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SOME TOPOLOGICAL PROPERTIES OF ω-COVERING SETS

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Abstract. We prove the following theorems:

1. There exists an ω-covering with the property s_0.
2. Under $\mathrm{cov}(\mathcal{N}) = 2^\omega$ there exists X such that $\forall_{B\in\mathcal{B}or}[B \cap X$ is not an ω-covering or $X \setminus B$ is not an ω-covering].
3. Also we characterize the property of being an ω-covering.

Keywords: ω-covering set, $\mathcal{E}$, hereditarily nonparadoxical set

MSC 2000: 03E15, 03E20, 28E15

Notation and Definitions

Our set theoretical and topological notation is standard and follows [BJ] and [E], respectively.

We denote by $\mathcal{E}$ the σ-ideal generated by closed, measure zero sets.

If H is an additive subgroup of $\mathbb{R}$ then we denote this fact by $H \leqslant \mathbb{R}$.

We will use the following well known notion of countable equidecomposability:

Definition 1. Given two sets $A, B \subseteq \mathbb{R}$, we say that A and B are countably equidecomposable if they can be partitioned into at most countably many Tr-congruent pieces (where Tr denotes the group of all translations of $\mathbb{R}$). In this case we write $A \approx_\infty B$.

Definition 2. ([P], Definition 0.1.(iii)) A set $A \subseteq \mathbb{R}$ is *paradoxical* if there are two disjoint subsets A_1 and A_2 of A such that $A_1 \approx_\infty A$ and $A_2 \approx_\infty A$.

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Definition 3. ([P], Definition 2.1.) A set $A \subseteq \mathbb{R}$ is *hereditarily nonparadoxical* if A has no uncountable paradoxical subset.

Lemma 1. ([P], Lemma 2.5) *For every subset A of $\mathbb{R}$ the following assertions are equivalent:*

1. *A is hereditarily nonparadoxical;*
2. *for every countable $G \leqslant \mathbb{R}$,*

$$|\{x \in \mathbb{R}\colon\ |Gx \cap A| = \omega\}| \leqslant \omega.$$

Definition 4. [C] Suppose κ is a cardinal. A subset X of a group G (2^ω or $\mathbb{R}$) is a κ-covering if every subset Y of G of size κ is contained in a translate of X.

We denote by UC the family of all ω-coverings.

The symbol $\mathcal{S}el(H)$ denotes the class of all selectors of the subgroup H, i.e., selectors from the class $\{x + H\colon\ x \in \mathbb{R}\}$ of cosets of H. We define also

$$\mathcal{S}el(\leqslant \omega) = \bigcup\{\mathcal{S}el(H)\colon\ H \leqslant \mathbb{R} \wedge |H| \leqslant \omega\}.$$

Recall here an old result of Marczewski: There exists (in ZFC) a set of measure zero and of the first category which is an ω-covering. P. Komjáth proved ([K2]) that assuming MA for every $\lambda < 2^\omega$ there exists a set, which is both of measure zero and the first category, and which is a λ-covering. However, these sets are Borel, so none of them has the Marczewski s_0 property. It is a natural question, whether there exists an ω-covering with the s_0 property. Clearly assuming CH or MA the answer is yes (under CH (MA) it is easy to construct a Luzin (generalized Luzin, respectively) set which is an ω-covering). We show the existence of an s_0 ω-covering in ZFC only.

Theorem 1. *There exists an ω-covering with the property s_0.*

Lemma 2. *There exists a family of disjoint Borel sets:*

$$\{B_\alpha\}_{\alpha<2^\omega}$$

such that for every $\alpha < 2^\omega$, B_α is an ω-covering.

P r o o f. Consider the topological space

$$X = \prod_{\alpha<2^\omega} \omega.$$

From the Hewitt-Marczewski-Pondiczery theorem (see [E], Theorem 2.3.15) we obtain that in this space X there exists a dense countable family $(h_n)_{n\in\omega}$ of functions

$$h_n\colon\ 2^\omega \to \omega.$$

Let

$$\{A_n\}_{n<\omega}$$

be disjoint, infinite subsets of ω. Let

$$\{a_n^{(i)}\}_{i<\omega}$$

be an increasing enumeration of elements of A_n. Let

$$\{\chi_\alpha\}_{\alpha<2^\omega}$$

be characteristic functions of all subsets of ω. We define the sets B_α in the following way:

$$B_\alpha := \{x \in 2^\omega\colon\ \exists_{n\in\omega}\forall_{m\in\omega}[h_m(\alpha) = n \to \forall_{i\in\omega} x(a_m^{(i)}) = \chi_\alpha(i)]\}.$$

We check that $B_\alpha \cap B_\beta = \emptyset$ for $\alpha \neq \beta$. To obtain a contradiction suppose that there exists $x \in B_\alpha \cap B_\beta$, and $\alpha \neq \beta$. Fix $n_\alpha, n_\beta \in \omega$ such that

$$\forall_{m\in\omega}[h_m(\alpha) = n_\alpha \to \forall_{i<\omega} x(a_m^{(i)}) = \chi_\alpha(i)],$$
$$\forall_{m\in\omega}[h_m(\beta) = n_\beta \to \forall_{i<\omega} x(a_m^{(i)}) = \chi_\beta(i)].$$

Choose $m < \omega$ such that

$$h_m(\alpha) = n_\alpha,$$
$$h_m(\beta) = n_\beta.$$

Thus

$$\forall_{i\in\omega}\chi_\alpha(i) = x(a_m^{(i)}) = \chi_\beta(i).$$

Therefore

$$\chi_\alpha = \chi_\beta,$$

which is a contradiction.

Let us check that for every $\alpha < 2^\omega$, B_α is an ω-covering.

Let $Y \subseteq 2^\omega$ be a countable set. Let $Y = \{y_l\colon\ l < \omega\}$. Fix $\alpha < 2^\omega$. For each $n \in \omega$ find $t_n\colon\ A'_n \to 2$ such that

$$(t_n + y_n)(a_m^{(i)}) = \chi_\alpha(i)$$

for each $i < \omega$ and $m < \omega$ such that $h_m(\alpha) = n$, where $A'_n = \bigcup_{m\in\{m\colon\ h_m(\alpha)=n\}} A_m$. Choose an element $t \in 2^\omega$ such that $\forall_{n\in\omega} t \restriction A'_n = t_n$. Hence $t + Y \subseteq B_\alpha$. $\square$

We will frequently use the following theorem:

Theorem 2. (see [M] Theorem 1) *Suppose $X \subseteq 2^\omega$ is an ω-covering. Then $\forall_{|Z|<2^\omega} X \setminus Z$ is an ω-covering.*

Next we will modify the classical construction of an s_0 set with the cardinality 2^ω. Let

$$\{P_\alpha\}_{\alpha<2^\omega}$$

be an enumeration of all perfect sets such that

$$\forall_{\beta<2^\omega} |B_\beta \cap P_\alpha| \leqslant \omega.$$

Let

$$\{C_\alpha\}_{\alpha<2^\omega}$$

be an enumeration of all sets from $[2^\omega]^\omega$. Assume that the numbers $\{s_\alpha\}_{\alpha<\theta}$ are defined. Lemma 2 now yields $\exists_{s\in 2^\omega} C_\theta + s \subseteq B_\theta \setminus \bigcup\limits_{\mu<\theta} P_\mu$. Take an $s_\theta \in 2^\omega$ such that $C_\theta + s_\theta \subseteq B_\theta \setminus \bigcup\limits_{\mu<\theta} P_\mu$. Define

$$S = \bigcup_{\alpha<2^\omega} C_\alpha + s_\alpha.$$

It is easy to see that S is an s_0 set. Indeed, let P be a perfect set. We consider two cases:

If $P = P_\theta$ for some $\theta < 2^\omega$ then $|P \cap S| < 2^\omega$, so one can find a perfect subset of P disjoint with S.

If $|P \cap B_\theta| > \omega$ for some $\theta < 2^\omega$ then we have $|S \cap B_\theta| \leqslant \omega$ so one can find a perfect subset of P disjoint with S.

K. Muthuvel proved (see [M] Theorem 1) that if $X \in \mathrm{UC}$ and F is a measure zero or a first category additive subgroup of the reals $\mathbb{R}$, or $|F| < 2^\omega$, then $A \setminus F \in \mathrm{UC}$. In the next theorem we characterize sets F with this property.

Theorem 3. *Suppose A is a set of real numbers. The following conditions are equivalent:*

$$\forall_{X\in\mathrm{UC}} X \setminus A \in \mathrm{UC}, \tag{1}$$

$$\forall_{G\leqslant\mathbb{R}} \big[|G| \leqslant \omega\big] \Rightarrow G + A \notin \mathrm{UC}, \tag{2}$$

$$\forall_{C\subseteq\mathbb{R}} \big[|C| \leqslant \omega\big] \Rightarrow C + A \notin \mathrm{UC}. \tag{3}$$

P r o o f. (1) $\Rightarrow$ (2) Let $G \leqslant \mathbb{R}$ be a countable subgroup of $\mathbb{R}$. To obtain a contradiction, suppose that $G+A \in \mathrm{UC}$. From (1) we obtain that $(A+G)\setminus A \in \mathrm{UC}$. Thus there exists $t \in \mathbb{R}$ such that $G+t \subseteq (A+G)\setminus A$. Therefore, $t \in (A+G)-G = A$, a contradiction.

(3) $\Rightarrow$ (2) The proof is immediate.

(2) $\Rightarrow$ (1) Suppose $A \subseteq \mathbb{R}$ is such that

$$\forall_{G\leqslant\mathbb{R}}\big[|G| \leqslant \omega\big] \Rightarrow A+G \notin \mathrm{UC}$$

and $X \in \mathrm{UC}$. It suffices to show that for every $H \leqslant R$, $|H| \leqslant \omega$ there exists $s \in \mathbb{R}$ such that $s+H \subseteq X \setminus A$. Let $H \leqslant \mathbb{R}$, $|H| \leqslant \omega$. By assumption, $H+A \notin \mathrm{UC}$. Hence there exists $|G| \leqslant \omega$, $G \leqslant \mathbb{R}$ such that

$$\forall_t G+t \not\subseteq H+A. \tag{4}$$

Since $X \in \mathrm{UC}$, we can find $s_0 \in \mathbb{R}$ such that $s_0+H+G \subseteq X$. From (4) we see that there exists $g_0 \in G$ such that

$$g_0+s_0 \notin H+A. \tag{5}$$

We show that

$$s_0+g_0+H \subseteq X \setminus A. \tag{6}$$

Observe that $s_0+g_0+H \subseteq X$. Let $y \in s_0+g_0+H$. Thus, there is $h \in H$ such that $y = s_0+g_0+h$. As $g_0+s_0 \notin H+A$, we have $g_0+s_0+h \notin A$. This establishes the formula (6). $\square$.

(2) $\Rightarrow$ (3) Let $C \subseteq \mathbb{R}$, $|C| \leqslant \omega$. Define $G = \langle C\rangle$ (additive subgroup generated by C). Then $|G| \leqslant \omega$. From the assumption (2) we have $G+A \notin \mathrm{UC}$. Observe that $C+A \subseteq G+A$. Thus $C+A \notin \mathrm{UC}$, which completes the proof of (2) $\Rightarrow$ (3). $\square$

Corollary 1. *Suppose $H \leqslant \mathbb{R}$ and $|\mathbb{R}/H| > \omega$. Then $\forall_{X\in\mathrm{UC}} X \setminus H \in \mathrm{UC}$.*

P r o o f. If we prove that $G+H \notin \mathrm{UC}$ for every countable $G \leqslant \mathbb{R}$, the assertion follows. Let $G \leqslant \mathbb{R}$ be a countable subgroup of $\mathbb{R}$. Therefore $G+H$ is a subgroup of $\mathbb{R}$. Note that $G+H \neq \mathbb{R}$ by $|\mathbb{R}/H| > \omega$. To see that $G+H \notin \mathrm{UC}$ take any $x \in G+H$, $y \notin G+H$. Hence we conclude that there exists no t such that $t+\{x,y\} \subseteq G+H$. Thus $G+H \notin \mathrm{UC}$. This completes the proof of Corollary 1. $\square$

Corollary 2. *Suppose that $A \subseteq \mathbb{R}$ is such that $\forall_{X\in\mathrm{UC}} X \setminus A \in \mathrm{UC}$. Suppose that $B \approx_\infty A$. Then also $\forall_{X\in\mathrm{UC}} X \setminus B \in \mathrm{UC}$.*

P r o o f. Let $A = \bigcup_{n<\omega} A_n$, where $(A_n)_{n<\omega}$ pairwise disjoint. Let $(r_n)_{n<\omega}$ be a sequence of real numbers such that $B = \bigcup_{n<\omega} A_n + r_n$ and the sets $\{A_n + r_n\}_{n<\omega}$ are pairwise disjoint. We define $G = \langle\{r_n\colon\ n \in \omega\}\rangle$. We will start with showing that $G + A = G + B$. Let $g \in G$ and $a \in A$. Then there is $n \in \omega$ such that $a \in A_n$. Thus $g + r_n + a \in G + B$. But this implies that $g + a \in G + B$. On the other hand, if $g \in G$ and $b \in B$, then there is $n \in \omega$ such that $b \in B_n$. Therefore $g - r_n + b \in G + A$. But this implies that $g + b \in G + A$. We shall have established Corollary 2 if we prove

$$\forall_{H \leqslant \mathbb{R}} |H| \leqslant \omega \Rightarrow H + B \in \mathrm{UC}.$$

Let $H \leqslant \mathbb{R}$ be a countable subgroup of $\mathbb{R}$. Observe that $A + (H + G) = (A + G) + H = (B + G) + H = B + (H + G)$. It is evident that $|H + G| \leqslant \omega$ and $H + G \leqslant \mathbb{R}$. But this implies that $A + (H + G) \notin \mathrm{UC}$, so $B + (H + G) \notin \mathrm{UC}$ and finally $B + H \notin \mathrm{UC}$, proving Corollary 2. $\square$

Theorem 4. *Suppose $X \subseteq \mathbb{R}$. The following conditions are equivalent:*

$$X \in \mathrm{UC}, \tag{7}$$

$$\forall_{S \in \mathcal{S}el(\leqslant\omega)} S \cap X \neq \emptyset. \tag{8}$$

P r o o f. (7) $\Rightarrow$ (8) Let $S \in \mathcal{S}el(H)$, where $H \leqslant \mathbb{R}$. Suppose, contrary to our claim, that $S \cap X = \emptyset$. Hence $(t + H) \cap S \neq \emptyset$ for every $t \in \mathbb{R}$. Thus $t + H \nsubseteq X$ for every t. This contradicts our assumption (7). This completes the proof of (7) $\Rightarrow$ (8).

(8) $\Rightarrow$ (7). It is sufficient to show that for every countable subgroup H of $\mathbb{R}$ there exists t such that $H + t \subseteq X$. To obtain a contradiction, suppose that there exists a countable subgroup $H \leqslant \mathbb{R}$ such that $\forall_{t\in\mathbb{R}} \exists_{s\in\mathbb{R}} s \in (H + t) \setminus X$. From this we see that there exists $S \in \mathcal{S}el(H)$ such that $S \cap X = \emptyset$, contrary to our assumption (8). $\square$

Theorem 5. *Let $X \subseteq \mathbb{R}$ be a hereditarily nonparadoxical set. Then $\forall_{Y\in\mathrm{UC}} Y \setminus X \in \mathrm{UC}$. Thus, in particular, no hereditarily nonparadoxical set is an ω-covering set.*

P r o o f. It is sufficient to prove that for every countable $H \leqslant \mathbb{R}$, $H + A \notin \mathrm{UC}$. Let $H \leqslant \mathbb{R}$ be a countable subgroup of $\mathbb{R}$. Choose $H' \leqslant \mathbb{R}$ such that $|H'| = \omega$ and $H \cap H' = \{0\}$. Define $H_1 = H + H'$. We first prove

$$\{x_0\colon\ (x_0 + H_1) \nsubseteq X + H\} \supseteq \{x_0\colon\ |(x_0 + H_1) \cap X| < \omega\}. \tag{9}$$

Suppose, contrary to (9), that there exists x_0 such that $|(x_0 + H_1) \cap X| < \omega$ and $x_0 + H_1 \subseteq X + H$. For every $h \in H'$ find $x_h \in X$, $k_h \in H$ such that $x_0 + h = x_h + k_h$.

Suppose that $h, g \in H'$ and that $x_g = x_h$. Therefore

$$\begin{cases} x_0 + h = x_h + k_h, \\ x_0 + g = x_g + k_g. \end{cases}$$

Thus $h - g = k_h - k_g$, $h - g \in H'$ and $k_h - k_g \in H$. Since $H \cap H' = \{0\}$, the last equality shows that $h = g$. By assumption, $|H'| = \omega$. Hence $|(x_0 + H' - H) \cap X| = \omega$, a contradiction. This establishes the inclusion (9). By assumption, X is a hereditarily nonparadoxical set. Therefore $|\{x_0\colon\ |(x_0 + H_1) \cap X| = \omega\}| \leqslant \omega$.

It follows from (9) that $|\{x_0\colon\ x_0 + H_1 \subseteq X + H\}| \leqslant \omega$. Suppose that $H + X \in \mathrm{UC}$. By Theorem 1 from [M],

$$H + X \setminus \{x_0\colon\ x_0 + H_1 \subseteq X + H\} \in \mathrm{UC}.$$

Then there is $x_1 \in \mathbb{R}$ such that

$$x_1 + H_1 \subseteq H + X \setminus \{x_0\colon\ x_0 + H_1 \subseteq X + H\}.$$

Thus $x_1 \in \{x_0\colon\ x_0 + H_1 \subseteq X + H\}$, which is impossible. This completes the proof of Theorem 5. □

Theorem 6. *Assume* $\mathrm{cov}(\mathcal{M}) = 2^\omega$ $(\mathrm{cov}(\mathcal{N}) = 2^\omega)$. *Then there exists* X, *a generalized Luzin set (Sierpiński set) such that*

$$\forall_{B \in \mathcal{B}or} B \cap X \notin \mathrm{UC} \vee X \setminus B \notin \mathrm{UC}.$$

P r o o f. We give the proof only for the case of a generalized Sierpiński set; the other case is similar. Assume $\mathrm{cov}(\mathcal{N}) = 2^\omega$. Let $(C_\theta)_{\theta < 2^\omega}$ be an enumeration of all countable sets in $\mathbb{R}$. Let $(B_\theta)_{\theta < 2^\omega}$ be an enumeration of all Borel sets in $\mathbb{R}$. Now define by induction a sequence $(t_\theta)_{\theta < 2^\omega}$ of real numbers and a sequence $(Z_\theta)_{\theta < 2^\omega}$ of measure zero sets in the following way. Assume that the sets $(Z_\alpha)_{\alpha < \theta}$ and the real numbers $(t_\alpha)_{\alpha < \theta}$ are defined.

Consider two cases:

Case 1

$$\mu(B_\theta) > 0. \tag{10}$$

We first observe that $\{x\colon\ x + \mathbb{Q} \subseteq B^c\} = (B + \mathbb{Q})^c$ for every $B \subseteq \mathbb{R}$. From this we obtain $(B_\theta + \mathbb{Q})^c \in \mathcal{N}$ (this follows easily from the Steinhaus property of the

Lebesgue measure). Define $Z_\theta = (B_\theta + \mathbb{Q})^c$. By the assumption $\operatorname{cov}(\mathcal{M}) = 2^\omega$, there exists t_θ such that $(C_\theta + t_\theta) \cap \bigcup_{\alpha \leqslant \theta} Z_\alpha = \emptyset$.

Case 2

$$\mu(B_\theta^c) > 0. \tag{11}$$

From this we obtain $(B_\theta^c + \mathbb{Q})^c \in \mathcal{N}$. Define $Z_\theta = (B_\theta^c + \mathbb{Q})^c$. Thus there exists $t_\theta \in \mathbb{R}$ such that $(C_\theta + t_\theta) \cap \bigcup_{\alpha \leqslant \theta} Z_\alpha = \emptyset$.

We define $X = \bigcup_{\theta < 2^\omega} (C_\theta + t_\theta)$. Obviously, $X \in \mathrm{UC}$.

We shall now show that for every $\theta \in 2^\omega$, $X \setminus B_\theta \notin \mathrm{UC}$ or $X \cap B_\theta \notin \mathrm{UC}$.

Consider an arbitrary $\theta < 2^\omega$.

Case 1

$$\mu(B_\theta) > 0. \tag{12}$$

Thus

$$\begin{aligned}
\{x\colon\ x + \mathbb{Q} \subseteq X \setminus B_\theta\} &= \{x\colon\ x + \mathbb{Q} \subseteq X\} \cap \{x\colon\ x + \mathbb{Q} \subseteq B_\theta^c\} \\
&= \{x\colon\ x + \mathbb{Q} \subseteq X\} \cap (B_\theta + \mathbb{Q})^c \\
&= \{x\colon\ x + \mathbb{Q} \subseteq X\} \cap Z_\theta \\
&\subseteq X \cap Z_\theta \subseteq \bigcup_{\alpha \leqslant \theta} C_\alpha + t_\alpha.
\end{aligned}$$

Therefore $|\{x\colon\ x + \mathbb{Q} \subseteq X \setminus B_\theta\}| < 2^\omega$. Suppose $X \setminus B_\theta \in \mathrm{UC}$. Then by Theorem 1 from [M] we have $(X \setminus B_\theta) \setminus \{x\colon\ x + \mathbb{Q} \subseteq X \setminus B_\theta\} \in \mathrm{UC}$. Then there is $x_0 \in \mathbb{R}$ such that $x_0 + \mathbb{Q} \subseteq (X \setminus B_\theta) \setminus \{x\colon\ x + \mathbb{Q} \subseteq X \setminus B_\theta\}$, which is impossible. Hence $X \setminus B_\theta \notin \mathrm{UC}$.

Case 2

$$\mu(B_\theta^c) > 0. \tag{13}$$

Thus

$$\begin{aligned}
&\{x\colon\ x + \mathbb{Q} \subseteq X \cap B_\theta\} \\
&\quad = \{x\colon\ x + \mathbb{Q} \subseteq X\} \cap \{x\colon\ x + \mathbb{Q} \subseteq B_\theta\} \\
&\quad = \{x\colon\ x + \mathbb{Q} \subseteq X\} \cap (B_\theta^c + \mathbb{Q})^c \\
&\quad = \{x\colon\ x + \mathbb{Q} \subseteq X\} \cap Z_\theta \subseteq X \cap Z_\theta \\
&\quad \subseteq \bigcup_{\alpha \leqslant \theta} C_\alpha + t_\alpha.
\end{aligned}$$

Therefore $|\{x\colon\ x+\mathbb{Q} \subseteq X \cap B_\theta\}| < 2^\omega$. Suppose $X \cap B_\theta \in \mathrm{UC}$. Then by Theorem 1 from [M]

$$(X \cap B_\theta) \setminus \{x\colon\ x + \mathbb{Q} \subseteq X \cap B_\theta\} \in \mathrm{UC}.$$

Then there is $x_0 \in \mathbb{R}$ such that $x_0 + \mathbb{Q} \subseteq (X \cap B_\theta) \setminus \{x\colon\ x + \mathbb{Q} \subseteq X \cap B_\theta\}$, which is impossible. Hence $X \cap B_\theta \notin \mathrm{UC}$. It remains to prove that X is a generalized Sierpiński set. To show it let $N \in \mathcal{N}$ be a Borel set. Thus $\mathbb{Q} + N \in \mathcal{N} \cap \mathcal{B}or$. Therefore there exists $\theta \in 2^\omega$ such that $\mathbb{Q} + N = B_\theta$. Note that $\mu(B_\theta^c) > 0$. Using the definition of the set Z_θ, we get $Z_\theta = (B_\theta^c + Q)^c$, i.e. $Z_\theta = ((\mathbb{Q} + N)^c + \mathbb{Q})^c$.

Claim 3. $\big((\mathbb{Q} + N)^c + \mathbb{Q}\big)^c = \mathbb{Q} + N.$

Indeed, let $q_1 + n_1 \in \mathbb{Q} + N$, $q_1 \in \mathbb{Q}$, $n_1 \in N$. Suppose that $q_1 + n_1 = q_2 + m_2$ for some $m_2 \in (\mathbb{Q} + N)^c$, $q_2 \in \mathbb{Q}$. Therefore $m_2 = n_1 + (q_1 - q_2) \in N + \mathbb{Q}$, which is a contradiction. On the other hand, $(\mathbb{Q} + N)^c \subseteq (\mathbb{Q} + N)^c + \mathbb{Q}$. Hence $\big((\mathbb{Q} + N)^c + \mathbb{Q}\big)^c \subseteq \mathbb{Q} + N$. This proves Claim 1. As a consequence we have $Z_\theta = \mathbb{Q} + N$.

Since $Z_\theta = \mathbb{Q} + N$, it follows by the construction of X that $X \cap (\mathbb{Q} + N) \subseteq \bigcup\limits_{\alpha \leqslant \theta} C_\alpha + t_\alpha$. Since

$$\Big|\bigcup_{\alpha \leqslant \theta} C_\alpha + t_\alpha\Big| < 2^\omega,$$

it follows that $|X \cap (\mathbb{Q} + N)| < 2^\omega$. Thus $|X \cap N| < 2^\omega$. Note that $|X| = 2^\omega$. This completes the proof of Theorem 6. □

The following theorem can be found in [BJ]:

Theorem 7. (Theorem 6.3 [BJ]) *There exists a measure zero set $H \subseteq 2^\omega$ such that for every perfect set P, if $P + H \in \mathcal{N}$ then $\exists_{x \in 2^\omega} P + x \subseteq H$.*

In our next theorem we show that there is no such set $E \in \mathcal{E}$.

Theorem 8. *There is no $E \in \mathcal{E}$ such that*

$$\forall_{Q \in \mathrm{Perf}} Q + E \in \mathcal{N} \Rightarrow \exists_{t \in \mathbb{R}} Q + t \subseteq E. \tag{14}$$

P r o o f. We may assume that $E = \bigcup\limits_{n<\omega} K_n$, where K_n are compact, nowhere dense. We have the following lemma:

Lemma 4. *Let $K \subseteq \mathbb{R}$ be a compact, nowhere dense set, and suppose that $I \subseteq \mathbb{R}$ is an open interval. Then there are pairwise disjoint intervals $I_0, \ldots, I_k \subseteq I$ such that*

$$\forall_{t \in \mathbb{R}} \exists_{0 \leqslant i \leqslant k} I_i \cap (K + t) = \emptyset. \tag{15}$$

P r o o f. First note that there is a compact interval L such that $\forall_{t\in\mathbb{R}}(K+t)\cap I\neq\emptyset\Rightarrow t\in L$.

For every $x\in L$ there is an open set $U_x\ni x$ and a closed subinterval $I_x\subseteq I$ such that $(U_x+K)\cap I_x=\emptyset$. By the compactness of L we can choose numbers $\{x_i\}_{i=1}^{k}\subseteq L$ such that $\bigcup\limits_{i=1}^{k} U_{x_i}\supseteq L$. We may assume (after shrinking $I_{x_1},\ldots,I_{x_k}$, if necessary) that $I_{x_1},\ldots,I_{x_k}$ are pairwise disjoint. It is easy to check that (15) is satisfied. This completes the proof of Lemma 3. □

Choose an enumeration $(n_k)_{k\in\omega}=\omega$ such that for each $n\in\omega$,

$$|\{k\colon\ n_k=n\}|=\omega.$$

We will construct a system of closed intervals as follows:

0. Set $I_\emptyset$—any closed interval.

1. From Lemma 3 we see that there are pairwise disjoint closed intervals $I_{\langle 1\rangle},\ldots,I_{\langle k_0\rangle}\subseteq I_\emptyset$ such that

$$\forall_{t\in\mathbb{R}}\exists_{1\leqslant i\leqslant k_0} I_{\langle i\rangle}\cap(K_{n_0}+t)=\emptyset.$$

Without loss of generality we may assume (after shrinking $I_{\langle 1\rangle},\ldots,I_{\langle k_0\rangle}$, if necessary) that

$$\mu\Big[\Big(\bigcup_{i=1,\ldots,k_0} I_{\langle i\rangle}\Big)+K_{n_0}\Big]\leqslant\frac{1}{0+1}.$$

2. Again from Lemma 3 we see that for each $i\in\{1,\ldots k_0\}$ there are pairwise disjoint closed intervals $I_{\langle i,1\rangle},\ldots,I_{\langle i,k_1\rangle}\subseteq I_{\langle i\rangle}$ (we may assume that k_1 is the same for different i) such that

$$\forall_{t\in\mathbb{R}}\exists_{1\leqslant j\leqslant k_1} I_{\langle i,j\rangle}\cap(K_{n_1}+t)=\emptyset.$$

Without loss of generality we may assume (after shrinking $I_{\langle i,j\rangle}$,if necessary) that

$$\mu\Big[\bigcup_{i=1,\ldots,k_0}\ \bigcup_{j=1,\ldots,k_1} I_{\langle i,j\rangle}+K_{n_1}\Big]\leqslant\frac{1}{1+1}.$$

In general:

1+2. From Lemma 3 we see that for each

$$(i_0,\ldots,i_l)\in\{1,\ldots,k_0\}\times\{1,\ldots,k_1\}\times\ldots\times\{1,\ldots,k_l\}$$

there are pairwise disjoint intervals $I_{\langle i_0,\dots,i_l,1\rangle},\dots,I_{\langle i_0,\dots,i_l,k_{l+1}\rangle}$ (we may assume that k_{l+l} is the same for different i) such that

$$\forall_{t\in\mathbb{R}}\exists_{1\leqslant j\leqslant k_{l+1}} I_{\langle i_0,\dots,i_l,j\rangle}\cap(K_{n_{l+1}}+t)=\emptyset.$$

We may assume (after shrinking $I_{\langle i_0,\dots,i_l,j\rangle}$, if necessary) that

$$\mu\Big[\bigcup_{i_0=1,\dots,k_0}\ \bigcup_{i_1=1,\dots,k_1}\cdots\bigcup_{i_{l+1}=1,\dots,k_{l+1}} I_{\langle i_0,\dots,i_{l+1}\rangle}+K_{n_{l+1}}\Big]\leqslant\frac{1}{(l+1)+1}.$$

Define $H=\prod_{i=0}^{\infty}\{1,\dots,k_l\}$ and

$$Q=\bigcup_{x\in H}\bigcap_{n=0}^{\infty} I_{x\restriction n}.$$

It is clear that Q is a perfect set. We show that Q is as required:

A. Let $m\in\omega$, then $\exists_l^\infty n_l=m$. By the construction of $I_{\langle i_0,\dots,i_l\rangle}$,

$$Q\subseteq\bigcup_{i_0=1,\dots,k_0}\cdots\bigcup_{i_l=1,\dots,k_l} I_{\langle i_0,\dots,i_l\rangle}.$$

Thus $Q+K_{n_l}\subseteq\Big[\bigcup_{i_0=1,\dots,k_0}\cdots\bigcup_{i_l=1,\dots,k_l} I_{\langle i_0,\dots,i_l\rangle}+K_{n_l}\Big]$. Therefore $\mu(Q+K_{n_l})\leqslant\frac{1}{l+1}$. Note that we have actually proved that for each $l\in\{l\colon n_l=m\}$, $\mu(Q+K_m)\leqslant\frac{1}{l+1}$. Thus $\mu(Q+K_m)=0$. This completes the proof of $E+Q\in\mathcal{N}$.

B. To obtain a contradiction, suppose that there exists $t_0\in\mathbb{R}$ such that $Q+t_0\subseteq\bigcup_{n\in\omega}K_n$. Since $\{K_n\}_{n\in\omega}$ are closed, we conclude that there is an open set W and a natural number $m\in\omega$ such that $W\cap Q\neq\emptyset$ and

$$(W\cap Q)+t_0\subseteq K_m. \tag{16}$$

Thus there exists an interval $I_{i_0,\dots,i_l}$ such that

$$Q\cap I_{i_0,\dots,i_l}\subseteq Q\cap W. \tag{17}$$

Therefore there is an interval $I_{i_0,\dots,i_l,\dots,i_p}\subseteq I_{i_0,\dots,i_l}$ such that $n_{p+1}=m$. From the construction of the intervals $\{I_{i_0,\dots,i_p}\}_{0\leqslant j\leqslant k_{p+1}}$ we see that there exists $1\leqslant j'\leqslant k_{p+1}$ such that

$$I_{i_0,\dots,i_p,j'}\cap(K_{n_{p+1}}-t_0)=\emptyset. \tag{18}$$

But $n_{p+1} = m$, thus

$$I_{i_0,\ldots,i_p,j'} \cap (K_m - t_0) = \emptyset. \tag{19}$$

Note that

$$Q \cap I_{i_0,\ldots,i_p,j'} \subseteq Q \cap W. \tag{20}$$

Therefore

$$(I_{i_0,\ldots,i_p,j'} + t_0) \cap K_m = \emptyset. \tag{21}$$

From (16) we obtain

$$(Q \cap I_{i_0,\ldots,i_p,j'}) + t_0 \subseteq K_m. \tag{22}$$

Thus $Q \cap I_{i_0,\ldots,i_p,j'} = \emptyset$, contrary to the definition of the set Q. This completes the proof of Theorem 8. □

Theorem 9. *There exists a set $E \in \mathcal{E}$ such that $\forall_{N \in \mathcal{N}^*} \exists_t N + t \subseteq E$.*

P r o o f. For each $n \in \omega$ pick $\beta_n \in \omega$ such that

$$\begin{cases} n | \beta_n \\ 2^{-\beta_n} < \frac{2^{-n^2}}{n^n}. \end{cases}$$

Let $(I_n)_{n\in\omega}$ be any partition of ω into finite, disjoint intervals such that $\forall_{n\in\omega} |I_n| = \beta_n$. For each $n \in \omega$ divide I_n into pairwise disjoint intervals of size $\frac{\beta_n}{n}$, $\{J_i^{(n)}\colon 1 \leqslant i \leqslant n\}$. Put

$$E = \{x\colon \forall_n^\infty \exists_{1\leqslant i \leqslant n} x \restriction J_i^{(n)} = 0 \restriction J_i^{(n)}\}.$$

First observe that $E \in F_\sigma$. Define

$$H_n = \{u \in 2^{I_n}\colon \exists_{1\leqslant i\leqslant n} u \restriction J_i^{(n)} = 0 \restriction J_i^{(n)}\}. \tag{23}$$

By (23) we have $|H_n| \leqslant \sum_{i=1}^{n} |\{u \in 2^{I_n}\colon u \restriction J_i^{(n)} = 0 \restriction J_i^{(n)}\}| = \sum_{i=1}^{n} \frac{2^{|I_n|}}{2^{|J_i^{(n)}|}} = n \cdot \frac{2^{\beta_n}}{2^{\frac{\beta_n}{n}}} = n \cdot 2^{\beta_n\left(\frac{n-1}{n}\right)}$. Therefore $\sum_{n=1}^{\infty} \frac{|H_n|}{2^{|I_n|}} \leqslant \sum_{n=1}^{\infty} \frac{n\cdot 2^{\beta_n\left(\frac{n-1}{n}\right)}}{2^{\beta_n}} = \sum_{n=1}^{\infty} n \cdot 2^{-\frac{\beta_n}{n}} = \sum_{n=1}^{\infty} n \cdot (2^{-\beta_n})^{\frac{1}{n}} \leqslant \sum_{n=1}^{\infty} n \cdot \left(\frac{2^{-n^2}}{n^n}\right)^{\frac{1}{n}} = \sum_{n=1}^{\infty} n \cdot \frac{2^{-n}}{n} < \infty$. Since $E = \{x\colon \forall_n^\infty x \restriction I_n \in H_n\}$ we have $E \in \mathcal{N}$. Thus $E \in \mathcal{E}$.

We show that E is as required. Let $N \in \mathcal{N}^*$. By Theorem 3.2 from [BJ] there exists a sequence $(T_n)_{n\in\omega}$ such that $\forall_n |T_n| \leqslant n$ and

$$N \subseteq \{x\colon \forall_n^\infty x \restriction I_n \in T_n\}. \tag{24}$$

Let $T_n = \{t_1^{(n)}, \dots, t_n^{(n)}\}$. Pick $t \in 2^\omega$ such that

$$\forall_n \forall_{1\leqslant i\leqslant n} t \restriction J_i^{(n)} = t_i^{(n)} \restriction J_i^{(n)}. \tag{25}$$

To complete the proof it is enough to show that $N + t \subseteq E$. Let $x \in N$. From (24) we conclude that $\forall_n^\infty x \restriction I_n \in T_n$, i.e. $\forall_{n>N_0} x \restriction I_n \in T_n$ for some $N_0 \in \omega$. Let $n > N_0$. Then $x \restriction I_n \in T_n$. So there is $1 \leqslant i \leqslant n$ such that $x \restriction I_n = t_i^{(n)}$. By the definition of t, we know that $t \restriction J_i^{(n)} = t_i^{(n)} \restriction J_i^{(n)}$. Thus $(t + x) \restriction J_i^{(n)} = 0 \restriction J_i^{(n)}$. Therefore, $\forall_{n>N_0} \exists_{1\leqslant i\leqslant n} (t + x) \restriction J_i^{(n)} = 0 \restriction J_i^{(n)}$. Thus, we have shown that $\forall_{x\in N} x + t \in E$. This completes the proof of the theorem. □

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