big[D_{p_m^\ell}Y_{m+j,\alpha}x^\alpha+Y_{m+j,\alpha}(\mathfrak{p}_m^\ell)\sum_{k=1}^m\alpha_kD_{p_m^\ell}(y_k-y_{k0})x^{\alpha-\varepsilon_k}\Big]\\&=\sum_{|\alpha|\leqslant\ell}\tfrac{1}{\alpha!}D_{p_m^\ell}Y_{m+j,\alpha}x^\alpha+\sum_{\substack{|\beta|\leqslant\ell-1\\1\leqslant k\leqslant m}}\tfrac{1}{\beta!}Y_{m+j,\beta+\varepsilon_k}(\mathfrak{p}_m^\ell)\sum_{1\leqslant|\gamma|\leqslant\ell}\tfrac{1}{\gamma!}F_{k,\gamma}(\mathfrak{p}_m^\ell)x^{\beta+\gamma}.\end{aligned}$$

On the other hand,

$$D_{p_m^\ell}y_{m+j}=f_{m+j}(p_m^\ell)=\sum_{|\alpha|\leqslant\ell}\frac{1}{\alpha!}F_{m+j,\alpha}(\mathfrak{p}_m^\ell)x^\alpha,\tag{2.7}$$

therefore, if $D_{\mathfrak{p}_m^\ell}$ is the projection of $D_{p_m^\ell}$ to $T_{\mathfrak{p}_m^\ell}\mathcal{J}_m^\ell(M)$, then (2.6)–(2.7) yields

$$D_{\mathfrak{p}_m^\ell}Y_{m+j,\alpha}=F_{m+j,\alpha}(\mathfrak{p}_m^\ell)-\sum_{\substack{1\leqslant k\leqslant m\\\beta+\gamma=\alpha\\|\gamma|>0}}\frac{\alpha!}{\beta!\gamma!}Y_{m+j,\beta+\varepsilon_k}(\mathfrak{p}_m^\ell)F_{k,\gamma}(\mathfrak{p}_m^\ell),\tag{2.8}$$

hence the prolongation of D can be written locally as:

$$\sum_{i=1}^{m} f_i \frac{\partial}{\partial y_{i0}} + \sum_{\substack{1\leqslant j\leqslant n-m \\ |\alpha|\leqslant \ell}} \Big[F_{m+j,\alpha} - \sum_{\substack{1\leqslant k\leqslant m \\ \beta+\gamma=\alpha \\ |\gamma|>0}} \frac{\alpha!}{\beta!\gamma!} Y_{m+j,\beta+\varepsilon_k} F_{k,\gamma}\Big] \frac{\partial}{\partial Y_{m+j,\alpha}}. \tag{2.9}$$

2.5. The vertical tangent space at a point of $\mathcal{J}^\ell M$.

Definition 2.5.1. Let $\pi\colon M \to X$ be a fibred bundle, $p \in M$ and $x = \pi(p)$. The *vertical tangent space to M at p* is the kernel V_pM of the linear mapping $\pi_*\colon T_pM \to T_xX$.

We know that, if $\mathfrak{p}^\ell \in \mathcal{J}^\ell M$ and $x = \pi^\ell(\mathfrak{p}^\ell)$, then $\mathfrak{p}^\ell$ establishes a ring isomorphism $C^\infty(M)/\mathfrak{p}^\ell \approx C^\infty(X)/\mathfrak{m}_x^{\ell+1}$; therefore, according to the previus paragraph $T_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ can be identified with

$$\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(X)/\mathfrak{m}_x^{\ell+1}\right)/\mathrm{Der}_{\mathbb{R}}\left(C^\infty(X)/\mathfrak{m}_x^{\ell+1}, C^\infty(X)/\mathfrak{m}_x^{\ell+1}\right).$$

As $\mathrm{Der}_{C^\infty(X)}\left(C^\infty(M), C^\infty(M)/\mathfrak{p}^\ell\right) \cap \mathrm{Der}_{\mathbb{R}}\left(C^\infty(X)/\mathfrak{m}_x^{\ell+1}, C^\infty(X)/\mathfrak{m}_x^{\ell+1}\right) = (0)$, there exists an injective mapping

$$\mathrm{Der}_{C^\infty(X)}\left(C^\infty(M), C^\infty(M)/\mathfrak{p}^\ell\right) \to T_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$$

whose image is a subspace of $V_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$; as the two vector spaces have the same dimension, this injection is an isomorphism. This proves

Theorem 2.5.2. *The vertical tangent space $V_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ is canonically isomorphic to the vector space* $\mathrm{Der}_{C^\infty(X)}\left(C^\infty(M), C^\infty(X)/\mathfrak{m}_x^{\ell+1}\right)$.

Remark. If D is a vertical vector field on M, then its prolongation to $\mathcal{J}^\ell M$ is a vertical vector field whose value at each point $\mathfrak{p}^\ell$ is the $C^\infty(X)$–derivation $D_{\mathfrak{p}^\ell} = \mathfrak{p}^\ell \circ D$. It can be easily shown that each vertical vector tangent to $\mathcal{J}^\ell M$ at $\mathfrak{p}^\ell$ is the value at $\mathfrak{p}^\ell$ of some tangent vector field in M vertical for the surjective submersion $\pi\colon M \to X$.

The above theorem allows us to translate to jet spaces the main results of paragraphs 2.2 and 2.3.

Proposition 2.5.3. *For each point $\mathfrak{p}^\ell \in \mathcal{J}^\ell M$ the kernel $Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ of the canonical projection*

$$V_{\mathfrak{p}^\ell}\mathcal{J}^\ell M \to V_{\mathfrak{p}^{\ell-1}}\mathcal{J}^{\ell-1} M$$

is identified by Theorem 2.5.2 *with the set of $C^\infty(X)$-derivations from $C^\infty(M)$ into $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$ whose images are contained in $\mathfrak{m}_x^\ell/\mathfrak{m}_x^{\ell+1}$.*

In much the same way as in Theorem 2.2.3, for each $r \leqslant \ell$ we obtain the mapping (2.10), called the *fundamental identification* by Kuranishi [8, p. 8]. Our characterization of the elements of $Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ as $C^\infty(X)$–derivations from $C^\infty(M)$ into $\mathfrak{m}_x^\ell / \mathfrak{m}_x^{\ell+1}$ allows us to give an elementary algebraic definition of the fundamental identification, which Kuranishi defines by means of local one-parametric groups.

Theorem 2.5.4. *For each $r \leqslant \ell$ there exists a canonical injection*

$$i_\ell^{\ell-r}\colon\ Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M \to S^r T_x^*(X) \otimes Q_{\mathfrak{p}^{\ell-r}}\mathcal{J}^{\ell-r} M \tag{2.10}$$

which maps the vertical tangent vector $D_{\mathfrak{p}^\ell}$ into the symmetric multilinear mapping defined by

$$i_\ell^{\ell-r}\left(D_{\mathfrak{p}^\ell}\right)\left(\overline{Z}_1, \ldots, \overline{Z}_r\right) = Z_r \circ \ldots \circ Z_1 \circ D_{\mathfrak{p}^\ell}, \tag{2.11}$$

where $D_{\mathfrak{p}^\ell}$ is understood as a $C^\infty(X)$-derivation from $C^\infty(M)$ into $C^\infty(X) / \mathfrak{m}_x^{\ell+1} \approx C^\infty(M) / \mathfrak{p}^\ell$ and each $Z_i\colon C^\infty(X) / \mathfrak{m}_x^{\ell-i+2} \to C^\infty(X) / \mathfrak{m}_x^{\ell-i+1}$ is any derivation whose "value" at x is the tangent vector $\overline{Z}_i$. In particular, when $r = \ell$ then the injection (2.10) *is an isomorphism*

$$Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M \approx S^\ell T_x^*(X) \otimes_{\mathbb{R}} V_p M. \tag{2.12}$$

Let $\mathfrak{p}^\ell \in \mathcal{J}^\ell M$; as we have shown there is a bijective correspondence between $Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ and the $C^\infty(X)$–derivations from $C^\infty(M)$ into $C^\infty(X) / m_x^{\ell+1}$ whose values are in $\mathfrak{m}_x^\ell / \mathfrak{m}_x^{\ell+1}$; this fact implies that, if for $r \leqslant \ell$ we write $\mathfrak{p}^r = \pi_\ell^r\left(\mathfrak{p}^\ell\right)$, each $D \in Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ is a derivation not only at $\mathfrak{p}^\ell$, but also at all points of the fibre at $\mathfrak{p}^r$, hence the vector space $Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell M$ does not depend on $\mathfrak{p}^\ell$, but only of $\mathfrak{p}^r$; consequently, the projection $Q\left(\mathcal{J}^\ell M\right) \to \mathcal{J}^r M$ is a vector fibre bundle. In particular, the projection $Q\left(\mathcal{J}^\ell M\right) \to M$ is a vector fibre bundle. The fibre of $\mathfrak{p}^r$ in the projection $Q\left(\mathcal{J}^\ell M\right) \to \mathcal{J}^r M$ will be denoted by $Q_{\mathfrak{p}^r}\mathcal{J}^\ell M$.

Let $\mathfrak{p}^\ell, \mathfrak{q}^\ell \in \mathcal{J}^\ell M$ such that $\mathfrak{p}^{\ell-1} = \mathfrak{q}^{\ell-1}$. If they are viewed as $C^\infty(X)$-algebra homomorphisms from $C^\infty(M)$ into $C^\infty(X) / \mathfrak{m}_x^{\ell+1}$, then $\mathfrak{p}^\ell - \mathfrak{q}^\ell$ is a $C^\infty(X)$-derivation from $C^\infty(M)$ into $C^\infty(X) / \mathfrak{m}_x^{\ell+1}$ whose image is contained in $\mathfrak{m}_x^\ell / \mathfrak{m}_x^{\ell+1}$, that is to say, it belongs to $Q_{\mathfrak{p}^{\ell-1}}\mathcal{J}^\ell M$.

Conversely, given $\mathfrak{p}^\ell \in \mathcal{J}^\ell M$ and $D \in Q_{\mathfrak{p}^{\ell-1}}\mathcal{J}^\ell M$, its sum as mappings from $C^\infty(M)$ into $C^\infty(X) / \mathfrak{m}_x^{\ell+1}$ is an element of $\mathcal{J}^\ell M$ at $\mathfrak{p}^{\ell-1}$.

The above discussion allows us to state the analogue of Theorem 2.3.1 for jets of cross-sections of a fibred bundle:

Theorem 2.5.5. *Let $\pi\colon M \to X$ be a fibred bundle. For each $\ell \geqslant 0$ there exists a mapping*

$$Q\left(\mathcal{J}^\ell M\right) \times_{\mathcal{J}^{\ell-1}M} \mathcal{J}^\ell M \to \mathcal{J}^\ell M \tag{2.13}$$

such that the canonical projection $\mathcal{J}^\ell M \to \mathcal{J}^{\ell-1}M$ is an affine fibred bundle modelled over the vector fibre bundle

$$Q(\mathcal{J}^\ell M) \to \mathcal{J}^{\ell-1}M.$$

Furthermore, the action (2.13) *is functorial: If $\varphi\colon M \to N$ is a fibred mapping over X, the following diagram is commutative:*

$$\begin{array}{ccc} Q(\mathcal{J}^\ell M) \times_{\mathcal{J}^{\ell-1}M} \mathcal{J}^\ell M & \longrightarrow & \mathcal{J}^\ell M \\ {\scriptstyle \varphi_*\times\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ Q(\mathcal{J}^\ell N) \times_{\mathcal{J}^{\ell-1}N} \mathcal{J}^\ell N & \longrightarrow & \mathcal{J}^\ell N \end{array}$$

Remarks. (1) Let $\varphi\colon M \to N$ be a morphism of fibred bundles over X. It follows from the above theorem that, for each $\ell \geqslant 0$, $j^\ell\varphi\colon \mathcal{J}^\ell M \to \mathcal{J}^\ell N$ is a mapping between affine fibred bundles over $j^{\ell-1}\varphi$ whose associated vector bundle morphism is the restriction of φ_* to $Q(\mathcal{J}^\ell M)$.

In particular, the Taylor imbedding

$$\mathcal{J}^{\ell+r}M \to \mathcal{J}^r(\mathcal{J}^\ell M)$$

is a morphism of affine fibred bundles over the imbedding

$$\mathcal{J}^{\ell+r-1}M \to \mathcal{J}^{r-1}(\mathcal{J}^\ell M).$$

Following Goldschmidt [3] we denote by $\Delta_{\ell,r}$ its attached morphism of fibred vector bundles over $\mathcal{J}^\ell M$. The image of each vector $D_{\mathfrak{p}^{\ell+r}} \in Q_{\mathfrak{p}^{\ell+r}}\mathcal{J}^{\ell+r}M$, considered as a derivation from $C^\infty(M)$ into $\mathfrak{m}_x^{\ell+r}/\mathfrak{m}_x^{\ell+r+1}$, is its composition with the Taylor homomorphism (2.14). $\Delta_{\ell,r}$ and the fundamental identification 2.10 agree up to the factor $\frac{1}{r!}$, which makes easy to operate with this mapping.

(2) Let us fix a local coordinate system $y_1, \dots, y_m \in C^\infty(X)$ on a neighborhood of $x \in X$, vanishing at x. For each $k \geqslant 0$ the ring $C^\infty(X)/\mathfrak{m}_x^{k+1}$ is isomorphic to $\mathbb{R}[y_1, \dots, y_m]/(y_1, \dots, y_m)^{k+1}$, hence the Taylor homomorphism

$$T\colon \mathbb{R}[y_1, \dots, y_m] \to \mathbb{R}[y_1, \dots, y_m] \otimes \mathbb{R}[y_1, \dots, y_m]$$
$$y_i \to y_i \otimes 1 + 1 \otimes y_i$$

factorizes as a homomorphism

$$C^\infty(X)/\mathfrak{m}_x^{\ell+r+1} \to C^\infty(X)/\mathfrak{m}_x^{\ell+1} \otimes C^\infty(X)/\mathfrak{m}_x^{r+1}$$

whose definition depends on the local coordinates; nevertheless, its specialization to $\mathfrak{m}_x^{\ell+r}/\mathfrak{m}_x^{\ell+r+1}$.

$$\mathfrak{m}_x^{\ell+r}/\mathfrak{m}_x^{\ell+r+1} \to \mathfrak{m}_x^{\ell}/\mathfrak{m}_x^{\ell+1} \otimes \mathfrak{m}_x^{r}/\mathfrak{m}_x^{r+1}, \tag{2.14}$$

does not depend on the coordinates chosen.

References

[1] *R. J. Alonso Blanco*: D-modules, contact valued calculus and Poincaré-Cartan form. Submitted for publication.

[2] *C. Ehresmann*: Les prolongements d'une varieté différentiable. I: Calcul del jets, prolongement principal; II: L'espace des jets d'ordre r de V_n dans V_m. C. R. Acad. Sci. Paris *233* (1951), 598–600, 777–779.

[3] *H. Goldschmidt*: Integrability criteria for systems of non-linear partial differential equations. J. Differential Geom. *1* (1967), 269–307.

[4] *D. R. Grigore and D. Krupka*: Invariants of velocities, and higher order Grassmann bundles. J. Geom. Phys *24* (1998), 244–264.

[5] *N. Jacobson*: Lie Algebras. John Wiley & Sons, Inc., New York, 1962.

[6] *I. Kolář*: An infinite dimensional motivation in higher order geometry. Proc. Conf. Aug. 28–Sep. 1 1995. Brno, Czech Republic, 1996, pp. 151–159.

[7] *I. Kolář, P. W. Michor and J. Slovák*: Natural Operations in Differential Geometry. Springer-Verlag, New York, 1993.

[8] *M. Kuranishi*: Lectures on Involutive Systems. Publicações Soc. Math., São Paulo, 1969.

[9] *S. Lie*: Theorie der Transformationsgruppen. Chelsea Publishing, New York, 1970.

[10] *A. Morimoto*: Prolongation of connections to bundles of infinitely near points. J. Differential Geom. *11* (1976), 479–498.

[11] *J. Muñoz, F. J. Muriel and J. Rodríguez*: Integrability of lie equations and pseudogroups. To appear in J. Math. Anal. Appl.

[12] *V. V. Shurygin*: Manifolds over algebras and their application to the geometry of jet bundles. Russian Math. Surveys *48* (1993), 75–104.

[13] *A. Weil*: Théorie des points proches sur les variétés différentiables. Colloque de Géometrie Différentielle, C.N.R.S. (1953), 111–117.

Authors' addresses: J. Muñoz, J. Rodriguez, Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, E-37008 Salamanca, Spain, e-mails: `clint@gugu.usal.es`, `jrl@gugu.usal.es`; F. J. Muriel, Departamento de Matemáticas, Universidad de Extremadura, Avda. de la Universidad s/n, E-10004 Cáceres, Spain, e-mail: `fjmuriel@unex.es`.