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LINEARIZED OSCILLATION RESULTS FOR EVEN-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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1. Introduction

Consider the even-order nonlinear neutral equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(x(t) - P(t)g\big(x(t-\tau)\big)\big) - Q(t)h\big(x(t-\sigma)\big) = 0 \tag{1}$$

where

$$P, Q \in C\big([t_0, \infty), \mathbb{R}\big), \quad g, h \in C(\mathbb{R}, \mathbb{R}) \quad \text{and } \tau > 0,\ \sigma \geqslant 0. \tag{2}$$

Recently, a linearized oscillation result for Eq. (1) has been established by Ladas et al. [3, 4]; for some further results, we refer to [1, 2, 5–7]. As we see in [3, 4], it seems that the

$$\operatorname*{Lim\,sup}_{t\to\infty} P(t) = P_0 \in (0,1), \quad \operatorname*{Lim\,inf}_{t\to\infty} P(t) = p_0 \in (0,1)$$

is always assumed to hold. However, the case $P(t) < 0$ or $P(t) \geqslant 1$ has not yet been handled. Therefore, Györi and Ladas put forth the following open problem in [4, problem 10.10.4]: Obtain linearized oscillation results for Eq. (1) when the coefficients $P(t) < 0$ for $t \geqslant t_0$ or $P(t) \geqslant 1$ for $t \geqslant t_0$.

Our aim in this paper is to answer the above problem when $P(t) \leqslant -1$ for $t \geqslant t_0$ and $P(t) \geqslant 1$ for $t \geqslant t_0$. Our main results are the following two theorems:

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Theorem 1. *Assume that* (2) *holds and that*

$$\operatorname*{Lim\,sup}_{t\to\infty} P(t) = -P_0 \in (-\infty, -1), \quad \operatorname*{Lim\,inf}_{t\to\infty} P(t) = -p_0 \in (-\infty, -1), \tag{3}$$

$$\operatorname*{Lim}_{t\to\infty} Q(t) = q \in (0, \infty), \tag{4}$$

$$\frac{g(u)}{u} \geqslant 1 \quad \text{for } u \neq 0 \quad \text{and} \quad \operatorname*{Lim}_{u\to 0} \frac{g(u)}{u} = 1, \tag{5}$$

$$uh(u) > 0 \quad \text{for } u \neq 0 \quad \text{and} \quad \operatorname*{Lim}_{u\to 0} \big(h(u)/u\big) = 1. \tag{6}$$

If every bounded solution of the linear equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(y(t) + p_0 y(t-\tau)\big) - q y(t-\sigma) = 0 \tag{7}$$

oscillates, then every bounded solution of Eq. (1) *also oscillates.*

Theorem 2. *Assume that* (2) *and* (4) *hold and that*

$$\operatorname*{Lim\,sup}_{t\to\infty} P(t) = P_0 \in (1, \infty), \quad \operatorname*{Lim\,inf}_{t\to\infty} P(t) = p_0 \in (1, \infty), \tag{8}$$

$$g(u)/u \geqslant 1 \quad \text{for } u \neq 0, \tag{9}$$

$$uh(u) > 0 \quad \text{for } u \neq 0 \quad \text{and} \quad \operatorname*{Lim\,inf}_{|u|\to\infty} |h(u)| > 0. \tag{10}$$

Then every bounded solution of Eq. (1) *oscillates.*

The proof of Theorems 1 and 2 will be given in Section 2.

Let $\varrho = \max\{\tau, \sigma\}$. By a solution of Eq. (1) we mean a function $x \in C\big([t_1 - \varrho, \infty), \mathbb{R}\big)$ for some $t_1 \geqslant t_0$, such that $x(t) - P(t)g\big(x(t-\tau)\big)$ is n times continuously differentiable on $[t_1, \infty)$ and (1) is satisfied for $t \geqslant t_1$.

Let $t_1 \geqslant t_0$ and let $\varphi \in C([t_1 - \varrho, t_1], \mathbb{R})$ be a given initial function, and let z_k, $k = 0, 1, \ldots, n-1$, be given initial constants. Using the method of steps one can see that Eq. (1) has a unique solution $x \in C\big([t_1 - \varrho, \infty), \mathbb{R}\big)$ such that

$$x(t) = \varphi(t) \quad \text{for } t \in [t_1 - \varrho, t_1]$$

and

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(\varphi(t) - P(t)g\big(\varphi(t-\tau)\big)\big)_{t=t_1} = z_k \quad \text{for } k = 0, 1, 2, \ldots, n-1.$$

As usual, a solution of Eq. (1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In the sequel, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Proof of Theorems 1 and 2

The following lemmas will be useful in the proof of Theorem 1.

Lemma 1. *Let n be even and assume that*

$$p \in (1, \infty), \quad \tau, q \in (0, \infty) \quad \text{and } \sigma \in [0, \infty). \tag{11}$$

If every bounded solution of the linear equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(x(t) + px(t-\tau)\big) - qx(t-\sigma) = 0 \tag{12}$$

oscillates, then there exists an $\varepsilon \in (0, q)$ such that every bounded solution of the equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(x(t) + (p+\varepsilon)x(t-\tau)\big) - (q-\varepsilon)x(t-\sigma) = 0 \tag{13}$$

also oscillates.

Proof. By Lemma 4 in [3], the hypothesis that every bounded solution of Eq. (12) oscillates implies that the characteristic equation of Eq. (12),

$$f(\lambda) = \lambda^n + p \cdot \lambda^n \mathrm{e}^{-\lambda t} - q\mathrm{e}^{-\lambda\sigma} = 0$$

has no real roots in $(-\infty, 0)$. This and $f(0) = -q < 0$ imply that

$$f(\lambda) < 0 \quad \text{for all } \lambda \in (-\infty, 0]$$

and hence $\tau < \sigma$. Clearly, $f(-\infty) = -\infty$ and so

$$f(\lambda) \leqslant \sup_{\xi \in (-\infty, 0]} f(\xi) := m < 0 \quad \text{for all } \lambda \in (-\infty, 0].$$

Next we set

$$\delta = \tfrac{1}{3}q \quad \text{and} \quad g(\lambda) = \delta(-\lambda^n \mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda\sigma}).$$

Then it is easy to see that

$$f(\lambda) - q(\lambda) = \lambda^n\big(1 + (p+\delta)\mathrm{e}^{-\lambda t}\big) - (q-\delta)\mathrm{e}^{-\lambda\sigma} \to -\infty \quad \text{as } \lambda \to -\infty,$$

which implies that there exists a $\lambda_0 < 0$ such that

$$f(\lambda) - q(\lambda) \leqslant \tfrac{1}{2}m \quad \text{for } \lambda \leqslant \lambda_0.$$

Let

$$\mu = \sup_{\lambda \in [\lambda_0, 0]} (\lambda^n \mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-\lambda\sigma})$$

and set

$$\varepsilon = \min\{\delta, -\tfrac{1}{2} m\mu\}.$$

To complete the proof, by Lemma 4 in [3] it suffices to show that the characteristic equation

$$\lambda^n + (p+\varepsilon)\lambda^n \mathrm{e}^{-\lambda\tau} - (q-\varepsilon)\mathrm{e}^{-\lambda\sigma} = 0 \tag{14}$$

has no real roots in $(-\infty, 0]$. In fact, because n is even, we have for $\lambda \leqslant \lambda_0$

$$\begin{aligned}\lambda^n + (p+\varepsilon)\lambda^n \mathrm{e}^{-\lambda\tau} - (q-\varepsilon)\mathrm{e}^{-\lambda\sigma} &= f(\lambda) + \varepsilon(\lambda^n \mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-\lambda\sigma}) \\ &\leqslant f(\lambda) + \delta(\lambda^n \mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-\lambda\sigma}) = f(\lambda) - g(\lambda) \leqslant \tfrac{1}{2} m < 0\end{aligned}$$

and for $\lambda_0 \leqslant \lambda \leqslant 0$

$$\begin{aligned}\lambda^n + (p+\varepsilon)\lambda^n \mathrm{e}^{-\lambda\tau} - (q-\varepsilon)\mathrm{e}^{-\lambda\sigma} &= f(\lambda) + \varepsilon(\lambda^n \mathrm{e}^{-\lambda\tau} + \mathrm{e}^{-\lambda\sigma}) \\ &\leqslant m + \mu\varepsilon \leqslant m - \tfrac{1}{2} m = \tfrac{1}{2} m < 0.\end{aligned}$$

The proof is complete. □

Lemma 2. *Consider the NDDE*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(x(t) - P(t)x(t-\tau)\big) - Q(t)x(t-\sigma) = 0 \tag{15}$$

where n is even, and

$$P, Q \in C\big(t_0, \infty), \mathbb{R}\big), \quad Q(t) \geqslant 0 \quad \text{for } t \geqslant t_0 \quad \text{and } \tau > 0,\ \sigma \geqslant 0. \tag{16}$$

Assume that there are numbers p_1 and p_2 such that

$$p_1 \leqslant P(t) \leqslant p_2 < -1 \tag{17}$$

and that

$$\int_{t_0}^{\infty} Q(s)\,\mathrm{d}s = \infty. \tag{18}$$

Let $x(t)$ be an eventually bounded positive solution of Eq. (15) *and set*

$$y(t) = x(t) - P(t)x(t-\tau).$$

Then eventually

$$y^{(n)}(t) \geqslant 0, \quad (-1)^i y^{(n-i)}(t) > 0 \qquad \textit{for } i = 1, 2, \ldots, n, \tag{19}$$

$$\operatorname*{Lim}_{t\to\infty} y^{(i)}(t) = 0 \qquad \textit{for } i = 0, 1, \ldots, n-1. \tag{20}$$

P r o o f. From (15) we have

$$y^{(n)}(t) = Q(t)x(t-\sigma) \geqslant 0 \tag{21}$$

and because $x(t)$ and $P(t)$ are bounded it follows that

$$\operatorname*{Lim}_{t\to\infty} y^{(n-1)}(t) = h \in \mathbb{R}$$

exists.

Hence for each $i = 0, 1, \ldots, n-1$, $y^{(i)}(t)$ is eventually monotonic and so

$$\operatorname*{Lim}_{t\to\infty} y(t) = r \in \mathbb{R}$$

exists.

We claim that $r = 0$. To this end, integrating both sides of (21) from t_1 to t and then letting $t \to \infty$ we obtain

$$h - y^{(n-1)}(t_1) = \int_{t_1}^{\infty} Q(s)x(s-\sigma)\,\mathrm{d}s.$$

This, in view of (18), implies that

$$\operatorname*{Lim\,inf}_{t\to\infty} x(t) = 0.$$

Then by Lemma 1 in [8] we get $r = 0$. For this and the monotonic nature of $y^{(i)}(t)$ it is easy to see that the consecutive derivatives of $y(t)$ alternate in sign, that is, (19) holds. It is now clear that (20) also holds, and the proof is complete. □

Now we are ready to prove Theorem 1 by using the Banach Contraction Principle.

Proof of Theorem 1. Assume that Eq. (1) has a bounded nonoscillatory solution $x(t)$. We will assume that $x(t)$ is eventually positive. The case when $x(t)$ is eventually negative is similar and will be omitted. Choose $t_1 \geqslant t_0$ to be such that

$$x(t-\tau) > 0, \quad x(t-\sigma) > 0 \quad \text{for } t \geqslant t_1.$$

Set

$$Z(t) = x(t) - P(t)g\big(x(t-\tau)\big). \tag{22}$$

Then $Z(t) > 0$ and

$$Z^{(n)}(t) = Q(t)h\big(x(t-\sigma)\big) \geqslant 0 \quad \text{for } t \geqslant t_1. \tag{23}$$

So, $Z^{(i)}(t)$ $(i = 0, 1, \ldots, n-1)$ are eventually positive or eventually negative and so either

$$Z^{(n-1)}(t) < 0, \tag{24}$$

or

$$Z^{(n-1)}(t) > 0. \tag{25}$$

We claim that (24) holds. Otherwise (25) holds which implies that there exists $\beta > 0$ such that eventually

$$Z^{(n-1)}(t) \geqslant \beta.$$

This yields $Z(t) \to \infty$, which is a contradiction because of the bounded nature of $x(t)$ and $P(t)$. Hence (24) holds. Let

$$\operatorname*{Lim}_{t\to\infty} Z^{(n-1)}(t) = \alpha \in (-\infty, 0].$$

Integrating (23) from $t \geqslant t_1$ to ∞, we have

$$\alpha - Z^{(n-1)}(t) = \int_t^\infty Q(s)h\big(x(s-\sigma)\big)\,\mathrm{d}s$$

which, together with (4) and (6), yields

$$\operatorname*{Lim\,inf}_{t\to\infty} x(t) = 0. \tag{26}$$

Now we claim that

$$\operatorname*{Lim\,sup}_{t\to\infty} x(t) = 0. \tag{27}$$

Indeed, let $\operatorname*{Lim}_{t\to\infty} Z(t) = L$, then $L \in [0,\infty)$ and from the definition of $Z(t)$ we have

$$\begin{aligned} L &\geqslant \operatorname*{Lim\,sup}_{t\to\infty} \big(-P(t)g\big(x(t-\tau)\big)\big) \\ &\geqslant \operatorname*{Lim\,sup}_{t\to\infty} \big(-P(t)x(t-\tau)\big) \geqslant P_0 \operatorname*{Lim\,sup}_{t\to\infty} x(t-\tau). \end{aligned}$$

This means

$$\operatorname*{Lim\,sup}_{t\to\infty} x(t) \leqslant L/P_0. \tag{28}$$

In view of (26), there exists a sequence $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$ and $x(s_n - \tau) \to 0$ as $n \to \infty$. Noting that $g\big(x(s_n-\tau)\big) \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \operatorname*{Lim\,sup}_{t\to\infty} x(t) &\geqslant \operatorname*{Lim\,sup}_{n\to\infty} x(s_n) \\ &= \operatorname*{Lim}_{n\to\infty} \big(x(s_n) - P(s_n)g\big(x(s_n-\tau)\big)\big) = \operatorname*{Lim}_{n\to\infty} Z(s_n) = L, \end{aligned}$$

which, together with (28), yields $L/P_0 \geqslant L$. Since $P_0 > 1$, it follows that $L = 0$ and so (27) holds. Form (26) and (27) we get

$$\operatorname*{Lim}_{t\to\infty} x(t) = 0. \tag{29}$$

Next we rewrite Eq. (1) in the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(x(t) + P^*(t)x(t-\tau)\big) - Q^*(t)x(t-\sigma) = 0 \tag{30}$$

where

$$P^*(t) = -P(t)g\big(x(t-\tau)\big)/x(t-\tau), \quad Q^*(t) = Q(t)h\big(x(t-\sigma)\big)/x(t-\sigma).$$

From (3)–(6) and (29) we have

$$\operatorname*{Lim\,sup}_{t\to\infty} P^*(t) \leqslant p_0, \quad \operatorname*{Lim}_{t\to\infty} Q^*(t) = q. \tag{31}$$

According to the definition of $Z(t)$, we can rewrite Eq. (30) in the form

$$Z^{(n)}(t) + P^*(t-\sigma)\frac{Q^*(t)}{Q^*(t-\tau)}Z^{(n)}(t-\tau) = Q^*(t)Z(t-\sigma). \tag{32}$$

Since every bounded solution of Eq. (7) oscillates, by Lemma 1 it follows that there is an $\varepsilon \in (0, q)$ such that

$$\lambda^n + (p_0 + \varepsilon)\lambda^n e^{-\lambda\tau} - (q - \varepsilon)e^{-\lambda\sigma} < 0 \quad \text{for all } \lambda \in (-\infty, 0]. \tag{33}$$

For this $\varepsilon > 0$, let $\alpha \in (0, 1)$ be such that $\alpha q > q - \varepsilon$, and let $\beta > 1$ be such that

$$\alpha q > \beta(q - \varepsilon) \quad \text{or} \quad q/\beta > (q - \varepsilon)/\alpha. \tag{34}$$

From (31) we see that there exists $t_2 > t_1 + \sigma$ such that

$$P^*(t - \sigma) \cdot \frac{Q^*(t)}{Q^*(t - \tau)} < p_0 + \varepsilon, \quad Q^*(t) > q/\beta \quad \text{for } t \geqslant t_2.$$

Substituting this into (32), we get

$$Z^{(n)}(t) + (p_0 + \varepsilon)Z^{(n)}(t - \tau) > \frac{q}{\beta} Z(t - \sigma), \quad t \geqslant t_2. \tag{35}$$

Set

$$G(t) = \big(Z^{(n)}(t) + p_0 + \varepsilon)Z^{(n)}(t - \tau)\big)/Z(t - \sigma), \tag{36}$$

then we have by (35)

$$G(t) > q/\beta \quad \text{for } t \geqslant t_2. \tag{37}$$

From (36) we see that

$$Z^{(n)}(t) + (p_0 + \varepsilon)Z^{(n)}(t - \tau) = G(t)Z(t - \sigma). \tag{38}$$

Integrating both sides of (38) from $t \geqslant t_2$ to ∞ $n - 1$ times and using Lemma 2, we get

$$Z'(t) + (p_0 + \varepsilon)Z'(t - \tau) + \frac{1}{(n - 2)!} \int_t^\infty (s - t)^{n-2} G(s)Z(s - \sigma)\, ds = 0.$$

In what follows, for the sake of convenience, we set

$$a = p_0 + \varepsilon, \quad H(t) = \frac{1}{(n - 2)!} \int_t^\infty (s - t)^{n-2} G(s)Z(s - \sigma)\, ds.$$

Then we have

$$Z'(t) + aZ'(t-\tau) + H(t) = 0.$$

Integrating this from t to ∞, we get

$$Z(t) + aZ(t-\tau) = \int_t^\infty H(u)\,\mathrm{d}u,$$

or equivalently

$$Z(t) = -\frac{1}{a}Z(t+\tau) + \frac{1}{a}\int_{t+\tau}^\infty H(u)\,\mathrm{d}u.$$

Integrating, we obtain

$$Z(t) = \sum_{i=1}^{k}(-1)^{i+1}a^{-i}\int_{t+i\tau}^\infty H(u)\,\mathrm{d}u + (-1)^k a^{-k} Z(t+k\tau).$$

Since $a > 1$ and $Z(t) \to 0$ as $t \to \infty$, we let $k \to \infty$ to obtain

$$\begin{aligned}
Z(t) &= \sum_{i=1}^{\infty}(-1)^{i+1}a^{-i}\int_{t+i\tau}^\infty H(u)\,\mathrm{d}u \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}(-1)^{j+1}a^{-j}\int_{t+i\tau}^{t+(i+1)\tau} H(u)\,\mathrm{d}u \\
&= \sum_{i=1}^{\infty}\int_{t+i\tau}^{t+(i+1)\tau}\frac{1-(-a)^{-i}}{1+a}H(u)\,\mathrm{d}u \\
&= \sum_{i=1}^{\infty}\int_{t+i\tau}^{t+(i+1)\tau}\frac{1}{1+a}\{1-(-a)^{-[(u-t)/\tau]}\}H(u)\,\mathrm{d}u \\
&= \frac{1}{1+a}\int_{t+\tau}^\infty \{1-(-a)^{-[(u-t)/\tau]}\}H(u)\,\mathrm{d}u.
\end{aligned}$$

That means

$$\begin{aligned}
Z(t) = \frac{1}{(1+p_0+\varepsilon)(n-2)!}&\int_{t+\tau}^\infty \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \\
&\times \int_u^\infty (s-u)^{n-2}G(s)Z(s-\sigma)\,\mathrm{d}s\,\mathrm{d}u
\end{aligned}$$

where [.] denotes the greatest integer function.

This together with (37) and (34) yields

$$Z(t) \geqslant \frac{q-\varepsilon}{\alpha(1+p_0+\varepsilon)(n-2)!} \int_{t+\tau}^{\infty} \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \times \int_{u}^{\infty} (s-u)^{n-2} Z(s-\sigma)\,\mathrm{d}s\,\mathrm{d}u, \quad t \geqslant t_2. \tag{39}$$

From (33) we know that $\tau < \sigma$. Now, let X be the set of all continuous and bounded functions on $[t_2+\tau-\sigma, \infty)$ with the sup-norm. Then X is a Banach space. Set

$$A = \{w \in X\colon\ 0 \leqslant w(t) \leqslant 1,\ \text{for } t \geqslant t_2+\tau-\sigma\}.$$

Clearly, A is a bounded, closed and convex subset of X. Define a mapping $S\colon\ A \to X$ as follows:

$$(Sw)(t) = \begin{cases} \dfrac{q-\varepsilon}{(1+p_0+\varepsilon)(n-2)!Z(t)} \displaystyle\int_{t+\tau}^{\infty} \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \\ \qquad \times \displaystyle\int_{u}^{\infty} (s-u)^{n-2} Z(s-\sigma) w(s-\sigma)\,\mathrm{d}s\,\mathrm{d}u, \quad t \geqslant t_2 \\ (Sw)(t_2) + \mathrm{e}^{r(t_2-t)} - 1, \quad t_2+\tau-\sigma \leqslant t \leqslant t_2 \end{cases}$$

where $r = \big(\ln(2-\alpha)\big)/(\sigma-\tau) > 0$.

Since for any $w \in A$ and $t \geqslant t_2$ we have by (39)

$$0 \leqslant (Sw)(t) \leqslant \frac{q-\varepsilon}{(1+p_0+\varepsilon)(n-2)!Z(t)} \int_{t+\tau}^{\infty} \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \times \int_{u}^{\infty} (s-u)^{n-2} Z(s-\sigma)\,\mathrm{d}s\,\mathrm{d}u \leqslant \alpha < 1,$$

it follows that $0 \leqslant (Sw)(t) \leqslant 1$ for all $t \geqslant t_2+\tau-\sigma$ and so S maps A into itself. Next we claim that S is a contradiction on A. In fact, for any w_1, $w_2 \in A$ and $t \geqslant t_2$

we have

$$
\begin{aligned}
&|(Sw_1)(t)-(Sw_2)(t)| \\
&\quad \leqslant \frac{q-\varepsilon}{(1+p_0+\varepsilon)(n-2)!Z(t)} \int\limits_{t+\tau}^{\infty} \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \\
&\qquad \times \int\limits_{u}^{\infty} (s-u)^{n-2} Z(s-\sigma)|w_1(s-\sigma)-w_2(s-\sigma)| \,\mathrm{d}s\,\mathrm{d}u \\
&\quad \leqslant \alpha\|w_1-w_2\|,
\end{aligned}
$$

and for $t_2+\tau-\sigma \leqslant t \leqslant t_2$ we have

$$|(Sw_1)(t)-(Sw_2)(t)|-|(Sw_1)(t_2)-(Sw_2)(t_2)| \leqslant \alpha\|w_1-w_2\|.$$

Hence

$$\|Sw_1-Sw_2\| = \sup_{t\geqslant t_2+\tau-\sigma} |(Sw_1)(t)-(Sw_2)(t)| \leqslant \alpha\|w_1-w_2\|.$$

Since $0<\alpha<1$, it follows that S is a contradiction on A. Therefore, by the Banach Contradiction Principle S has a fixed point $w \in A$, i.e.

$$
\begin{aligned}
w(t) &= \frac{q-\varepsilon}{(1+p_0+\varepsilon)(n-2)!Z(t)} \int\limits_{t+\tau}^{\infty} \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \\
&\quad \times \int\limits_{u}^{\infty} (s-u)^{n-2} Z(s-\sigma) w(s-\sigma) \,\mathrm{d}s\,\mathrm{d}u, \quad t \geqslant t_2,
\end{aligned}
\tag{40}
$$

and for $t_2+\tau-\sigma \leqslant t < t_2$ we have

$$w(t) = w(t_2) + \mathrm{e}^{r(t_2-t)} - 1 > 0$$

which, together with (40) and the continuity of $w(t)$ yields

$$w(t) > 0 \quad \text{for all } t \geqslant t_2+\tau-\sigma.$$

Now, we set

$$y(t) = Z(t)w(t).$$

Then $y(t)$ is a positive continuous function on $[t_2+\tau-\sigma,\infty)$ and satisfies for $t \geqslant t_2$

$$
\begin{aligned}
y(t) &= \frac{q-\varepsilon}{(1+p_0+\varepsilon)(n-2)!} \int\limits_{t+\tau}^{\infty} \{1-(-p_0-\varepsilon)^{-[(u-t)/\tau]}\} \\
&\quad \times \int\limits_{u}^{\infty} (s-u)^{n-2} y(s-\alpha) \,\mathrm{d}s\,\mathrm{d}u.
\end{aligned}
$$

This implies that for $t \geqslant t_2 + \tau$

$$y(t) + (p_0 + \varepsilon)y(t-\tau) = \frac{q-\varepsilon}{(n-2)!} \int_t^{\infty}\int_u^{\infty} (s-u)^{n-2} y(s-\sigma)\,\mathrm{d}s\,\mathrm{d}u.$$

Differentiating it n times, we get

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(y(t) + (p_0+\varepsilon)y(t-\tau)\big) = (q-\varepsilon)y(t-\sigma), \quad t \geqslant t_2 + \tau,$$

which contradicts (33) and so the proof is complete. □

P r o o f of Theorem 2. Assume, by way of contradiction, that Eq. (1) has a bounded eventually positive solution $x(t)$. Let $t_1 \geqslant t_0$ be such that $x(t-\tau) > 0$, $x(t-\sigma) > 0$ for $t \geqslant t_1$. Set

$$y(t) = x(t) = P(t)g\big(x(t-\tau)\big). \tag{41}$$

Then $y(t)$ is bounded and satisfies

$$y^{(n)}(t) = Q(t)h\big(x(t-\sigma)\big) > 0 \quad \text{for } t \geqslant t_1.$$

Clearly, noting that n is even, we eventually have

$$y^{(n-1)}(t) < 0, \ \ldots, \ y''(t) > 0, \ y'(t) < 0.$$

We consider the following two possible cases:

Case 1. $y(t) > 0$ eventually. Let $t_2 \geqslant t_1$ be such that $y(t) > 0$ for $t \geqslant t_2$, that is,

$$x(t) > P(t)g\big(x(t-\tau)\big) \quad \text{for } t \geqslant t_2.$$

This together with (8) and (9) yields $x(t) \to \infty$ as $t \to \infty$. This is a contradiction.

Case 2. $y(t) < 0$ eventually. Let $t_2^* \geqslant t_1$ be such that $y(t) < 0$ for $t \geqslant t_2^*$. By the nonincreasing nature of $y(t)$, we have

$$y(t) \leqslant y(t_2^*) \quad \text{for } t \geqslant t_2^*,$$

that is,

$$x(t) - P(t)g\big(x(t-\tau)\big) \leqslant y(t_2^*) < 0 \quad \text{for } t \geqslant t_2^*.$$

We claim that

$$\beta := \inf_{t \geqslant t_2^*} x(t) > 0.$$

Otherwise, $\beta = 0$ and hence there exists a sequence $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$ and $x(s_n) \to 0$ as $n \to \infty$. Noting that $g\bigl(x(s_n)\bigr) \to 0$ as $n \to \infty$, we have

$$0 \leqslant \operatorname*{Lim\,inf}_{n \to \infty} x(s_n + \tau) \leqslant \operatorname*{Lim}_{n \to \infty} \bigl(P(S_n + \tau) g\bigl(x(s_n)\bigr) + y(t_2^*)\bigr) = y(t_2^*) < 0,$$

which is a contradiction and so $\beta > 0$. Therefore,

$$x(t) \geqslant \beta \quad \text{ for } t \geqslant t_2^*.$$

From (10) we see that

$$\alpha \colon = \min\{h(u) \colon\ u \geqslant \beta\} > 0$$

which, together with (42), yields

$$h\bigl(x(t - \sigma)\bigr) \geqslant \alpha \quad \text{ for } t \geqslant t_2^* + \sigma.$$

Substituting this into Eq. (1), we get

$$y^{(n)}(t) \geqslant \alpha Q(t) \quad \text{ for } t \geqslant t_2^* + \sigma.$$

This implies that $y^{(n-1)}(t) \to \infty$ as $t \to \infty$, which is a contradiction and so the proof is complete. □

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