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ON VECTOR VALUED MEASURE SPACES OF BOUNDED  
 $\Phi$ -VARIATION CONTAINING COPIES OF  $\ell_\infty$ 

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*Abstract.* Given a Young function  $\Phi$ , we study the existence of copies of  $c_0$  and  $\ell_\infty$  in  $\text{cabv}_\Phi(\mu, X)$  and in  $\text{cabsv}_\Phi(\mu, X)$ , the countably additive,  $\mu$ -continuous, and  $X$ -valued measure spaces of bounded  $\Phi$ -variation and bounded  $\Phi$ -semivariation, respectively.

## 1. INTRODUCTION

The interest in Lebesgue's and Bochner's integration theory in Analysis has been a powerful incentive in the study of the Young functions and the Orlicz spaces. In fact the Orlicz theory of measurable functions and measures appears in literature as a natural attempt to generalize the classical theory of vector measures and integration which was restricted to the  $L_p$  spaces, and also because of the characterization of the uniformly integrable sets in  $L_1(\mu)$  given by de la Vallée Poussin in 1915 [1] in terms of Orlicz spaces. Again the classical Banach sequence spaces, especially the non reflexive ones, play a central role in the study of the Banach spaces. In this way, we present some results related to the existence of copies of  $c_0$  and  $\ell_\infty$  in Orlicz spaces of vector valued measures. This problem has been studied

- (a) in [2] for  $\text{cabv}(\mu, X)$ , the space of the countably additive,  $\mu$ -continuous and  $X$ -valued measures of bounded variation endowed with the topology of the variation norm,
- (b) in [4] for  $\text{ba}(\Sigma, X)$ , the space of bounded  $X$ -valued vector measures and for  $\text{ca}(\Sigma, X)$ , the space of countably additive and  $X$ -valued vector measures, both equipped with the semivariation norm.

Clearly this paper is a natural continuation of the results of a) and b).

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## 2. DEFINITIONS, NOTATION AND BASIC FACTS

The notation is standard, see [6] and [8] for details.

A Young function is a convex function  $\Phi: \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $\Phi(-x) = \Phi(x)$ ,  $\Phi(0) = 0$  and  $\lim_{x \rightarrow \infty} \Phi(x) = \infty$ .

From now on,  $(\Omega, \Sigma, \mu)$  will denote an atomless abstract finite measure space, where  $\Sigma$  is a  $\sigma$ -algebra on which  $\mu$  is a  $\sigma$ -additive and nonnegative measure. For every Banach space  $X$ ,  $L_\Phi(\mu, X)$  is the space of classes of  $\mu$ -measurable and  $X$ -valued functions  $f: \Omega \rightarrow X$ , such that there is a real constant  $H > 0$  such that  $\int_\Omega \Phi(H\|f(x)\|) d\mu < \infty$  (with the identification of functions that coincide a.e.), which is a Banach space with the norm

$$NV_\Phi(f) := \inf \left\{ K > 0: \int_\Omega \Phi(\|f(x)\|/K) d\mu \leq 1 \right\}.$$

For every convex function  $\Phi$  on  $A$ , we say that  $y = ax + b$  is a support line of  $\Phi$  if  $\Phi(x) \geq ax + b, \forall x \in A$ . The properties of the Young functions imply the existence of support lines with  $a > 0$  and  $b \leq 0$ . This fact can be used to prove that  $L_\Phi(\mu, X)$  is continuously embedded in  $L_1(\mu, X)$ . We denote by  $\chi(\mu, X)$  the set of step functions of  $L_1(\mu, X)$ .

Let  $F$  be a countably additive,  $X$ -valued and  $\mu$ -continuous measure on  $(\Omega, \Sigma, X)$ . The  $\Phi$ -variation of  $F$ , denoted by  $I_\Phi(F)$ , is defined by

$$I_\Phi(F) := \sup_\pi \left\{ \sum_n \Phi \left( \frac{\|F(A_n)\|}{\mu(A_n)} \right) \mu(A_n) \right\}$$

where the supremum is taken over all partitions  $\pi = \{A_n\}$  of  $\Omega$  in  $\Sigma$ . If  $I_\Phi(F) < \infty$ ,  $F$  is said to be of bounded  $\Phi$ -variation.

We denote by  $\text{cabv}_\Phi(\mu, X)$  the space of  $\mu$ -continuous countably additive and  $X$ -valued measures  $F$  such that there is a  $K > 0$  with  $I_\Phi(F/K) \leq 1$ , which is a Banach space with the norm

$$NV_\Phi(F) := \inf \{ K > 0: I_\Phi(F/K) \leq 1 \}.$$

The space  $L_\Phi(\mu, X)$  is an isometric subspace of  $\text{cabv}_\Phi(\mu, X)$  by the map  $G: L_\Phi(\mu, X) \rightarrow \text{cabv}_\Phi(\mu, X)$  such that  $G(f)(E) = \int_E f d\mu, \forall f \in L_\Phi(\mu, X)$  and for every  $E \in \Sigma$ , see [8].

If  $x' \in X'$  and  $F$  is an  $X$ -valued measure, we denote by  $x'F$  the scalar measure such that  $x'F(E) = \langle x', F(E) \rangle$ . The  $\Phi$ -semivariation of  $F$  is

$$IS_\Phi(F) := \sup \{ I_\Phi(x'F): x' \in X', \|x'\| \leq 1 \}.$$

If  $IS_{\Phi}(F) < \infty$ , then  $F$  is said to be of bounded  $\Phi$ -semivariation. We denote by  $\text{cabsv}_{\Phi}(\mu, X)$  the Banach space of countably additive and  $\mu$ -continuous  $X$ -valued measures  $F$  such that there is a  $K > 0$  with  $IS_{\Phi}(F/K) \leq 1$ , endowed with the norm

$$NS_{\Phi}(F) := \inf\{K > 0: IS_{\Phi}(F/K) \leq 1\}.$$

It is clear that  $\text{cabv}_{\Phi}(\mu, X) \subset \text{cabsv}_{\Phi}(\mu, X)$ , with  $NS_{\Phi}(F) \leq NV_{\Phi}(F)$  for every  $F \in \text{cabv}_{\Phi}(\mu, X)$ . We denote by  $J$  the canonical injection of  $\text{cabv}_{\Phi}(\mu, X)$  into  $\text{cabsv}_{\Phi}(\mu, X)$ .

Finally, we need the following result of Rosenthal:

**Lemma 1** ([7] Proposition 1.2 and Remark 1). *Let  $T: \ell_{\infty} \rightarrow X$  be a linear and continuous map such that  $\{\|T(e_n)\|\}$  does not converge to zero, where  $(e_n)$  is the unit vector sequence in  $\ell_{\infty}$ . Then there is an infinite subset  $D$  of  $\mathbb{N}$  such that  $T|_{\ell_{\infty}(D)}$  is an isomorphism.*

### 3. MAIN RESULTS

**Theorem 1.** *Let  $\{f_n\}$  be a  $\sigma(L_1(\mu), \chi(\mu))$ -null sequence in  $L_1(\mu)$  with the following properties:*

- (1)  $\exists M > 0$  such that  $\mu(\{\omega \in \Omega: |f_n(\omega)| > M\}) = 0, \forall n \in \mathbb{N}$ .
- (2)  $\exists B > 0$  and  $\exists S > 0$  such that  $\forall n \in \mathbb{N}, \exists A_n \in \Sigma$  with  $\mu(A_n) \geq S$  and  $f_n(\omega) \geq B, \forall \omega \in A_n$ .

*Let  $\Phi$  be a Young function such that  $0 < \Phi(x) < \infty$  if  $0 < x < \infty$  and  $\exists x_0 > 0: \Phi(x_0) \leq 1/\mu(\Omega)$ , and let  $X$  be a Banach space containing a copy of  $c_0$ . Then  $\text{cabv}_{\Phi}(\mu, X)$  and  $\text{cabsv}_{\Phi}(\mu, X)$  contain the respective subspaces  $S$  and  $S'$  isomorphic to  $\ell_{\infty}$ . Moreover,  $S \cap L_{\Phi}(\mu, X)$  contains a subspace isomorphic to  $c_0$ .*

**P r o o f.** It is enough to prove the theorem if  $X = c_0$ . For every  $n \in \mathbb{N}$ , let  $\Lambda_n$  be the scalar measure

$$\Lambda_n(E) = \langle f_n, \chi_E \rangle$$

for every  $E \in \Sigma$ . We define  $G: \ell_{\infty} \rightarrow \text{cabv}_{\Phi}(\mu, X)$  such that  $G((\xi_i)) = F_{(\xi_i)}$ , where  $F_{(\xi_i)}(E) = (\xi_i \Lambda_i(E))$ . If  $(\xi_i) \neq 0$ , it is clear that  $(\xi_i \Lambda_i(E)) \in c_0$  for every  $E \in \Sigma$ , and  $F_{(\xi_i)}$  is a countably additive and  $\mu$ -continuous  $c_0$ -valued measure. For every partition  $\{E_n\}$  of  $\Omega$  contained in  $\Sigma$  and for every  $K > 0$ , we have

$$\sum_{n \in \mathbb{N}} \Phi\left(\frac{\|F_{(\xi_i)}(E_n)\|_{c_0}}{K \mu(E_n)}\right) \mu(E_n) \leq \Phi(M \|(\xi_i)\|_{\ell_{\infty}} / K) \mu(\Omega) < \infty.$$

Consequently,  $F_{(\xi_i)}$  has bounded  $\Phi$ -variation and

$$NV_{\Phi}(F_{(\xi_i)}) \leq \inf\{K > 0: \Phi(M\|(\xi_i)\|_{\ell_{\infty}}/K)\mu(\Omega) \leq 1\}.$$

If we take  $K_0 > 0$  such that  $\forall \varepsilon > 0$ ,  $\Phi(M\|(\xi_i)\|_{\ell_{\infty}}/K_0)\mu(\Omega) \leq 1$  then

$$\Phi(M\|(\xi_i)\|_{\ell_{\infty}}/(K_0 - \varepsilon))\mu(\Omega) > 1 \geq \Phi(x_0)\mu(\Omega),$$

$M\|(\xi_i)\|_{\ell_{\infty}}/(K_0 - \varepsilon) \geq x_0$  and  $NV_{\Phi}(F_{(\xi_i)}) - \varepsilon \leq K_0 - \varepsilon \leq M\|(\xi_i)\|_{\ell_{\infty}}/x_0$ , which implies that  $NV_{\Phi}(F_{(\xi_i)}) \leq M\|(\xi_i)\|_{\ell_{\infty}}/x_0$  and hence  $G$  and  $JG$  are continuous. If  $(\xi_i) = 0$ , the conclusion follows directly.

If  $(e_n)$  is the unit basis in  $c_0$ , then  $F_{e_n} \in L_{\Phi}(\mu, X)$  with Radon-Nikodym derivative  $(0, \dots, 0, f_n(\cdot), 0, \dots)$  for every  $n \in \mathbb{N}$ . Moreover, fix  $n$ , take the partition  $\{E_k\}_{k=1}^2: E_1 = A_n, E_2 = \Omega \setminus A_n$ . For every  $x = (x_i) \in \ell_1$  we have  $|\langle x, F_{e_n}(E_1) \rangle| = |x_n \Lambda_n(E_1)| \geq |x_n| B \mu(E_1)$ , and then

$$\sum_{k=1}^2 \Phi\left(\frac{|\langle x, F_{e_n}(E_k) \rangle|}{K \mu(E_k)}\right) \mu(E_k) \geq \Phi(|x_n| B / K) \mu(E_1) \geq \Phi(|x_n| B / K) S.$$

Hence  $I_{\Phi}(x F_{e_n} / K) \geq \Phi(|x_n| B / K) S$ , therefore

$$NV_{\Phi}(x F_{e_n}) \geq \inf\{K > 0: \Phi(|x_n| B / K) S \leq 1\}.$$

For every support line  $y = ax + b$  of  $\Phi$  with  $a > 0$ ,  $b \leq 0$ , taking  $x = e_n$ , we obtain

$$\begin{aligned} NS_{\Phi}(F_{e_n}) &\geq \inf\{K > 0: \Phi(B/K) S \leq 1\} \\ &\geq \inf\{K > 0: (aB/K + b) S \leq 1\} = aBS / (1 - bS) > 0. \end{aligned}$$

Hence

$$\inf\{NV_{\Phi}(F_{e_n}), n \in \mathbb{N}\} \geq \inf\{NS_{\Phi}(F_{e_n}), n \in \mathbb{N}\} \geq aBS / (1 - bS) > 0.$$

Then we use Lemma 1 to conclude that there are infinite subsets  $DV$  and  $DS$  of  $\mathbb{N}$  such that  $G|_{\ell_{\infty}(DV)}$  and  $JG|_{\ell_{\infty}(DS)}$  are isomorphisms.  $\square$

### Remarks.

1) Every Rademacherlike sequence in  $\Omega$ , i.e., every orthogonal sequence  $\{r_n\}$  such that  $\mu(\{\omega \in \Omega: r_n(\omega) = 1\}) = \mu(\{\omega \in \Omega: r_n(\omega) = -1\}) = 1/2$  verifies the required condition. If  $\mu$  is the Lebesgue measure in  $[0, 1]$ , we also can take  $f_n(\omega) = \sin(n\pi\omega)$ .

2) Every continuous Young function such that  $0 < \Phi(x) < \infty$  if  $0 < x < \infty$  verifies the hypothesis of Theorem 1 for all finite measure spaces  $(\Omega, \Sigma, \mu)$ .

3) A Young function satisfies  $\Phi \in \Delta_2$  if  $\exists H > 0: \forall x > 0, \Phi(2x) \leq H\Phi(x)$ . Many properties of the space  $L_\Phi(\mu, X)$  and  $\text{cabv}_\Phi(\mu, X)$  with  $\Phi(x) = \|x\|^p, 1 \leq p < \infty$ , are fulfilled for  $\Phi \in \Delta_2$  and the corresponding proofs are also valid in this setting. This happens mainly because if  $\Phi \in \Delta_2$ , the simple functions are dense in  $L_\Phi(\mu, X)$ . Moreover, if  $(\Omega, \Sigma, \mu)$  is separable, then  $L_\Phi(\mu)$  is separable, [6]. For example, if  $(\Omega, \Sigma, \mu)$  is separable and  $\Phi \in \Delta_2$  then

- a)  $L_\Phi(\mu, X)$  contains a copy of  $\ell_\infty$  if and only if  $X$  does, Mendoza [5];
- b) if  $X$  contains a copy of  $c_0$ , then  $L_\Phi(\mu, X)$  contains a complemented copy of  $c_0$ , see Emmanuelle [3]. If moreover  $X$  contains no copies of  $\ell_\infty$ , a consequence of Theorem 1 is that  $L_\Phi(\mu, X)$  is an uncomplemented subspace of  $\text{cabv}_\Phi(\mu, X)$ , see Drewnowski and Emmanuelle [2].

**Theorem 2.** *Let  $\Phi$  be a continuous Young function such that  $\Phi(x) = 0$  iff  $x = 0$ . Then for every separable finite measure space  $(\Omega, \Sigma, \mu)$ , the space  $\text{cabv}_\Phi(\mu, X)$  (or  $\text{cabsv}_\Phi(\mu, X)$ ) contains a copy of  $c_0$  iff it contains a copy of  $\ell_\infty$ .*

*P r o o f.* We only prove the theorem for  $\text{cabv}_\Phi(\mu, X)$  (the proof in the case of  $\text{cabsv}_\Phi(\mu, X)$  is analogous). By virtue of Theorem 1 and the above remarks, it is enough to prove the statement if  $X$  contains no copies of  $c_0$  and  $\text{cabv}_\Phi(\mu, X)$  contains a copy of  $c_0$ . Let  $J: c_0 \rightarrow \text{cabv}_\Phi(\mu, X)$  be an isomorphism. First of all we will see that  $\sum_{i=1}^{\infty} \xi_i J(e_i)(E) \in X$  for every  $(\xi_i) \in \ell_\infty$  and for every  $E \in \Sigma$ . We know that the formal series  $\sum_{i=1}^{\infty} J(e_i)$  is weakly unconditionally Cauchy in  $\text{cabv}_\Phi(\mu, X)$ . For every  $E \in \Sigma$ , we consider the map  $H_E: \text{cabv}_\Phi(\mu, X) \rightarrow X$  such that  $H_E(F) = F(E)$  for every  $F \in \text{cabv}_\Phi(\mu, X)$ . If  $y = ax + b$  is a support line of  $\Phi$  with  $a > 0, b \leq 0$ , for every  $K > 0$  we have

$$I_\Phi(F/K) \geq \Phi\left(\frac{\|F(E)\|}{K\mu(E)}\right)\mu(E) \geq a\|F(E)\|/K + b\mu(E).$$

Then

$$N_\Phi(F) \geq \inf\{K > 0: a\|F(E)\|/K + b\mu(E) \leq 1\} = \frac{a}{1 - b\mu(E)}\|F(E)\|$$

therefore  $H_E$  is both continuous and weakly continuous. Hence the series  $\sum_{i=1}^{\infty} J(e_i)(E)$  is a weakly unconditionally Cauchy series in  $X$ , and as  $X$  does not contain copies of  $c_0$ , by virtue of a classical result of Bessaga and Pelczynski, the series is unconditionally convergent in  $X$ , and then  $\sum_{i=1}^{\infty} \xi_i J(e_i)(E) \in X$ . For every  $u = (\xi_i) \in \ell_\infty$ , we

define the measure

$$F_u: \Sigma \rightarrow X: F_u(E) = \sum_{i=1}^{\infty} \xi_i J(e_i)(E) \quad \forall E \in \Sigma.$$

Let  $(F_{u_n})_{n \in \mathbb{N}}$  be a sequence in  $\text{cabv}_{\Phi}(\mu, X)$  with  $F_{u_n}(E) = \sum_{i=1}^n \xi_i J(e_i)(E)$ ,  $\forall E \in \Sigma$ . It is clear that  $F_u(E) = \lim_n F_{u_n}(E) \quad \forall E \in \Sigma$ , and then by the Vitali-Hahn-Saks theorem  $F_u$  is  $\mu$ -continuous and countably additive. Moreover,  $NV_{\Phi}(F_{u_n}) \leq \|J\| \|u\|$ ,  $\forall n \in \mathbb{N}$ . Given a partition  $\mathcal{P}$  of  $\Omega$  by elements of  $\Sigma$  and a  $\varepsilon > 0$ , there is  $n_{\mathcal{P}, \varepsilon} \in \mathbb{N}$  such that

$$\sum_{E \in \mathcal{P}} \Phi\left(\frac{\|F_u(E)\|}{\|J\| \|u\| \mu(E)}\right) \mu(E) \leq \sum_{E \in \mathcal{P}} \Phi\left(\frac{\sum_{i=1}^{n_{\mathcal{P}, \varepsilon}} \|F_{u_n}(E)\| + \varepsilon}{\|J\| \|u\| \mu(E)}\right) \mu(E).$$

Thus  $I_{\Phi}\left(\frac{F_u}{\|J\| \|u\|}\right) \leq \sup_{n \in \mathbb{N}} I_{\Phi}\left(\frac{F_{u_n}}{\|J\| \|u\|}\right) \leq 1$ , and  $F_u \in \text{cabv}_{\Phi}(\mu, X)$  with  $NV_{\Phi}(F_u) \leq \|J\| \|u\|$ . This implies that  $G: \ell_{\infty} \rightarrow \text{cabv}_{\Phi}(\mu, X)$  such that  $G(u) = F_u$  is a well defined, linear and continuous map. As  $G|_{c_0} = J$  and  $\inf_{n \in \mathbb{N}} NV_{\Phi}G(e_n) > 0$ , we can use Lemma 1 to conclude that  $\text{cabv}_{\Phi}(\mu, X)$  contains a subspace isomorphic to  $\ell_{\infty}$ .  $\square$

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