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_Czechoslovak Mathematical Journal_, Vol. 51 (2001), No. 1, 39–44

Persistent URL: [http://dml.cz/dmlcz/127624](http://dml.cz/dmlcz/127624)

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A GRADIENT ESTIMATE FOR SOLUTIONS OF THE
HEAT EQUATION II

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(Received March 20, 1997)

Abstract. The author obtains an estimate for the spatial gradient of solutions of the heat equation, subject to a homogeneous Neumann boundary condition, in terms of the gradient of the initial data. The proof is accomplished via the maximum principle; the main assumption is that the sufficiently smooth boundary be convex.

Keywords: gradient estimate, heat equation, maximum principle

MSC 2000: 35K05

1. Introduction

In [1] the writer obtained an estimate for the spatial gradient of the solution \( u(x,t) \) of the following initial-boundary value problem for the heat equation:

\[
\begin{aligned}
&u_t = \Delta u & \text{in } \Omega \times (0, \infty) \\
&u = 0 & \text{on } \partial \Omega \times (0, \infty) \\
&u(x,0) = f(x) & \text{in } \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 2 \). Assuming that \( f(x) \in C^1(\Omega) \) and vanished on \( \partial \Omega \); and that \( \partial \Omega \) was \( C^3 \) and satisfied an appropriate mean curvature condition (see (1.6) in [1]), the estimate

\[
|\text{grad } u(x,t)| \leq \max_{\Omega} |\text{grad } f(x)|, \quad (x,t) \in \partial \Omega \times (0, \infty)
\]

(1.2)

was obtained as a consequence of the maximum principle. (Here \( \text{grad } u(x,t) \) denotes the gradient with respect to the spatial variables \( x \)).
The purpose of this paper is to obtain the same estimate for solutions of the problem (1.1) in which \( u \) satisfies a homogeneous Neumann boundary condition rather than a homogeneous Dirichlet boundary condition.

In order to obtain this result we need a stronger assumption on \( \partial \Omega \) than the mean curvature assumption (1.6) made in [1]. In fact we need to assume that \( \partial \Omega \) satisfies a convexity condition.

To describe this condition let \( p \) be a typical point on \( \partial \Omega \) and suppose that after suitable rotation and translation of our coordinate system placing \( p \) at the origin of the system, the portion of \( \partial \Omega \) lying in a neighbourhood of \( p \) is the surface corresponding to the function

\[
x_n = g(x_1, \ldots, x_{n-1})
\]

where \((x_1, \ldots, x_{n-1})\) varies over a neighbourhood of \((x_1 = 0, \ldots, x_{n-1} = 0)\), with \(g(0, \ldots, 0) = 0\) and with the positive \( x_n \) direction corresponding to the outward normal direction from \( \partial \Omega \) at \( p \). Then the convexity condition that we shall assume \( \partial \Omega \) to satisfy is that

\[
\sum_{1 \leq j, k \leq n-1} g_{x_j x_k}(0, \ldots, 0) \eta_j \eta_k \leq 0
\]

for any \( \eta = (\eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^{n-1} \).

We can now state the result we wish to prove as follows:

**Theorem 1.** Assume

\[
\begin{cases}
  u_t = \Delta u & \text{in } \Omega \times (0, \infty) \\
  \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty) \\
  u(x, 0) = f(x) & \text{in } \Omega
\end{cases}
\]

with \( f(x) \in C^1(\bar{\Omega}) \) and satisfying the boundary condition

\[
\frac{\partial f}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]

Suppose further that \( \partial \Omega \in C^3 \) and satisfies the convexity condition (1.4). Then

\[
|\nabla u(x, t)| \leq \max_{\Omega} |\nabla f(x)|, \quad (x, t) \in \Omega \times (0, \infty).
\]

The proof of the theorem will be presented in the following section of the paper.
The proof of Theorem 1 will be conducted along the same general lines as the proof of the same estimate (1.2) for problem (1.1) given in [1]. As in that proof it suffices, in view of the maximum principle (see Proposition 2.1 and Theorem 2.2 of [1]), to show that

\begin{equation}
\frac{\partial}{\partial n}|\text{grad } u|^2\bigg|_{\partial \Omega \times (0, \infty)} \leq 0.
\end{equation}

However, unlike that proof, where to establish (2.1) we used the fact that \( u \) was a solution of the heat equation in \( \Omega \times (0, \infty) \), we don’t use the equation here. Rather, the conclusion (2.1) stems in the present case from the boundary condition \( \frac{\partial u}{\partial n} = 0 \) satisfied by \( u \) on \( \partial \Omega \times (0, \infty) \) and the convexity condition (1.4) satisfied by \( \partial \Omega \). This result is of independent interest and we state it separately as:

**Theorem 2.** Suppose that \( u(x) \) is a \( C^2(\bar{\Omega}) \) function which satisfies the boundary condition

\begin{equation}
\frac{\partial u}{\partial n}\big|_{\partial \Omega} = 0;
\end{equation}

and suppose that \( \partial \Omega \) is \( C^3 \) and satisfies the convexity condition (1.4). Then \( |\text{grad } u(x)|^2 \) satisfies the boundary condition

\begin{equation}
\frac{\partial}{\partial n}|\text{grad } u(x)|^2\big|_{\partial \Omega} \leq 0.
\end{equation}

**Preliminaries.** To prove Theorem 2 we are going to show that for a typical point \( p \) of \( \partial \Omega \)

\begin{equation}
\frac{\partial}{\partial n}|\text{grad } u(x)|^2\big|_p \leq 0.
\end{equation}

For this purpose we introduce the same coordinate change used in [1] and delineated in Section 3 of that paper.

Recapitulating, that coordinate change was based on the function

\[ x_n = g(x_1, \ldots, x_{n-1}) \]

which described the surface constituting that portion of \( \partial \Omega \) lying in a sufficiently small neighbourhood of the point \( p \), with \( p \) placed at the origin of our coordinate system, and so

\begin{equation}
g(0, \ldots, 0) = 0.
\end{equation}
We also assumed the positive $x_n$ direction to correspond to the outward normal direction on $\partial \Omega$ at $p$, which implies that $x_n = 0$ is the tangent plane to $\partial \Omega$ at $p$; so that necessarily

$$g_{x_j}(0, \ldots, 0) = 0 \quad \text{for } j = 1, \ldots, n - 1. \quad (2.6)$$

Starting from the point $(\xi_1, \ldots, \xi_{n-1}, g(\xi_1, \ldots, \xi_{n-1}))$ on the surface describing $\partial \Omega$, we then proceeded $\xi_n$ units in the outward normal direction arriving at the point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Accordingly, the coordinates of the resulting point $x = (x_1, \ldots, x_n)$ are connected to the coordinates of $\xi = (\xi_1, \ldots, \xi_n)$ through the formulas

$$\left\{ \begin{array}{rl}
x_j &= \xi_j - g_{\xi_j}(\xi_1, \ldots, \xi_{n-1}) \left(1 + \sum_{k=1}^{n-1} g_{\xi_k}^2(\xi_1, \ldots, \xi_{n-1})\right)^{-\frac{1}{2}} \xi_n \\
x_n &= g(\xi_1, \ldots, \xi_{n-1}) + \left(1 + \sum_{k=1}^{n-1} g_{\xi_k}^2(\xi_1, \ldots, \xi_{n-1})\right)^{-\frac{1}{2}} \xi_n.
\end{array} \right. \quad (2.7)$$

And it is these equations, abbreviated as $x = x(\xi)$, which describe the coordinate change from $\xi$ to $x$ that we are going to use prove (2.4).

Clearly, from the way we arrived at (2.7), the outward normal derivative in the $x$ coordinates on $\partial \Omega$ corresponds to differentiation with respect to $\xi_n$ in the $\xi$ coordinates when $\xi_n = 0$. More precisely if $\varphi(x)$ represents a function in the $x$ coordinates and $\psi(\xi)$ represents the corresponding function in the $\xi$ coordinates, i.e. $\psi(\xi) = \varphi(x(\xi))$, then

$$\left. \frac{\partial \varphi(x)}{\partial n} \right|_{\partial \Omega} = \left. \frac{\partial \psi(\xi)}{\partial \xi_n} \right|_{\xi_n=0}; \quad (2.8)$$

in particular

$$\left. \frac{\partial \varphi(x)}{\partial n} \right|_p = \left. \frac{\partial \psi(\xi)}{\partial \xi_n} \right|_{\xi=0}. \quad (2.9)$$

The differentiability properties of the transformation $x = x(\xi)$ defined by (2.7) are described in Propositions 3.1 and 3.2 of [1] and we summarize them here.

Most importantly, if $g(\xi_1, \ldots, \xi_{n-1})$ is $C^2$ in a neighbourhood of $\xi_1 = 0, \ldots, \xi_{n-1} = 0$, then $x = x(\xi)$ is a $C^1$ transformation in a neighbourhood of $\xi = 0$, sending $\xi = 0$ into $x = 0$, whose Jacobian at the origin is the identity matrix:

$$\left. \frac{\partial x}{\partial \xi} \right|_{\xi=0} = I. \quad (2.10)$$
Consequently, the inverse transformation $\xi = \xi(x)$ exists in a neighbourhood of $x = 0$, is $C^1$ there and its Jacobian at the origin is also the identity matrix:

\begin{equation}
\frac{\partial \xi}{\partial x} \bigg|_{x=0} = I.
\end{equation}

Moreover, if $g(\xi_1, \ldots, \xi_{n-1})$ is $C^3$ in a neighbourhood of $(\xi_1 = 0, \ldots, \xi_{n-1} = 0)$, then both $x = x(\xi)$ and $\xi = \xi(x)$ are $C^2$ transformations in neighbourhoods of $\xi = 0$ and $x = 0$, respectively; with the following identities holding for their second derivatives at the origin $\xi = x = 0$:

\begin{equation}
\frac{\partial}{\partial \xi_m} \left( \frac{\partial \xi_j}{\partial x_l} \right) \bigg|_{\xi = 0} = - \frac{\partial}{\partial \xi_m} \left( \frac{\partial x_j}{\partial \xi_l} \right) \bigg|_{\xi = 0},
\end{equation}

$j, l, m = 1, \ldots, n$ (see equation (3.9) of [1]).

Proof of Theorem 2. We are now prepared to establish Theorem 2 by showing that the function $u(x)$ which that theorem concerns satisfies the condition (2.4). Our first step in doing so is to introduce the coordinate transformation $x = x(\xi)$ defined by (2.7) and then to consider the function $u(x)$ referred to $\xi$ coordinates which we denote by $v(\xi)$, i.e. $v(\xi) = u(x(\xi))$. Expressing $|\text{grad } u(x)|^2$ in terms of $v(\xi)$ we obtain

\begin{equation}
|\text{grad } u(x)|^2 = \sum_{1 \leq j, k \leq n} b_{jk} \frac{\partial v}{\partial \xi_j} \frac{\partial v}{\partial \xi_k}.
\end{equation}

where

\begin{equation}
b_{jk} = \sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i} \frac{\partial \xi_k}{\partial x_i}, \quad j, k = 1, \ldots, n.
\end{equation}

Hence, in view of the correspondence (2.8) between differentiation in the normal direction on $\partial \Omega$ in the $x$ coordinates and differentiation with respect to $\xi_n$ when $\xi_n = 0$ in the $\xi$ coordinates, we have

\begin{equation}
\frac{\partial}{\partial n} |\text{grad } u|^2 \bigg|_{\partial \Omega} = \frac{\partial}{\partial \xi_n} \left( \sum_{1 \leq j, k \leq n} b_{jk} \frac{\partial v}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} \right) \bigg|_{\xi_n = 0}
\end{equation}

\begin{equation}
= \sum_{1 \leq j, k \leq n} \frac{\partial}{\partial \xi_n} (b_{jk}) \frac{\partial v}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} \bigg|_{\xi_n = 0} + \sum_{1 \leq j, k \leq n} \frac{2 b_{jk}}{\partial \xi_n \partial \xi_j \partial \xi_k} \bigg|_{\xi_n = 0}.
\end{equation}

But now in terms of $v(\xi)$, our hypotheses $\frac{\partial v}{\partial \xi_n} \bigg|_{\xi_n = 0} = 0$, asserts, again because of (2.8), that $\frac{\partial^2 v}{\partial \xi_n^2} \bigg|_{\xi_n = 0} = 0$ for $j \neq n$; and consequently $\frac{\partial^2 v}{\partial \xi_n \partial \xi_j} \bigg|_{\xi_n = 0} = 0$ for $j \neq n$; thus the preceding becomes

\begin{equation}
\frac{\partial}{\partial n} |\text{grad } u|^2 \bigg|_{\partial \Omega} = \sum_{1 \leq j, k \leq n-1} \frac{\partial}{\partial \xi_n} (b_{jk}) \frac{\partial v}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} \bigg|_{\xi_n = 0} + \sum_{k=1}^{n-1} \frac{2 b_{nk}}{\partial \xi_n \partial \xi_j \partial \xi_k} \bigg|_{\xi_n = 0}.
\end{equation}
Specializing down to the point \( x = p \) on \( \partial \Omega \), which corresponds to \( \xi = 0 \), we then find, on account of \( b_{nk|x=0} = 0 \) for \( k \neq n \) (see equation (4.6) of [1]), that

\[
\frac{\partial}{\partial n} |\text{grad } u|^2_p = \sum_{1 \leq j, k \leq n-1} \frac{\partial}{\partial \xi_n} (b_{jk}) \frac{\partial v}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} |_{\xi=0}.
\]

Finally, from the evaluation

\[
\frac{\partial}{\partial \xi_n} (b_{jk}) |_{\xi=0} = 2 g_{\xi_j \xi_k} (0, \ldots , 0), \quad 1 \leq j, k \leq n - 1,
\]

which we will establish in a moment, (2.16) then yields

\[
\frac{\partial}{\partial n} |\text{grad } u|^2_p = \sum_{1 \leq j, k \leq n-1} 2 g_{\xi_j \xi_k} (0, \ldots , 0) \frac{\partial v}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} |_{\xi=0} \leq 0
\]

because of the assumed convexity condition (1.4) regarding \( \partial \Omega \). This proves (2.4) and with it Theorem 2.

It remains to establish the evaluation (2.17). For this purpose we differentiate the defining formula (2.13) for \( b_{jk} \) with respect to \( \xi_n \) and evaluate at \( \xi = 0 \):

\[
\frac{\partial}{\partial \xi_n} (b_{jk}) |_{\xi=0} = \sum_{i=1}^n \frac{\partial}{\partial \xi_n} \left( \frac{\partial \xi_i}{\partial x_j} \frac{\partial \xi_k}{\partial x_i} \right) |_{\xi=0} + \sum_{i=1}^n \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_n} \left( \frac{\partial \xi_k}{\partial x_i} \right) |_{\xi=0}.
\]

In view of (2.11), \( \frac{\partial \xi_k}{\partial x_i} |_{\xi=0} = \delta_{ki} \), where \( \delta_{ki} \) is the Kronecker delta, \( i.e. \delta_{ki} = 1 \) if \( k = i \) and is zero otherwise. Hence

\[
\frac{\partial}{\partial \xi_n} (b_{jk}) |_{\xi=0} = \frac{\partial}{\partial \xi_n} \left( \frac{\partial \xi_j}{\partial x_k} \right) |_{\xi=0} + \frac{\partial}{\partial \xi_n} \left( \frac{\partial \xi_k}{\partial x_j} \right) |_{\xi=0}.
\]

Making use of (2.12) this becomes

\[
\frac{\partial}{\partial \xi_n} (b_{jk}) |_{\xi=0} = - \frac{\partial}{\partial \xi_n} \left( \frac{\partial x_j}{\partial \xi_k} \right) |_{\xi=0} - \frac{\partial}{\partial \xi_n} \left( \frac{\partial x_k}{\partial \xi_j} \right) |_{\xi=0}.
\]

The derivatives on the right are then evaluated directly by differentiating the expressions (2.7) defining the \( x_j \)'s in terms of the \( \xi_k \)'s; taking (2.6) into account, this yields

\[
\frac{\partial}{\partial \xi_n} \left( \frac{\partial x_j}{\partial \xi_k} \right) |_{\xi=0} = \frac{\partial}{\partial \xi_k} \left( \frac{\partial x_j}{\partial \xi_n} \right) |_{\xi=0} = - g_{\xi_j \xi_k} (0, \ldots , 0) \quad \text{for } j, k = 1, \ldots , n - 1;
\]

and (2.17) follows. \( \square \)

References


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