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# A GRADIENT ESTIMATE FOR SOLUTIONS OF THE HEAT EQUATION II

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*Abstract.* The author obtains an estimate for the spatial gradient of solutions of the heat equation, subject to a homogeneous Neumann boundary condition, in terms of the gradient of the initial data. The proof is accomplished via the maximum principle; the main assumption is that the sufficiently smooth boundary be convex.

*Keywords*: gradient estimate, heat equation, maximum principle

*MSC 2000*: 35K05

## 1. Introduction

In [1] the writer obtained an estimate for the spatial gradient of the solution $u(x,t)$ of the following initial-boundary value problem for the heat equation:

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = f(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $R^n$, $n \geqslant 2$. Assuming that $f(x) \in C^1(\overline{\Omega})$ and vanished on $\partial\Omega$; and that $\partial\Omega$ was $C^3$ and satisfied an appropriate mean curvature condition (see (1.6) in [1]), the estimate

$$|\operatorname{grad} u(x,t)| \leqslant \max_{\overline{\Omega}} |\operatorname{grad} f(x)|, \quad (x,t) \in \partial\Omega \times (0,\infty) \tag{1.2}$$

was obtained as a consequence of the maximum principle. (Here $\operatorname{grad} u(x,t)$ denotes the gradient with respect to the spatial variables $x$).

The purpose of this paper is to obtain the same estimate for solutions of the problem (1.1) in which $u$ satisfies a homogeneous Neumann boundary condition rather than a homogeneous Dirichlet boundary condition.

In order to obtain this result we need a stronger assumption on $\partial\Omega$ than the mean curvature assumption (1.6) made in [1]. In fact we need to assume that $\partial\Omega$ satisfies a convexity condition.

To describe this condition let $p$ be a typical point on $\partial\Omega$ and suppose that after suitable rotation and translation of our coordinate system placing $p$ at the origin of the system, the portion of $\partial\Omega$ lying in a neighbourhood of $p$ is the surface corresponding to the function

$$x_n = g(x_1, \ldots, x_{n-1}) \tag{1.3}$$

where $(x_1, \ldots, x_{n-1})$ varies over a neighbourhood of $(x_1 = 0, \ldots, x_{n-1} = 0)$, with $g(0, \ldots, 0) = 0$ and with the positive $x_n$ direction corresponding to the outward normal direction from $\partial\Omega$ at $p$. Then the convexity condition that we shall assume $\partial\Omega$ to satisfy is that

$$\sum_{1 \leqslant j,k \leqslant n-1} g_{x_j x_k}(0, \ldots, 0)\eta_j \eta_k \leqslant 0 \tag{1.4}$$

for any $\eta = (\eta_1, \ldots, \eta_{n-1}) \in R^{n-1}$.

We can now state the result we wish to prove as follows:

**Theorem 1.** *Assume*

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0, \infty) \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = f(x) & \text{in } \Omega \end{cases}$$

*with $f(x) \in C^1(\overline{\Omega})$ and satisfying the boundary condition*

$$\frac{\partial f}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

*Suppose further that $\partial\Omega \in C^3$ and satisfies the convexity condition* (1.4). *Then*

$$|\operatorname{grad} u(x, t)| \leqslant \max_{\overline{\Omega}} |\operatorname{grad} f(x)|, \qquad (x, t) \in \Omega \times (0, \infty). \tag{1.5}$$

The proof of the theorem will be presented in the following section of the paper.

## Section 2

The proof of Theorem 1 will be conducted along the same general lines as the proof of the same estimate (1.2) for problem (1.1) given in [1]. As in that proof it suffices, in view of the maximum principle (see Proposition 2.1 and Theorem 2.2 of [1]), to show that

$$\frac{\partial}{\partial n}|\operatorname{grad} u|^2\Big|_{\partial\Omega\times(0,\infty)} \leqslant 0. \tag{2.1}$$

However, unlike that proof, where to establish (2.1) we used the fact that $u$ was a solution of the heat equation in $\Omega\times(0,\infty)$, we don't use the equation here. Rather, the conclusion (2.1) stems in the present case from the boundary condition $\frac{\partial u}{\partial n}=0$ satisfied by $u$ on $\partial\Omega\times(0,\infty)$ and the convexity condition (1.4) satisfied by $\partial\Omega$. This result is of independent interest and we state it separately as:

**Theorem 2.** *Suppose that $u(x)$ is a $C^2(\overline{\Omega})$ function which satisfies the boundary condition*

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0; \tag{2.2}$$

*and suppose that $\partial\Omega$ is $C^3$ and satisfies the convexity condition* (1.4). *Then $|\operatorname{grad} u(x)|^2$ satisfies the boundary condition*

$$\frac{\partial}{\partial n}|\operatorname{grad} u(x)|^2\Big|_{\partial\Omega} \leqslant 0. \tag{2.3}$$

**Preliminaries.** To prove Theorem 2 we are going to show that for a typical point $p$ of $\partial\Omega$

$$\frac{\partial}{\partial n}|\operatorname{grad} u(x)|^2\Big|_{p} \leqslant 0. \tag{2.4}$$

For this purpose we introduce the same coordinate change used in [1] and delineated in Section 3 of that paper.

Recapitulating, that coordinate change was based on the function

$$x_n = g(x_1,\dots,x_{n-1})$$

which described the surface constituting that portion of $\partial\Omega$ lying in a sufficiently small neighbourhood of the point $p$, with $p$ placed at the origin of our coordinate system, and so

$$g(0,\dots,0)=0. \tag{2.5}$$

We also assumed the positive $x_n$ direction to correspond to the outward normal direction on $\partial\Omega$ at $p$, which implies that $x_n = 0$ is the tangent plane to $\partial\Omega$ at $p$; so that necessarily

$$g_{x_j}(0, \ldots, 0) = 0 \quad \text{for } j = 1, \ldots, n-1. \tag{2.6}$$

Starting from the point $(\xi_1, \ldots, \xi_{n-1}, g(\xi_1, \ldots, \xi_{n-1}))$ on the surface describing $\partial\Omega$, we then proceeded $\xi_n$ units in the outward normal direction arriving at the point $(x_1, \ldots, x_n)$ in $R^n$. Accordingly, the coordinates of the resulting point $x = (x_1, \ldots, x_n)$ are connected to the coordinates of $\xi = (\xi_1, \ldots, \xi_n)$ through the formulas

$$\begin{cases} x_j = \xi_j - g_{\xi_j}(\xi_1, \ldots, \xi_{n-1}) \left(1 + \sum\limits_{k=1}^{n-1} g^2_{\xi_k}(\xi_1, \ldots, \xi_{n-1})\right)^{-\frac{1}{2}} \xi_n \\ \hfill j = 1, \ldots, n-1, \text{ and} \\ x_n = g(\xi_1, \ldots, \xi_{n-1}) + \left(1 + \sum\limits_{k=1}^{n-1} g^2_{\xi_k}(\xi_1, \ldots, \xi_{n-1})\right)^{-\frac{1}{2}} \xi_n. \end{cases} \tag{2.7}$$

And it is these equations, abbreviated as $x = x(\xi)$, which describe the coordinate change from $\xi$ to $x$ that we are going to use prove (2.4).

Clearly, from the way we arrived at (2.7), the outward normal derivative in the $x$ coordinates on $\partial\Omega$ corresponds to differentiation with respect to $\xi_n$ in the $\xi$ coordinates when $\xi_n = 0$. More precisely if $\varphi(x)$ represents a function in the $x$ coordinates and $\psi(\xi)$ represents the corresponding function in the $\xi$ coordinates, i.e. $\psi(\xi) = \varphi(x(\xi))$, then

$$\left.\frac{\partial\varphi(x)}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial\psi(\xi)}{\partial\xi_n}\right|_{\xi_n=0}; \tag{2.8}$$

in particular

$$\left.\frac{\partial\varphi(x)}{\partial n}\right|_{p} = \left.\frac{\partial\psi(\xi)}{\partial\xi_n}\right|_{\xi=0}. \tag{2.9}$$

The differentiability properties of the transformation $x = x(\xi)$ defined by (2.7) are described in Propositions 3.1 and 3.2 of [1] and we summarize them here.

Most importantly, if $g(\xi_1, \ldots, \xi_{n-1})$ is $C^2$ in a neighbourhood of ($\xi_1 = 0, \ldots, \xi_{n-1} = 0$), then $x = x(\xi)$ is a $C^1$ transformation in a neighbourhood of $\xi = 0$, sending $\xi = 0$ into $x = 0$, whose Jacobian at the origin is the identity matrix:

$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=0} = I. \tag{2.10}$$

Consequently, the inverse transformation $\xi = \xi(x)$ exists in a neighbourhood of $x = 0$, is $C^1$ there and its Jacobian at the origin is also the identity matrix:

$$\frac{\partial \xi}{\partial x}\Big|_{x=0} = I. \tag{2.11}$$

Moreover, if $g(\xi_1, \ldots, \xi_{n-1})$ is $C^3$ in a neighbourhood of $(\xi_1 = 0, \ldots, \xi_{n-1} = 0)$, then both $x = x(\xi)$ and $\xi = \xi(x)$ are $C^2$ transformations in neighbourhoods of $\xi = 0$ and $x = 0$, respectively; with the following identities holding for their second derivatives at the origin $\xi = x = 0$;

$$\frac{\partial}{\partial \xi_m}\Big(\frac{\partial \xi_j}{\partial x_l}\Big)\Big|_{\xi=0} = -\frac{\partial}{\partial \xi_m}\Big(\frac{\partial x_j}{\partial \xi_l}\Big)\Big|_{\xi=0}, \tag{2.12}$$

$j, l, m = 1, \ldots, n$ (see equation (3.9) of [1]).

Proof of Theorem 2. We are now prepared to establish Theorem 2 by showing that the function $u(x)$ which that theorem concerns satisfies the condition (2.4). Our first step in doing so is to introduce the coordinate transformation $x = x(\xi)$ defined by (2.7) and then to consider the function $u(x)$ referred to $\xi$ coordinates which we denote by $v(\xi)$, i.e. $v(\xi) = u(x(\xi))$. Expressing $|\operatorname{grad} u(x)|^2$ in terms of $v(\xi)$ we obtain

$$|\operatorname{grad} u(x)|^2 = \sum_{1 \leqslant j,k \leqslant n} b_{jk} \frac{\partial v}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}$$

where

$$b_{jk} = \sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i}\frac{\partial \xi_k}{\partial x_i} \quad j, k = 1, \ldots, n. \tag{2.13}$$

Hence, in view of the correspondence (2.8) between differentiation in the normal direction on $\partial\Omega$ in the $x$ coordinates and differentiation with respect to $\xi_n$ when $\xi_n = 0$ in the $\xi$ coordinates, we have

$$\begin{aligned}\frac{\partial}{\partial n}|\operatorname{grad} u|^2\Big|_{\partial\Omega} &= \frac{\partial}{\partial \xi_n}\Big(\sum_{1 \leqslant j,k \leqslant n} b_{jk}\frac{\partial v}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\Big)\Big|_{\xi_n=0} \\ &= \sum_{1 \leqslant j,k \leqslant n} \frac{\partial}{\partial \xi_n}(b_{jk})\frac{\partial v}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\Big|_{\xi_n=0} + \sum_{1 \leqslant j,k \leqslant n} 2b_{jk}\frac{\partial^2 v}{\partial \xi_n \partial \xi_j}\frac{\partial v}{\partial \xi_k}\Big|_{\xi_n=0}.\end{aligned} \tag{2.14}$$

But now in terms of $v(\xi)$, our hypotheses $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$, asserts, again because of (2.8), that $\frac{\partial v}{\partial \xi_n}|_{\xi_n=0} = 0$; and consequently $\frac{\partial^2 v}{\partial \xi_n \partial \xi_j}|_{\xi_n=0} = 0$ for $j \neq n$; thus the preceding becomes

$$\frac{\partial}{\partial n}|\operatorname{grad} u|^2\Big|_{\partial\Omega} = \sum_{1 \leqslant j,k \leqslant n-1} \frac{\partial}{\partial \xi_n}(b_{jk})\frac{\partial v}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\Big|_{\xi_n=0} + \sum_{k=1}^{n-1} 2b_{nk}\frac{\partial^2 v}{\partial \xi_n^2}\frac{\partial v}{\partial \xi_k}\Big|_{\xi_n=0}. \tag{2.15}$$

Specializing down to the point $x = p$ on $\partial\Omega$, which corresponds to $\xi = 0$, we then find, on account of $b_{nk}|_{\xi=0} = 0$ for $k \neq n$ (see equation (4.6) of [1]), that

$$\frac{\partial}{\partial n}|\operatorname{grad} u|^2\Big|_p = \sum_{1\leqslant j,k\leqslant n-1} \frac{\partial}{\partial \xi_n}(b_{jk}) \frac{\partial v}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\Big|_{\xi=0}. \tag{2.16}$$

Finally, from the evaluation

$$\frac{\partial}{\partial \xi_n}(b_{jk})\Big|_{\xi=0} = 2g_{\xi_j\xi_k}(0,\dots,0), \quad 1 \leqslant j,k \leqslant n-1, \tag{2.17}$$

which we will establish in a moment, (2.16) then yields

$$\frac{\partial}{\partial n}|\operatorname{grad} u|^2\Big|_p = \sum_{1\leqslant j,k\leqslant n-1} 2g_{\xi_j\xi_k}(0,\dots,0) \frac{\partial v}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\Big|_{\xi=0} \leqslant 0$$

because of the assumed convexity condition (1.4) regarding $\partial\Omega$. This proves (2.4) and with it Theorem 2.

It remains to establish the evaluation (2.17). For this purpose we differentiate the defining formula (2.13) for $b_{jk}$ with respect to $\xi_n$ and evaluate at $\xi = 0$:

$$\frac{\partial}{\partial \xi_n}(b_{jk})\Big|_{\xi=0} = \sum_{i=1}^{n} \frac{\partial}{\partial \xi_n}\Big(\frac{\partial \xi_j}{\partial x_i}\Big)\frac{\partial \xi_k}{\partial x_i}\Big|_{\xi=0} + \sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i}\frac{\partial}{\partial \xi_n}\Big(\frac{\partial \xi_k}{\partial x_i}\Big)\Big|_{\xi=0}.$$

In view of (2.11), $\frac{\partial \xi_k}{\partial x_i}|_{\xi=0} = \delta_{ki}$, where $\delta_{ki}$ is the Kronecker delta, *i.e.* $\delta_{ki} = 1$ if $k = i$ and is zero otherwise. Hence

$$\frac{\partial}{\partial \xi_n}(b_{jk})\Big|_{\xi=0} = \frac{\partial}{\partial \xi_n}\Big(\frac{\partial \xi_j}{\partial x_k}\Big)\Big|_{\xi=0} + \frac{\partial}{\partial \xi_n}\Big(\frac{\partial \xi_k}{\partial x_j}\Big)\Big|_{\xi=0}.$$

Making use of (2.12) this becomes

$$\frac{\partial}{\partial \xi_n}(b_{jk})\Big|_{\xi=0} = -\frac{\partial}{\partial \xi_n}\Big(\frac{\partial x_j}{\partial \xi_k}\Big)\Big|_{\xi=0} - \frac{\partial}{\partial \xi_n}\Big(\frac{\partial x_k}{\partial \xi_j}\Big)\Big|_{\xi=0}.$$

The derivatives on the right are then evaluated directly by differentiating the expressions (2.7) defining the $x_j$'s in terms of the $\xi_k$'s; taking (2.6) into account, this yields

$$\frac{\partial}{\partial \xi_n}\Big(\frac{\partial x_j}{\partial \xi_k}\Big)\Big|_{\xi=0} = \frac{\partial}{\partial \xi_k}\Big(\frac{\partial x_j}{\partial \xi_n}\Big)\Big|_{\xi=0} = -g_{\xi_j\xi_k}(0,\dots,0) \quad \text{for } j,k = 1,\dots,n-1;$$

and (2.17) follows. □

*References*


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