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A-PROJECTIVE RESOLUTIONS AND AN AZUMAYA THEOREM
FOR A CLASS OF MIXED ABELIAN GROUPS

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1. INTRODUCTION

One of the oldest problems in the theory of abelian groups is the question whether a given class of groups is closed with respect to direct summands. Simply presented p -groups and completely decomposable groups are examples of classes where this problem has been solved by an Azumaya-style theorem. No similar results can, however, be obtained for direct summands of sums taken over an arbitrary class \mathcal{A} of groups without imposing some immediate restrictions on the elements of \mathcal{A} . To simplify our notation while considering such decompositions, the class of A -projective groups consists of direct summands of direct sums of copies of a fixed group A . An abelian group G is \mathcal{A} -decomposable if it is of the form $G \cong \bigoplus_{A \in \mathcal{A}} P_A$ where each P_A is A -projective. Azumaya's original result describes the case that every $A \in \mathcal{A}$ has a local endomorphism ring [4]. Arnold, Hunter, and Richman extended his work in [6] where they showed that the class of \mathcal{A} -decomposable groups is closed with respect to direct summands if \mathcal{A} is a pseudo-rigid class of countable groups.

The goal of this paper is to establish the existence of an Azumaya-style theorem for the class of \mathcal{G} -decomposable groups where \mathcal{G} is the class of mixed abelian groups which was introduced by Glaz and Wickless in [11]. In order to define \mathcal{G} , we first consider the class Γ of mixed groups G with the property that G is isomorphic to a pure subgroup of $\prod_p G_p$ containing $\bigoplus_p G_p$. The symbol Γ_∞ denotes the groups in Γ which have finite torsion-free rank. Every $G \in \Gamma_\infty$ contains a finite independent subset X such that $F = \langle X \rangle$ is a free subgroup of G with G/F torsion. We view G as a pure subgroup of $\prod_p G_p$, and write $X = \{x_i = (x_{ip}) \mid i = 1, \dots, n\}$. Glaz and Wickless investigated the class \mathcal{G} of groups in Γ_∞ for which G_p is finite for all p and satisfies $G_p = \langle x_{1p}, \dots, x_{np} \rangle$ for all but finitely many p . Observe that every element of \mathcal{G} is either an honest mixed group of finite. They showed in [11] that

a group $G \in \Gamma_\infty$ such that G_p is finite for all p is in \mathcal{G} if and only if $\text{Hom}(G, tG)$ is a torsion group. In particular, $E(A)/tE(A)$ is a finite dimensional \mathbb{Q} -algebra for $A \in \mathcal{G}$. Goeters, Wickless, and the author continued the discussion of [11] in [3] by showing that the elements of \mathcal{G} are the mixed self-small abelian groups which have finite torsion-free rank.

Since \mathcal{G} is not a pseudo-rigid class, we cannot use the results of [6] directly to show that the class of \mathcal{G} -decomposable groups is closed with respect to direct summands. However, Corollary 4.4 and Theorem 5.2 yield that every group in \mathcal{G} is the finite direct sum of groups with local Walk-endomorphism ring where Walk is the category with abelian groups as objects, but whose morphisms are defined by $\text{Mor}_{\text{Walk}}(G, H) = \text{Hom}(G, H)/\text{Hom}(G, tH)$ ([13]). Therefore, every direct summand G of a \mathcal{G} -decomposable group is Walk-isomorphic to a \mathcal{G} -decomposable group. Going back to the category of abelian groups, this only gives us that there exists a torsion group T such that $G \oplus T$ is \mathcal{G} -decomposable [13, Theorem 12]. Since our proof that a direct summand G of a \mathcal{G} -decomposable group is \mathcal{G} -decomposable will not be simpler if G has a torsion complement, we prove the Azumaya-Theorem for \mathcal{G} directly by showing that the Walk-indecomposable groups in \mathcal{G} exhibit a behavior similar to that of modules with local endomorphism ring when they appear as direct summands of \mathcal{G} -decomposable groups (Lemma 5.4). This behavior is in stark contrast to the case that \mathcal{A} is the class of torsion-free groups of finite rank, where the class of \mathcal{A} -decomposable groups is not closed with respect to direct summands [10, Theorem 91.1].

The Azumaya Theorem for \mathcal{G} is a consequence of our discussion of the structure of A -generated groups in the first part of this paper. For a fixed $A \in \mathcal{G}$, call an abelian group (*finitely*) A -generated if it is an epimorphic image of a group of the form $\bigoplus_I A$ for some (finite) index-set I . Theorem 2.2 shows that every reduced A -generated group is in Γ and isomorphic to a subgroup of $\prod_I A$ for some index-set I . Furthermore, if $G \in \Gamma_\infty$ is A -generated, then $G = H + tG$ for some A -generated group $H \in \mathcal{G}$. Finally, an A -generated reduced group is in \mathcal{G} if and only if it is finitely A -generated.

Section 3 discusses finitely A -generated groups. Following [1], we say that an abelian group G is A -solvable if the evaluation map $\theta_G: \text{Hom}(A, G) \otimes_{E(A)} A \rightarrow G$ is an isomorphism. Theorem 3.2 and Corollary 3.4 characterize the finitely A -generated A -solvable groups. Applications of these results are given in Section 4 where we investigate when finitely A -generated A -solvable groups are essentially indecomposable (Proposition 4.1 and Theorem 4.2). We want to remind the reader that a group A is essentially indecomposable if, whenever $A = B \oplus C$, then B or C is bounded. In particular, we show that a group $A \in \mathcal{G}$ is essentially indecomposable if and only if $E(A)/tE(A)$ is a local ring (Corollary 4.4).

The results of Sections 2 and 3 not merely lay the ground-work for the proof of the Azumaya-Theorem. They also provide substantially deeper insight in the structure of A -generated groups in \mathcal{G} than the corresponding results did in the case of torsion-free groups. This is primarily due to the fact that we are able to combine the well-developed machinery for the discussion of A -solvable groups with a structure theory, which is significantly richer than that for torsion-free abelian groups of finite rank. In addition, we find that these characterizations can be obtained without imposing any immediate restrictions on the $E(A)$ -module structure of A . In contrast, this was necessary for the discussion of A -solvable groups when A is torsion-free.

This demonstrates that there are significant differences between \mathcal{G} and \mathcal{TF}_∞ , the category of torsion-free abelian groups of finite rank, in spite of Wickless' results in [18] which establish a high degree of similarity between these categories at the quasi-level. We want to point out that Wickless' results just like our proof of the Azumaya-Theorem does not use the category Walk .

2. A -GENERATED ABELIAN GROUPS

We begin this section with a summary of standard properties of reduced groups $G \in \Gamma$ which we will frequently use without reference. We view $G \in \Gamma$ as a pure subgroup of $\prod_p G_p$. For any finite number p_1, \dots, p_n of primes, $G = G_{p_1} \oplus \dots \oplus G_{p_n} \oplus G'$ where G' is a fully invariant subgroup of G such that multiplication by p_i is an automorphism of G' for $i = 1, \dots, n$. Moreover, a reduced group G is in Γ if and only if G_p is a direct summand of G for all primes p and G/tG is divisible. In particular, if $G \in \Gamma$ has bounded p -primary subgroups for all p , then $E(G)$ is a pure subring of $\prod_p E(G_p)$ and $tE(G) = \bigoplus_p E(G_p)$.

Given abelian groups A and G , composition of maps induces a right $E(A)$ -module-structure on $H_A(G) = \text{Hom}(A, G)$. For a right $E(A)$ -module M , the symbol $T_A(M)$ denotes $M \otimes_{E(A)} A$. Since the functors induced by H_A and T_A between the category of abelian groups, \mathcal{Ab} , and the category, $\mathcal{M}_{E(A)}$, of right $E(A)$ -modules form an adjoint pair, there exist natural maps $\theta_G: T_A H_A(G) \rightarrow G$ for $G \in \mathcal{Ab}$ and $\varphi_M: M \rightarrow H_A T_A(M)$ for $M \in \mathcal{M}_{E(A)}$.

Lemma 2.1. *Let $A \in \mathcal{G}$.*

- a) *tA is projective as a left $E(A)$ -module, and $\text{Tor}_{E(A)}^1(M, A)$ is torsion-free divisible for all right $E(A)$ -modules M . In particular, $\text{Tor}_{E(A)}^1(M, A) = 0$ whenever the additive group of M is torsion.*
- b) *Every reduced A -generated torsion group G is A -solvable and has the property that $H_A(G) \cong \bigoplus_p H_A(G_p)$ is torsion. Moreover, if $G \in \mathcal{G}$ is A -generated, then $tG \subseteq tA^n$ for some $n < \omega$.*

Proof. a) Let p be a prime of \mathbb{Z} , and write $A = A_p \oplus A^p$ where A^p is p -divisible and $A^p[p] = 0$. Since $E(A) = E(A_p) \times E(A^p)$, every right $E(A)$ -module M decomposes as $M = M_p \oplus M^p$ where M_p is an $E_p = E(A_p)$ -module and M^p is an $E^p = E(A^p)$ -module. In order to show that A_p is a projective $E(A)$ -module, it thus is enough to show that it is projective over E_p . For this, observe that every subgroup of A_p^n is A_p -generated whenever $n < \omega$ since A_p is finite. By Ulmer's Theorem [17], A_p is a flat E_p -module. Since E_p is finite, every finite flat E_p -module is projective.

Furthermore, $\text{Tor}_{E(A)}^1(M, A/tA)$ is torsion-free divisible, and fits into the induced exact sequence $0 = \text{Tor}_{E(A)}^1(M, tA) \rightarrow \text{Tor}_{E(A)}^1(M, A) \rightarrow \text{Tor}_{E(A)}^1(M, A/tA) \xrightarrow{\Delta} M \otimes_{E(A)} tA$, in which $M \otimes_{E(A)} tA$ is a torsion group with bounded p -components. Thus, $\text{Im } \Delta = 0$, and $\text{Tor}_{E(A)}^1(M, A) \cong \text{Tor}_{E(A)}^1(M, A/tA)$ is torsion-free divisible. If the additive group of M is torsion, then so is $\text{Tor}_{E(A)}^1(M, -)$. By what has been shown, $\text{Tor}_{E(A)}^1(M, A) = 0$.

b) Let G be a reduced A -generated torsion group. It remains to show that θ_G is one-to-one. Consider an exact sequence $\bigoplus_I A \rightarrow G_p \rightarrow 0$. Since $\text{Hom}(A^p, G_p) = 0$, the group G_p is an epimorphic image of $\bigoplus_I A_p$. If $k_p < \omega$ is minimal with $p^{k_p} A_p = 0$, then A has a direct summand U_p isomorphic to $\mathbb{Z}/p^{k_p}\mathbb{Z}$ and $p^{k_p} G = 0$. Therefore, we can find a monomorphism $\alpha: G_p \rightarrow \bigoplus_{I_p} U_p$ for some index-set I_p . Consequently, G is isomorphic to a subgroup of the A -projective torsion group $P = \bigoplus_p [\bigoplus_{I_p} U_p]$. Observe that $H_A(P) \cong \bigoplus_p [\bigoplus_{I_p} H_A(U_p)]$ since A is self-small. Thus, $H_A(P)$ is torsion and $\text{Tor}_{E(A)}^1(H_A(P)/H_A(G), A) = 0$ by a). Therefore, the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_A H_A(G) & \longrightarrow & T_A H_A(P) \\ & & \downarrow \theta_G & & \downarrow \theta_P \\ 0 & \longrightarrow & G & \longrightarrow & P \end{array}$$

has exact rows, from which it follows that θ_G is one-to-one.

If $G \in \mathcal{G}$ is A -generated, then every cyclic summand of G_p is isomorphic to a subgroup of A_p by what has been established. The statement follows once we have shown that there exists $m < \omega$ such that G_p is the direct sum of at most m cyclic groups. To see this, observe that, for all but finitely many primes, G_p is generated by at most n elements where $n = r_0(G)$. But then, G_p cannot be the direct sum of more than n non-zero cyclic subgroups. Since G_p is finite for all primes, the last statement in b) follows. \square

In the following, the A -radical of a group G is denoted by $R_A(G) = \bigcap \{ \ker \varphi \mid \varphi \in \text{Hom}(G, A) \}$. Clearly, $R_A(G) = 0$ if and only if $G \subseteq A^I$ for some index-set I .

Theorem 2.2. *Let $A \in \mathcal{G}$.*

- a) *The following conditions are equivalent for an A -generated group G :*
- i) G is reduced.
 - ii) $\ker \theta_G$ is torsion-free divisible, and $\mathbb{Q} \not\subseteq G$.
 - iii) $R_A(G) = 0$.
 - iv) $\mathbb{Q} \not\subseteq G$; and if $p^k A_p = 0$ for some $k < \omega$, then $p^k G_p = 0$.
- b) *A reduced A -generated group G is finitely A -generated if and only if $G \in \mathcal{G}$.*
- c) *Let G be a reduced A -generated group which has finite torsion-free rank. Then, G contains a finitely A -generated subgroup H such that $\ker \theta_H \cong \ker \theta_G$ and $G = H + tG$. In particular, G/H is A -solvable.*

Proof. For the sake of an easier reference, we first show that every reduced A -generated group G is in Γ : Observe that G/tG is an epimorphic image of $\bigoplus_I A$ for some index-set I . Under this isomorphism, $\bigoplus_I tA$ is mapped to zero. Hence G/tG is divisible as an epimorphic image of the divisible group $\bigoplus_I A/tA$.

Let k_p be the smallest positive integer such that $p^{k_p} A_p = 0$. If $p^{k_p} G_p \neq 0$, then G has a direct summand $U \cong \mathbb{Z}/p^n \mathbb{Z}$ for some integer $n > k_p$ because G_p is reduced. Since U is A -generated, there is an epimorphism $\varphi: A \rightarrow U$ whose kernel contains $p^n A$. If we write $A = A_p \oplus A^p$ with A^p is p -divisible and $A[p] = 0$, then $n > k_p$ yields $p^n A = A^p$, and $A/p^n A$ is bounded by p^{k_p} . Hence, U cannot be an image of A , a contradiction. Therefore, G_p is a bounded direct summand of G , and $G \in \Gamma$. Furthermore, tG is A -generated. We now prove the equivalences in part a) of the theorem:

iv) \Rightarrow ii): Since G has to be reduced, $G \in \Gamma$, and tG is an A -solvable group such that $H_A(tG) \cong \bigoplus_p H_A(G_p)$ is torsion (Lemma 2.1). The fact that $H_A(G)/H_A(tG)$ is isomorphic to a subgroup of the torsion-free divisible group $H_A(G/tG)$ yields $tH_A(G) = H_A(tG)$. On the other hand, if we view G as a pure subgroup of $\prod_p G_p$, then $H_A(G)$ is a pure subgroup of $H_A(\prod_p G_p)$, and the torsion subgroup of the latter group is $H_A(tG)$ too. Consequently, the right $E(A)$ -module $M = H_A(G)/H_A(tG)$ has a torsion-free divisible additive group. Consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker \theta_G & \xrightarrow{\lambda} & \ker \theta & \longrightarrow & 0 \\
 & & \downarrow \tau & & \downarrow e & & \\
 T_A H_A(tG) & \xrightarrow{T_A H_A(\alpha)} & T_A H_A(G) & \xrightarrow{T_A H_A(\beta)} & T_A(M) & \longrightarrow & 0 \\
 & & \downarrow \theta_G & & \downarrow \theta & & \\
 0 & \longrightarrow & tG & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & G/tG \longrightarrow 0
 \end{array}$$

where λ is onto by the Snake-Lemma. Since $T_A(M)$ and G/tG are torsion-free divisible, $\ker \theta_G \cong \ker \theta$ is torsion-free divisible.

ii) \Rightarrow i): Observe that G is isomorphic to a direct summand of the group $H = T_A H_A(G)$. Any free resolution of $H_A(G)$ induces an exact sequence $0 \rightarrow U \xrightarrow{\alpha} \bigoplus_I A \xrightarrow{\beta} H \rightarrow 0$ with $S_A(U) = U$. Let p be a prime of \mathbb{Z} . For $x \in H_p$, we can find $y \in \bigoplus_I A$ with $\beta(y) = x$. If $p^m x = 0$, then there are $a_1, \dots, a_n \in A$ and $\varphi_1, \dots, \varphi_n \in H_A(U)$ with $p^m y = \sum_{i=1}^n \alpha \varphi_i(a_i)$. Since A/tA is divisible, we can find $b_1, \dots, b_n \in tA$ and $c_1, \dots, c_n \in A$ with $a_i = p^m c_i + b_i$. Thus $p^m [y - \sum_{i=1}^n \alpha \varphi_i(c_i)] = \sum_{i=1}^n \alpha \varphi_i(b_i) \in \bigoplus_I tA$. Write $o(y - \sum_{i=1}^n \alpha \varphi_i(c_i)) = p^t q$ where $t \leq k_p$ and $(p, q) = 1$. Then, $p^t q x = \beta(p^t q (y - \sum_{i=1}^n \alpha \varphi_i(c_i))) = 0$. Since $x \in H_p$, we obtain $p^t x = 0$. Thus, H_p is bounded by p^{k_p} , and G_p is reduced.

i) \Rightarrow iii): Since G_p is bounded by p^{k_p} , we have $G_p \subseteq \prod_{I_p} A_p$ for some index-set I_p . But $G \in \Gamma$ implies $G \subseteq \prod_p G_p$.

iii) \Rightarrow iv) follows directly from the initial remarks of the proof.

b) Let G be a reduced A -generated group. If G admits an exact sequence $A^n \rightarrow G \rightarrow 0$, then $r_0(G) < \infty$. Using the arguments in the initial part of the proof of a), we obtain that $G \in \Gamma_\infty$, tG is A -generated, and $|G_p| < \infty$ for all p . It remains to show that $\text{Hom}(G, tG)$ is torsion. For this, observe that there is an induced exact sequence $0 \rightarrow \text{Hom}(G, tG) \rightarrow \bigoplus_n H_A(tG)$, in which $H_A(tG)$ is torsion by Lemma 2.1b. Conversely, assume that G is an A -generated group in \mathcal{G} . We can find $n < \omega$ and a map $\alpha: A^n \rightarrow G$ such that $G/\alpha(A^n)$ is torsion. We show that $G = tG + \alpha(A^n)$. For this observe that $[tG + \alpha(A^n)]/tG \cong \alpha(A^n)/[tG \cap \alpha(A^n)]$ is a torsion-free image of A^n , and hence an image of the divisible group A^n/tA^n . Therefore, $[tG + \alpha(A^n)]/tG$ is a direct summand of the torsion-free divisible group G/tG . Since $G/[tG + \alpha(A^n)]$ is torsion, we have $G = tG + \alpha(A^n)$.

Consider the exact sequence $0 \rightarrow \alpha(A^n) \rightarrow G \rightarrow T \rightarrow 0$ for some torsion group T . Since $T \cong [tG + \alpha(A^n)]/\alpha(A^n) \cong tG/[tG \cap \alpha(A^n)]$, we have that T_p is finite for all p . Moreover, $G[p] = 0$ implies $T[p] = 0$. If $T[p] \neq 0$ for all primes p in an infinite set P_1 of primes, then $\text{Hom}(T, tG) \supseteq \prod_{P_1} \text{Hom}(T_p, G_p)$, which cannot be torsion since T_p and G_p are non-zero finite p -groups for $p \in P_1$. On the other hand, the exact sequence $0 \rightarrow \text{Hom}(T, tG) \rightarrow \text{Hom}(G, tG)$ shows that this group is torsion. The resulting contradiction establishes that T is finite, from which it immediately follows that G is finitely A -generated.

c) There is nothing to show if G is torsion. Thus, we may assume that G contains a non-empty independent subset $X = \{x_1, \dots, x_n\}$ such that $\langle X \rangle$ is free and $G/\langle X \rangle$ is torsion. Since we can view G as a pure subgroup of $\prod_p G_p$, we set $X_p = \langle x_{1p}, \dots, x_{np} \rangle$ for each prime p . The inclusions $X_p \subseteq G_p$ coordinatewise induce a monomorphism $\lambda: \prod_p X_p \rightarrow \prod_p G_p$. Clearly, λ operates like the identity on $\bigoplus_p X_p$ and satisfies $\lambda(x_i) = x_i$ for all i . Observe that x_1, \dots, x_n generate a free subgroup of $\prod_p X_p$.

We set $tH = \bigoplus_p X_p$, and let H be the subgroup of $\prod_p X_p$ containing tH such that $H/tH = \bigoplus_{i=1}^n \mathbb{Q}(x_i + tH)$. By definition, $H \in \mathcal{G}$. For $h \in H$, we can find $r_1, \dots, r_n \in \mathbb{Z}$ and $y \in tH$ such that $mh = r_1x_1 + \dots + r_nx_n + y$ for some non-zero integer m . Then, $m\lambda(h) \in G$ since $tH \subseteq tG$. Since $(\prod_p G_p)/G$ is torsion-free, we have $\lambda(h) \in G$. We shall identify H with its image under λ in G . Since $x_1, \dots, x_n \in H + tG$, we have that $G/(H + tG)$ is torsion. On the other hand, $(H + tG)/tG \cong H/(H \cap tG) = H/tH$ is torsion-free divisible. Thus, $(H + tG)/tG$ is a direct summand of the torsion-free group G/tG whose complement is isomorphic to $G/(H + tG)$. By what has been shown, this is only possible if $G = H + tG$. In particular, $G/H = (H + tG)/H \cong tG/tH$ is a reduced A -generated torsion group which is A -solvable by Lemma 2.1b.

In the exact sequence $0 \rightarrow H_A(H) \rightarrow H_A(G) \rightarrow M \rightarrow 0$, the additive group of M is torsion as a subgroup of $H_A(G/H)$ by Lemma 2.1b. Because of Lemma 2.1a, $\text{Tor}_{E(A)}^1(M, A) = 0$, and the top-row of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_A H_A(H) & \longrightarrow & T_A H_A(G) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\
& & \downarrow \theta_H & & \downarrow \theta_G & & \downarrow \theta & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0
\end{array}$$

is exact. The induced map θ satisfies $\theta = \theta_{G/H} T_A(\iota)$ where $\iota: M \rightarrow H_A(G/H)$ is the inclusion map. Observe that $T_A(\iota)$ is one-to-one by Lemma 2.1a, from which we obtain that θ is an isomorphism. By the Snake-Lemma, θ_H is an epimorphism, and $\ker \theta_H \cong \ker \theta_G$. \square

Corollary 2.3. *The class of reduced A -solvable groups is closed with respect to direct sums whenever $A \in \mathcal{G}$.*

Proof. Let $\{G_i\}_{i \in I}$ be a family of reduced A -solvable groups. Suppose $\varphi \in H_A(\bigoplus_I G_i)$ satisfies $\pi_i \varphi \neq 0$ for infinitely many $i \in I$ where $\pi_j: \bigoplus_{i \in I} G_i \rightarrow G_j$ denotes the projection onto the j^{th} -coordinate. For each such i , there is a map $\alpha_i: G_i \rightarrow A$ with $\alpha_i \pi_i \varphi \neq 0$ by Theorem 2.2a. Coordinatewise, the maps α_i induce a map $\alpha: \bigoplus_I G_i \rightarrow \bigoplus_I A$. If $\delta_i: \bigoplus_I A \rightarrow A$ denotes the projection onto the i^{th} -coordinate, then $\delta_i \alpha \varphi = \alpha_i \varphi$ is non-zero for infinitely many i , which contradicts the fact that the groups in \mathcal{G} are self-small [3]. Thus, $\{G_i\}_{i \in I}$ is A -small. Since the class of A -solvable groups is closed with respect to A -small direct sums by [2], the proof is complete. \square

Furthermore, Theorem 2.2 allows to answer the question for which groups A there may exist A -solvable groups G with $R_A(G) \neq 0$:

Corollary 2.4. *Let A be a self-small abelian group of finite torsion-free rank such that A/tA is a faithfully flat $E(A)/tE(A)$ -module. Then, there exists a reduced A -solvable group G with $R_A(G) \neq 0$ if and only if A is torsion-free and reduced.*

Proof. Suppose that there exists a reduced A -solvable group G with $R_A(G) \neq 0$. By Theorem 2.2 together with [3], we obtain that A is either torsion or torsion-free. In the first case, the group A has to be finite, and every A -solvable group has a zero A -radical, which is not possible. In the second case, it remains to show that A is reduced. If $\mathbb{Q} \subseteq A$, then every torsion-free A -generated group is A -solvable as in [1]. But this yields that A is homogeneous completely decomposable by [1]. Therefore, $A \cong \mathbb{Q}^n$, and all A -solvable groups are A -projective. On the other hand, if A is torsion-free reduced, then there exists a right $E(A)$ -module M which is \aleph_1 -free and has $E(A)$ as its \mathbb{Z} -endomorphism-ring using the construction of [9] (e.g. see [2]). Let $G = T_A(M)$. Once we have shown that G is A -solvable, it will follow as in [2] that $E(G) = \text{Center}(E(A))$ and $\text{Hom}(G, A) = 0$. The A -solvability of G , however, follows from the fact that, for $\varphi_1, \dots, \varphi_k \in H_A(G)$, there is a free submodule U of M with $\sum_{i=1}^k \varphi_i(A) \subseteq T_A(U)$. \square

3. \mathcal{G}_A -PRESENTED ABELIAN GROUPS

A sequence $0 \rightarrow B \xrightarrow{\alpha} C \xrightarrow{\beta} G \rightarrow 0$ of abelian groups is *almost A -balanced* if $M = H_A(G)/\text{Im } H_A(\beta)$ is torsion.

Lemma 3.1. *Let $A \in \mathcal{G}$ and $0 \rightarrow B \rightarrow C \xrightarrow{\pi} G \rightarrow 0$ be an almost A -balanced sequence in which C is A -solvable and $G \in \mathcal{G}$. Then, G is A -solvable if and only if B is A -generated.*

Proof. By [1, Lemma 2.1], $M = \text{Im } H_A(\pi)$ fits into an exact sequence $T_A H_A(B) \xrightarrow{\theta_B} B \rightarrow T_A(M) \xrightarrow{\theta} G \rightarrow 0$, and it is enough to show that G is A -solvable iff θ is a monomorphism. If $\iota: M \rightarrow H_A(G)$ denotes the inclusion map, then $\theta = \theta_G T_A(\iota)$ and $\text{coker } \iota$ is torsion as an abelian group. Moreover, $\ker T_A(\iota) = 0$ as an epimorphic image of $\text{Tor}_{E(A)}^1(H_A(G)/M, A)$ which vanishes by Lemma 2.1a. Clearly, this gives that θ is one-to-one if G is A -solvable. Conversely, assume that θ is a monomorphism. For $x \in \text{Im } T_A(\iota) \cap \ker \theta_G$, there is $y \in T_A(M)$ such that $x = T_A(\iota)(y)$. Then, $\theta(y) = \theta_G T_A(\iota)(y) = 0$ yields $x = 0$. Therefore, $\ker \theta_G$ is isomorphic to a subgroup of the torsion group $\text{coker } T_A(\iota) \cong T_A(\text{coker } \iota)$ which results in a contradiction unless G is A -solvable since $\ker \theta_G$ is torsion-free divisible by Theorem 2.2. \square

Theorem 3.2. *Let $A \in \mathcal{G}$. The following conditions are equivalent for a reduced abelian group G :*

- a) G is a finitely A -generated A -solvable group.
- b) G is an A -solvable group in \mathcal{G} .
- c) G admits an almost A -balanced sequence $0 \rightarrow U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \rightarrow 0$ such that U is A -generated.

Moreover, the sequence in c) can be chosen in such a way that

$$T_A(H_A(G))/\text{Im } H_A(\beta) = 0.$$

Proof. The equivalence of a) and b) is an immediate consequence of Theorem 2.2a. Moreover, a reduced finitely A -generated group is in \mathcal{G} by Theorem 2.2b. By Lemma 3.1, the implication c) \Rightarrow b) is true.

b) \Rightarrow c): Since G has finite torsion-free rank, we can choose a finite independent subset $X = \{x_1, \dots, x_n\}$ of G such that $\langle X \rangle$ is free and $G/\langle X \rangle$ is torsion. Because $G \in \mathcal{G}$, we have $G_p = \langle x_{1p}, \dots, x_{np} \rangle$ for almost all primes p of \mathbb{Z} . There are maps $\varphi_1, \dots, \varphi_m \in H_A(G)$ such that $X \subseteq \varphi_1(A) + \dots + \varphi_m(A)$. We write $A = A_p \oplus A^p$ and $G = G_p \oplus G^p$ for each prime p such that A^p and G^p are fully invariant. There are $a_{1p}, \dots, a_{mp} \in A_p$ and $b_{1p}, \dots, b_{mp} \in A^p$ such that $x_i = \sum_{j=1}^m \varphi_j(a_{jp} + b_{jp})$. Since $\text{Hom}(A_p, G^p) = 0 = \text{Hom}(A^p, G_p)$, we have $x_{ip} = \sum_{j=1}^m \varphi_j(a_{jp})$; and $G_p \subseteq \varphi_1(A) + \dots + \varphi_m(A)$ for all but finitely many primes p . By adding finitely many maps to $\{\varphi_1, \dots, \varphi_m\}$ if necessary, we may assume that $tG \subseteq \varphi_1(A) + \dots + \varphi_m(A)$ and $G/[\varphi_1(A) + \dots + \varphi_m(A)]$ is torsion. Since $H_A(G)$ has finite torsion-free rank, no generality is lost if we assume that $M = \langle \varphi_1, \dots, \varphi_m \rangle$ has the additional property that $H_A(G)/M$ is torsion as an abelian group.

We define a map $\theta: T_A(M) \rightarrow G$ by $\theta(\alpha \otimes a) = \alpha(a)$, and observe $\theta = \theta_G T_A(\iota)$ where $\iota: M \rightarrow H_A(G)$ is the inclusion map. Since θ_G is an isomorphism and $\ker T_A(\iota) = 0$ as an image of $\text{Tor}_{E(A)}^1(H_A(G)/M, A)$ which vanishes by Lemma 2.1, θ is one-to-one. Furthermore, $\text{Im } \theta = \varphi_1(A) + \dots + \varphi_m(A)$ yields that $\text{coker } T_A(\iota) \cong G/\text{Im } \theta \cong (G/tG)/(\text{Im } \theta/tG)$ is torsion and divisible. In particular, $(G/\text{Im } \theta)_p$ either vanishes or is unbounded. On the other hand, since M and $H_A(G)$ are $E(A)$ -modules, the same holds for $(\text{coker } \iota)_p$. However, the $E(A)$ -module-structure of the latter module is completely determined by its E_p -module-structure. Since E_p is finite, $(\text{coker } \iota)_p$ is bounded, and $\text{coker } T_A(\iota) \cong T_A(\text{coker } \iota)$ has bounded p -components. This is only possible if $\text{coker } T_A(\iota) = 0$. Therefore, $T_A(\iota)$ is an isomorphism, and the same holds for θ .

Choose a projective resolution $0 \rightarrow V \xrightarrow{\lambda} E(A)^m \xrightarrow{\pi} M \rightarrow 0$ of M . Since $U = \ker T_A(\pi) = \text{Im } T_A(\lambda)$ is A -generated, it remains to show that $0 \rightarrow U \rightarrow T_A(E(A)^m) \xrightarrow{T_A(\pi)} T_A(M) \rightarrow 0$ is almost A -balanced. An application of the functor

H_A yields the diagram

$$\begin{array}{ccccc}
H_A T_A(E(A)^m) & \xrightarrow{H_A T_A(\pi)} & H_A T_A(M) & & \\
\uparrow \varphi_{E(A)^m} & & \uparrow \varphi_M & & \\
E(A)^m & \xrightarrow{\pi} & M & \longrightarrow & 0.
\end{array}$$

It gives $\text{Im}(H_A T_A(\pi)) = \text{Im } \varphi_M$, and the proof is complete once we have shown that φ_M has a torsion cokernel. For this, we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_A T_A(M) & \xrightarrow{H_A T_A(\iota)} & H_A T_A H_A(G) & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \uparrow \varphi_M & & \uparrow \varphi_{H_A(G)} & & \uparrow \varphi & & \\
0 & \longrightarrow & M & \xrightarrow{\iota} & H_A(G) & \longrightarrow & \text{coker } \iota & \longrightarrow & 0
\end{array}$$

whose first row is exact since $T_A(\iota)$ is an isomorphism. By the Snake-Lemma, $\text{coker } \varphi_M \cong \ker \varphi = \text{coker } \iota$. As we have shown before $\text{coker } \iota$ is torsion as abelian group. \square

We say that an A -generated group $G \in \mathcal{G}$ is \mathcal{G}_A -presented if there exists an almost A -balanced exact sequence $0 \rightarrow U \rightarrow A^n \rightarrow G \rightarrow 0$ in which U is A -generated and in \mathcal{G} . By Lemma 3.1, any \mathcal{G}_A -presented group is A -solvable.

Proposition 3.3. *For a group $A \in \mathcal{G}$, the class of \mathcal{G}_A -presented groups is closed with respect to direct summands.*

Proof. Let $G = B \oplus C$ be a \mathcal{G}_A -presented group. Choose an almost A -balanced exact sequence $0 \rightarrow U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \rightarrow 0$ with $U \in \mathcal{G}$, and let $\pi_B: G \rightarrow B$ be the projection along C . We consider the induced exact sequence $(\mathcal{E}) 0 \rightarrow V \xrightarrow{\lambda} A^n \xrightarrow{\pi_B \beta} B \rightarrow 0$ in which λ is the inclusion-map. If $\varphi \in H_A(B)$, then there is $\gamma \in H_A(G)$ with $\varphi = \pi_B \gamma = H_A(\pi_B)(\gamma)$. We can find a non-zero integer m and a map $\delta \in H_A(A^n)$ with $m\gamma = H_A(\beta)(\delta)$. Therefore, $m\varphi = H_A(\pi_B)(m\gamma) = H_A(\pi_B \beta)(\delta)$, and (\mathcal{E}) is almost A -balanced. Furthermore, B is A -solvable and in \mathcal{G} as a direct summand of an A -solvable group in \mathcal{G} . If $M = \text{Im } H_A(\pi_B \beta) \subseteq H_A(B)$, then the evaluation map $\theta: T_A(M) \rightarrow B$ satisfies $\theta = \theta_B T_A(\iota)$ where $\iota: M \rightarrow H_A(B)$ is the inclusion map. Since $\text{coker } \iota$ is torsion, we have that $T_A(\iota)$ is a monomorphism, and the same holds for θ . By [1, Lemma 2.1], $\text{coker } \theta_V \cong \ker \theta = 0$. Therefore, V is an A -generated subgroup of A^n , and it remains to show that $V \in \mathcal{G}$. Since $V \in \Gamma_\infty$ as in the proof of Theorem 2.2, it suffices to establish that $\text{Hom}(V, tV)$ is torsion:

Observe that $C = \beta(V)$ and fits into the exact sequence $0 \rightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta|_V} C \rightarrow 0$ in which U and C are A -generated groups in \mathcal{G} . By Theorem 2.2, U and C are

finitely A -generated. In particular, $\text{Hom}(U, tA)$ and $\text{Hom}(C, tA)$ are torsion. Since $V \subseteq A^n$, we have $\text{Hom}(V, tV) \subseteq \bigoplus_n \text{Hom}(V, tA)$. But $\text{Hom}(V, tA)$ fits into the exact sequence $0 \rightarrow \text{Hom}(C, tA) \rightarrow \text{Hom}(V, tA) \rightarrow \text{Hom}(U, tA)$ in which the first and third Hom-group are torsion. \square

Corollary 3.4. *Let A be in \mathcal{G} . The following conditions are equivalent for a reduced abelian group G :*

- a) G is an A -solvable group in \mathcal{G} .
- b) There exists an almost A -balanced exact sequence $0 \rightarrow T \rightarrow H \rightarrow G \rightarrow 0$ in which H is \mathcal{G}_A -presented and T is a reduced A -generated torsion group.

P r o o f. b) \Rightarrow a): Since \mathcal{G}_A -presented groups are finitely A -generated, G is in \mathcal{G} by Theorem 2.2b as a reduced finitely A -generated group. Because T is A -solvable, G is A -solvable by Theorem 3.2.

a) \Rightarrow b): Since G is A -solvable, we can find an almost A -balanced exact sequence $0 \rightarrow U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \rightarrow 0$ in which U is A -generated and α is an inclusion map. By Theorem 2.2c, there exists an A -generated subgroup V of U with $V \in \mathcal{G}$ and $U = V + tU$. We consider the induced sequence $(\mathcal{E}) 0 \rightarrow U/V \xrightarrow{\bar{\alpha}} A^n/V \xrightarrow{\bar{\beta}} G \rightarrow 0$. Observe that $U/V \cong tU/tV$ is a reduced A -generated torsion-group which is A -solvable.

Let $\pi_1: U \rightarrow U/V$ and $\pi_2: A^n \rightarrow A^n/V$ be the canonical projections. Since $\pi_2\alpha = \bar{\alpha}\pi_1$ and $\bar{\beta}\pi_2 = \beta$, we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_A(U) & \xrightarrow{H_A(\alpha)} & H_A(A^n) & \xrightarrow{H_A(\beta)} & H_A(G) \\ & & \downarrow H_A(\pi_1) & & \downarrow H_A(\pi_2) & & \downarrow 1_{H_A(G)} \\ 0 & \longrightarrow & H_A(U/V) & \xrightarrow{H_A(\bar{\alpha})} & H_A(A^n/V) & \xrightarrow{H_A(\bar{\beta})} & H_A(G). \end{array}$$

Given $x \in H_A(G)$, there is a non-zero integer m such that $mx = H_A(\beta)(y)$ for some $y \in H_A(A^n)$. Then, $mx = H_A(\bar{\beta})H_A(\pi_2)(y)$, and (\mathcal{E}) is almost A -balanced. Set $M = \text{Im } H_A(\bar{\beta})$, and let $\iota: M \rightarrow H_A(G)$ be the inclusion map. Since $\text{coker } \iota$ is torsion and the evaluation map $\theta: T_A(M) \rightarrow G$ satisfies $\theta = \theta_G T_A(\iota)$, the map $T_A(\iota)$ is a monomorphism, and the same holds for θ . The map θ fits into the commutative diagram

$$\begin{array}{ccccccccc} T_A H_A(U/V) & \longrightarrow & T_A H_A(A^n/V) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\ & & \downarrow \theta_{U/V} & & \downarrow \theta_{A^n/V} & & \downarrow \theta \\ 0 & \longrightarrow & U/V & \longrightarrow & A^n/V & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

It follows that $\theta_{A^n/V}$ is an isomorphism. In particular, A^n/V is reduced by Theorem 2.2a. An application of Theorem 2.2b gives $A^n/V \in \mathcal{G}$.

It remains to show that $0 \rightarrow V \rightarrow A^n \xrightarrow{\pi_2} A^n/V \rightarrow 0$ is almost A -balanced. Given $\varphi \in H_A(A^n/V)$, there is a non-zero integer ℓ such that $H_A(\overline{\beta})(\ell\varphi) = H_A(\beta)(\psi) = H_A(\overline{\beta})H_A(\pi_2)(\psi)$ for some $\psi \in H_A(A^n)$. Thus, $\ell\varphi - H_A(\pi_2)(\psi) \in H_A(\overline{\alpha})(H_A(U/V))$ which is a torsion group by Lemma 2.1b. Hence, $k\ell\varphi \in \text{Im } H_A(\pi_2)$ for some non-zero integer k . \square

Looking at the almost A -balanced sequences of the form $\cdot \xrightarrow{\beta} \cdot \rightarrow 0$ constructed in Theorem 3.2 and Corollary 3.4, we see that $N = \text{coker } H_A(\beta)$ always satisfies $T_A(N) = 0$. Since N is torsion, we obtain the exact sequence $0 = \text{Tor}_{E(A)}^1(N, A/tA) \rightarrow N \otimes_{E(A)} tA \rightarrow T_A(N) = 0$. It yields $N_p \otimes_{E_p} A_p \cong N_p \otimes_{E(A)} A_p = 0$ for all primes p . Since A_p is homogeneous if and only if it is faithfully flat as an E_p -module [3], we have $N = 0$ if A_p is homogeneous for all primes p . We thus have shown:

Corollary 3.5. *Let $A \in \mathcal{G}$ have homogeneous p -components for all primes p , and suppose that $G \in \mathcal{G}$ is A -solvable.*

- a) *There exists an A -balanced exact sequence $0 \rightarrow U \rightarrow A^n \rightarrow G \rightarrow 0$ with $S_A(U) = U$. Moreover, G is \mathcal{G}_A -presented if and only if the sequence can be chosen such that $U \in \mathcal{G}$.*
- b) *There exists an A -balanced exact sequence $0 \rightarrow T \rightarrow H \rightarrow G \rightarrow 0$ in which T is a torsion A -solvable group and H is \mathcal{G}_A -presented.*

4. DIRECT SUM DECOMPOSITIONS OF \mathcal{G}_A -PRESENTED GROUPS

In the following, projection modulo the torsion subgroup of a given abelian group will be indicated by an overscore. This section investigates how the existence of non-trivial direct sum decompositions of the right $E(A)/tE(A)$ -module $\overline{H_A(G)}$ is related to decompositions of the A -solvable group G . Observe that the $E(A)/tE(A)$ -module structure of \overline{M} coincides with its $E(A)$ -module structure for any right $E(A)$ -module M .

Proposition 4.1. *Let $A \in \mathcal{G}$, and G be an A -solvable group in \mathcal{G} . If $\overline{H_A(G)}$ is an indecomposable $E(A)/tE(A)$ -module, then G is essentially indecomposable.*

Proof. There is nothing to show if G is torsion. Hence, suppose that G is an honest mixed group. If it is not essentially indecomposable, then there exists an idempotent $e \in E(G)$ such that $e, 1 - e \notin tE(G)$. Since G is A -solvable, we

have $E(G) \cong \text{End}_{E(A)}(H_A(G))$ as has been shown in [2, Theorem 4.4]. Let f be the idempotent corresponding to e under this isomorphism. Clearly, f induces an idempotent endomorphism $\overline{f}: \overline{H_A(G)} \rightarrow \overline{H_A(G)}$ by $\overline{f}(\overline{x}) = \overline{f(x)}$. Since $\overline{H_A(G)}$ is indecomposable as an $E(A)/tE(A)$ -module, its endomorphism ring has no non-trivial idempotents. Without loss of generality, we have $\overline{f} = 0$. Thus, $f: H_A(G) \rightarrow tH_A(G)$. However, $tH_A(G) = H_A(tG)$ by Lemma 2.1 since $G \in \mathcal{G}$. We view $T_A(f)$ as a map from $T_A H_A(G)$ into $T_A H_A(tG)$. Since tG and G are A -solvable, $\theta_{tG} T_A(f) \theta_G^{-1}$ is an element of the torsion group $\text{Hom}(G, tG)$. We obtain that $mT_A(f) = 0$ for some non-zero integer m . Observe that $H_A T_A(f) \varphi_{H_A(G)} = \varphi_{H_A(tG)} f$ yields $\varphi_{H_A(tG)} m f = 0$. Since tG is A -solvable by Lemma 2.1b, we have that $\varphi_{H_A(tG)}$ is an isomorphism. Hence, $m f = 0$, which is not possible by the choice of f . \square

We now show that the converse of this result is true if G is \mathcal{G}_A -presented.

Theorem 4.2. *Let $A \in \mathcal{G}$. A \mathcal{G}_A -presented group G is essentially indecomposable if and only if $\overline{H_A(G)}$ is an indecomposable $E(A)/tE(A)$ -module.*

Proof. As before, it is enough to consider the case that G is an honest mixed group. Assume that G is essentially indecomposable. Let $0 \rightarrow U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \rightarrow 0$ be an almost A -balanced exact sequence where $U \in \mathcal{G}$ is A -generated. It induces the exact sequence $0 \rightarrow H_A(U) \xrightarrow{H_A(\alpha)} H_A(A^n) \xrightarrow{H_A(\beta)} M \rightarrow 0$ in which $M = \text{Im } H_A(\beta)$ is a submodule of $H_A(G)$ with $H_A(G)/M$ torsion. Observe that $G \cong T_A(M)$ by [1, Lemma 2.1]. Moreover, $K = H_A(G)/[M + tH_A(G)]$ is torsion as an abelian group and fits into the exact sequence $0 \rightarrow M/tM \rightarrow H_A(G)/tH_A(G) \rightarrow K \rightarrow 0$ because of $(M + tH_A(G))/tH_A(G) \cong M/(M \cap tH_A(G)) = M/tM$. Since M/tM and $H_A(G)/tH_A(G)$ are torsion-free divisible, the same has to hold for K . Therefore, $K = 0$, and $\overline{M} \cong \overline{H_A(G)}$. Once we have shown that

- (I) if $\varphi: \overline{M} \rightarrow \overline{M}$ is an $E(A)/tE(A)$ -morphism, then there is a map $\tau: M \rightarrow M$ such that $\varphi(\overline{x}) = \overline{\tau(x)}$ for all $x \in M$, and
- (II) if $\tau \in \text{End}_{E(A)}(M)$ satisfies $\overline{\tau(x)} = 0$ for all $x \in M$, then $\tau \in tE(A)$,

then the theorem is shown as follows:

Let $\pi: \overline{M} \rightarrow \overline{M}$ be an $E(A)$ -morphism with $\pi^2 = \pi$. There is a map $\lambda: M \rightarrow M$ with $\overline{\lambda(x)} = \pi(\overline{x})$ for all $x \in M$ by (I). Since $\overline{\lambda^2(x)} = \pi(\overline{\lambda(x)}) = \pi^2(\overline{x}) = \pi(\overline{x})$, we have $k\lambda^2 = k\lambda$ for some non-zero integer k using (II). Let P be the set of primes dividing k , and write $A = A_1 \oplus A_2$ with $A_1 = \bigoplus_{q \in P} A_q$ and $\text{Hom}(A_i, A_j) = 0$ if $i \neq j$. Then, $E(A) = E(A_1) \times E(A_2)$, and $M = M_1 \oplus M_2$ such that $M_i E(A_j) = 0$ for $i \neq j$. In particular, M_1 is bounded, and $\lambda(M_i) \subseteq M_i$ for $i = 1, 2$. Therefore, $\lambda|_{M_2}$ is an idempotent of $\text{End}_{E(A)}(M_2)$. Write $M_2 = \lambda(M_2) \oplus (1 - \lambda)(M_2)$, and observe $G \cong T_A(M) = T_A(M_1 \oplus (1 - \lambda)(M_2)) \oplus T_A(\lambda(M_2))$. Since G is essentially indecomposable, we have that one of the modules $T_A(\lambda(M_2))$ or $T_A(M_1 \oplus (1 - \lambda)(M_2))$ is bounded.

In the first case, $H_A T_A(\lambda(M_2))$ is bounded as an abelian group. The commutative diagram

$$\begin{array}{ccccc}
 H_A T_A(\lambda(M_2)) & \longrightarrow & H_A T_A H_A(G) & & \\
 \uparrow \varphi_{\lambda(M_2)} & & \wr \uparrow \varphi_{H_A(G)} & & \\
 0 & \longrightarrow & \lambda(M_2) & \longrightarrow & H_A(G)
 \end{array}$$

yields that $\lambda(M_2)$ is isomorphic to a subgroup of the bounded group $H_A T_A(\lambda(M_2))$. Consequently, $\lambda(M) = \lambda(M_1) \oplus \lambda(M_2)$ is bounded, and $\pi = \bar{\lambda} = 0$. On the other hand, if $T_A(M_1 \oplus (1 - \lambda)(M_2))$ is bounded, then the same argument as before gives that $M_1 \oplus (1 - \lambda)(M_2)$ is bounded, from which we obtain that $0 = \overline{1 - \lambda} = 1 - \bar{\lambda} = 1 - \pi$. In either case, \overline{M} is indecomposable.

In order to verify the two statements, we first show that $\text{Hom}(T_A H_A(U), T_A(tM))$ is torsion: For this, observe that $T_A(tM)$ is A -generated and has bounded p -components for each prime p because M_p is an E_p -module, and E_p is finite. By Lemma 2.1b, $T_A(tM)$ is A -solvable, and $0 \rightarrow T_A(tM) \rightarrow T_A(M) \rightarrow T_A(M/tM) \rightarrow 0$ is exact. Moreover, M/tM is torsion-free and divisible implies that $T_A(tM) = tT_A(M) \cong tG$. Since G is A -solvable and in \mathcal{G} , we have that tG is isomorphic to a subgroup of tA^m for some $m < \omega$ by Lemma 2.1b. By Theorem 2.2a, $\ker \theta_U$ is torsion-free and divisible, and $T_A H_A(U) \cong U \oplus \bigoplus_k \mathbb{Q}$ for some $k < \omega$. Therefore, $\text{Hom}(T_A H_A(U), T_A(tM)) \cong \text{Hom}(U, tG)$ which is isomorphic to a subgroup of $\text{Hom}(U, tA^m)$. The latter group is torsion, since $U \in \mathcal{G}$ implies that there is an epimorphism $A^s \rightarrow U \rightarrow 0$ for some $s < \omega$ which induces a monomorphism $0 \rightarrow \text{Hom}(U, tA^m) \rightarrow \text{Hom}(A^s, tA^m)$.

We view φ as a map $M \rightarrow \overline{M}$, and can find a map $\psi: H_A(A^n) \rightarrow M$ making the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_A(U) & \xrightarrow{H_A(\alpha)} & H_A(A^n) & \xrightarrow{H_A(\beta)} & M & \longrightarrow & 0 \\
 & & & & \downarrow \psi & & \downarrow \varphi & & \\
 0 & \longrightarrow & tM & \longrightarrow & M & \xrightarrow{\delta} & \overline{M} & \longrightarrow & 0
 \end{array}$$

commutative in which δ denotes the canonical projection. An easy diagram chase shows that $\psi H_A(\alpha) \in \text{Hom}_{E(A)}(H_A(U), tM)$ and, hence, $T_A(\psi)T_A H_A(\alpha): T_A H_A(U) \rightarrow T_A(tM)$. By the result of the previous paragraph, there is a non-zero integer r such that $rT_A(\psi)T_A H_A(\alpha) = 0$. Because $r\varphi_{tM}\psi H_A(\alpha) = H_A T_A(r\psi H_A(\alpha))\varphi_{H_A(U)} = 0$, we have $\varphi_{tM}r\psi H_A(\alpha) = 0$. But φ_{tM} is one-to-one: To see this, observe that $tM \subseteq H_A(G)$ and that $\varphi_{H_A(G)}$ is an isomorphism such that $\varphi_{H_A(G)}\iota = H_A T_A(\iota)\varphi_{tM}$ where $\iota: tM \rightarrow H_A(G)$ is the inclusion map. Therefore, $r\psi H_A(\alpha) = 0$.

We can write $E(A) = R_1 \times R_2$ where R_1 is finite and multiplication by r is an automorphism of R_2 . Given an $E(A)$ -module N , this ring-decomposition yields a corresponding decomposition $N = N_1 \oplus N_2$ such that $N_i R_j = 0$ for $i \neq j$. In particular, multiplication by r is an automorphism of N_2 . Therefore, $r\psi_{H_A(\alpha)}(H_A(U)_2) = 0$ yields $\psi_{H_A(\alpha)}(H_A(U)_2) = 0$.

We now define τ : Write $x \in H_A(A^n)$ as $x = x_1 + x_2$ with $x_i \in H_A(A^n)_i$, and define a map $\nu: H_A(A^n) \rightarrow M$ by $\nu(x) = \psi(x_2)$. Since $H_A(\alpha)(H_A(U)_1) \subseteq H_A(A^n)_1 \subseteq \ker \nu$, we have $H_A(\alpha)(H_A(U)) \subseteq \ker \nu$, and ν induces a map $\tau: M \rightarrow M$ in the following way: For $x \in M$, choose $y \in H_A(A^n)$ with $H_A(\beta)(y) = x$, and define $\tau(x) = \nu(y)$. Write $x = x_1 + x_2$ and $y = y_1 + y_2$, and obtain $\delta\tau(x) = \delta\nu(y) = \delta\psi(y_2) = \varphi_{H_A(\beta)}(y_2) = \varphi(x_2)$. Since $x_1 \in M_1 \subseteq tM$, we have $\varphi(x_1) = 0$, and τ is the desired map.

Moreover, if $\varrho: M \rightarrow M$ is a map with $\overline{\varrho(x)} = 0$ for all $x \in M$, then $\varrho(M) \subseteq tM$. However, this yields that $T_A(\varrho)$ is an element of $\text{Hom}(T_A(M), tT_A(M))$ which is isomorphic to the torsion group $\text{Hom}(G, tG)$. Since $\varphi_{tM}\varrho = H_A T_A(\varrho)\varphi_M$ is torsion, we obtain as before that ϱ has finite order. \square

Corollary 4.3. *Let $A \in \mathcal{G}$ and $G \in \mathcal{G}$ be an A -solvable group. If $0 \rightarrow T \rightarrow H \xrightarrow{\pi} G \rightarrow 0$ is an almost A -balanced exact sequence in which T an A -solvable torsion group and H is an essentially indecomposable \mathcal{G}_A -presented group, then G is essentially indecomposable.*

Proof. Let $M = \text{Im } H_A(\pi)$. Since $H_A(G)/M$ and $H_A(T)$ are torsion, we have $r_0(H_A(H)) = r_0(M) = r_0(H_A(G)) < \infty$. Thus, $\overline{H_A(H)} \cong \overline{M} \cong \overline{H_A(G)}$. By the last theorem, $\overline{H_A(H)}$ is indecomposable, and hence the same holds for $\overline{H_A(G)}$, from which it follows that G is essentially indecomposable by Proposition 4.1. \square

Corollary 4.4. *The following conditions are equivalent for an abelian group $A \in \mathcal{G}$:*

- a) A is essentially indecomposable.
- b) A is indecomposable in Walk.
- c) $E(A)/tE(A)$ is local.

Proof. To see that a) and c) are equivalent, observe that the fact that A is \mathcal{G}_A -presented yields that A is essentially indecomposable if and only if $E(A)/tE(A)$ is an indecomposable $E(A)/tE(A)$ -module. However, an Artinian ring without non-trivial idempotents is local.

For the equivalence of b) and c), observe that $\text{Hom}(A, tA) = tE(A)$. Hence, the Walk-endomorphism ring $E_W(A)$ of A coincides with $E(A)/tE(A)$, and nothing is to prove. \square

5. AN AZUMAYA THEOREM FOR GROUPS IN \mathcal{G}

Our first step toward showing that the class of \mathcal{G} -decomposable groups is closed with respect to direct summands is the verification of the fact that there is a Krull-Schmitt-Theorem for the groups in \mathcal{G} .

Lemma 5.1. *Let $A \in \mathcal{G}$. If $\{\bar{e}_1, \dots, \bar{e}_n\}$ is a family of orthogonal idempotents of $E(A)/tE(A)$, then there are orthogonal idempotents $e_1, \dots, e_n \in E(A)$ with $\bar{e}_i = e_i + tE(A)$ for $i = 1, \dots, n$.*

Proof. Write $\bar{e}_1 = f_1 + tE(A)$ for some $f_1 \in E(A)$. Then, $f_1^2 - f_1 \in \bigoplus_{p \in P_1} E(A_p)$ for some finite subset P_1 of the set P of all primes. As rings, $E(A) = (\times_{p \in P_1} E(A_p)) \times S$ for some subring S of $E(A)$. There is a central idempotent $g_1 \in E(A)$ with $E(A)g_1 = \times_{p \in P_1} E(A_p)$. Since $E(A_p)$ is torsion, and $(1 - g_1)(f_1^2 - f_1) = 0$, we have that $(1 - g_1)f_1$ is an idempotent of $E(A)$ with $\bar{e}_1 = f_1 + tE(A) = (1 - g_1)f_1 + tE(A)$. Hence, we can find a finite subset P_1 of P and an idempotent e_1 of $E(A)$ with $\bar{e}_1 = e_1 + tE(A)$ and $e_1(\bigoplus_{p \in P_1} A_p) = 0$.

Assume that we have found finite subsets $P_1 \subseteq \dots \subseteq P_n$ of P and orthogonal idempotents e_1, \dots, e_n of $E(A)$ with $\bar{e}_i = e_i + tE(A)$ and $e_i(\bigoplus_{p \in P_i} A_p) = 0$. If $n < m$, then we choose $f_{n+1} \in E(A)$ with $\bar{e}_{n+1} = f_{n+1} + tE(A)$. As before, we can find a finite subset $Q_{n+1} \supseteq P_n$ of P and a central idempotent $\tilde{g}_{n+1} \in E(A)$ such that $(1 - \tilde{g}_{n+1})(\bigoplus_{p \in Q_{n+1}} A_p) = 0$ and $h_{n+1} = (1 - \tilde{g}_{n+1})f_{n+1}$ is an idempotent of $E(A)$ with $\overline{e_{n+1}} = h_{n+1} + tE(A)$. Since $\bar{e}_i \bar{e}_{n+1} = \bar{e}_{n+1} \bar{e}_i = 0$, we can enlarge Q_{n+1} to a finite subset P_{n+1} of P such that $e_i h_{n+1}, h_{n+1} e_i \in \bigoplus_{p \in P_{n+1}} E(A_p)$ for $i = 1, \dots, n$. If we choose a central idempotent g_{n+1} in $E(A)$ with $E(A)g_{n+1} = \times_{p \in P_{n+1}} E(A_p)$, then $e_{n+1} = (1 - g_{n+1})h_{n+1}$ is an idempotent of $E(A)$ with the desired properties. \square

If A is essentially indecomposable and T is a bounded abelian group, then $A \oplus T$ is essentially indecomposable. To see this, write $A \oplus T = B \oplus C$, and let P_1 be the set of those primes p for which $T[p] \neq 0$. Since $A \in \mathcal{G}$, we can write $A = D \oplus E$ with E bounded and $\text{Hom}(D, T \oplus E) = \text{Hom}(E \oplus T, D) = 0$. We have $B = B_1 \oplus T_1$ and $C = C_1 \oplus T_2$ with $B_1, C_1 \subseteq D$ and $T_1, T_2 \subseteq E \oplus T$. Hence, $A = B_1 \oplus C_1 \oplus S$ for some bounded group S . Since A is essentially indecomposable, B_1 or C_1 is bounded. This shows that $A \oplus T$ is essentially indecomposable.

Theorem 5.2. *Let $A \in \mathcal{G}$.*

- a) *There are essentially indecomposable subgroups A_1, \dots, A_n of A with $A = A_1 \oplus \dots \oplus A_n$.*
- b) *If $A = A_1 \oplus \dots \oplus A_n = B_1 \oplus \dots \oplus B_m$ with A_i and B_j essentially indecomposable for all i and j , then $n = m$ and, after reindexing, there are bounded groups*

C_1, \dots, C_n and D_1, \dots, D_n with $A_i \oplus C_i = B_i \oplus D_i$. Moreover, if $1 \leq k \leq n$, then $A = B_1 \oplus \dots \oplus B_k \oplus A'_{k+1} \oplus \dots \oplus A'_n \oplus T$ where $A_j = A'_j \oplus S_j$ for bounded groups T, S_1, \dots, S_n .

Proof. a) Since $E(A)/tE(A)$ is a finite dimensional \mathbb{Q} -algebra, we can find a finite set $\{\bar{e}_1, \dots, \bar{e}_n\}$ of orthogonal primitive idempotents of $E(A)/tE(A)$ with $1_A = \bar{e}_1 + \dots + \bar{e}_n$. By Lemma 5.1, each \bar{e}_i is of the form $\bar{e}_i = e_i + tE(A)$ for orthogonal idempotents e_1, \dots, e_n of $E(A)$. Setting $e = e_1 + \dots + e_n$ yields a decomposition $A = e_1(A) \oplus \dots \oplus e_n(A) \oplus (1 - e)(A)$. We set $A_i = e_i(A)$ and $T = (1 - e)(A)$. Since $(1 - e) \in tE(A)$, we have $mT = 0$ for some non-zero integer m . Once we have shown that the A'_i 's are essentially indecomposable, $A_1 \oplus T$ is essentially indecomposable by the preceding remarks; and we have a decomposition of A as in a). By Theorem 4.2, it is enough to show that $\overline{H_A(A_i)}$ is indecomposable:

Since $E(A) = H_A(A_1) \oplus \dots \oplus H_A(A_n) \oplus H_A(T)$, we have $\overline{E(A)} = \overline{H_A(A_1)} \oplus \dots \oplus \overline{H_A(A_n)}$. In particular, $\overline{H_A(A_i)} = e_i E(A)/tH_A(A_i) = e_i E(A)/[e_i E(A) \cap tE(A)] \cong [e_i E(A) + tE(A)]/tE(A) = \bar{e}_i E(A)$ yields that $\overline{H_A(A_i)}$ is an indecomposable $E(A)$ -module.

b) Choose orthogonal idempotents f_1, \dots, f_m of $E(A)$ with $f_i(A) = B_i$ for $i = 1, \dots, m$. Since $\overline{H_A(B_i)}$ is an indecomposable $E(A)$ -module for $i = 1, \dots, m$, the classical Krull-Schmitt-Theorem yields $n = m$ and $\overline{H_A(A_i)} \cong \overline{H_A(B_i)}$ for $i = 1, \dots, n$ after a possible reindexing.

Inverse $E(A)$ -module isomorphisms $\bar{\sigma}: \overline{H_A(A_i)} \rightarrow \overline{H_A(B_i)}$ and $\bar{\tau}: \overline{H_A(B_i)} \rightarrow \overline{H_A(A_i)}$ can be lifted to a pair of maps $\sigma: H_A(A_i) \rightarrow H_A(B_i)$ and $\tau: H_A(B_i) \rightarrow H_A(A_i)$ such that $\bar{\sigma}\bar{\tau} = 1_{\overline{H_A(B_i)}}$ and $\bar{\tau}\bar{\sigma} = 1_{\overline{H_A(A_i)}}$ using property (I) in the proof of Theorem 4.2. Furthermore, using property (II) in the same proof, we can find a non-zero integer k with $k\sigma\tau = k1_{H_A(B_i)}$ and $k\tau\sigma = k1_{H_A(A_i)}$. By splitting off the p -components of A_i and B_i for those primes p which divide k , we obtain decompositions $A_i = T_i \oplus C_i$ and $B_i = S_i \oplus D_i$ into direct sums of fully invariant subgroups such that multiplication by k is an automorphism of C_i and D_i . In particular, the restriction of $T_A(\sigma)$ to $T_A H_A(C_i)$ induces an isomorphism between C_i and D_i . This shows $A_i \oplus T_i \cong B_i \oplus S_i$ as required.

Finally, $\overline{E(A)} = \overline{H_A(B_1)} \oplus \dots \oplus \overline{H_A(B_k)} \oplus \overline{H_A(A_{k+1})} \oplus \dots \oplus \overline{H_A(A_n)}$ by the classical Krull-Schmitt-Theorem. This gives $1_A - (f_1 + \dots + f_k) - (e_{k+1} + \dots + e_n) \in tE(A)$. By Lemma 5.1, we can extend $\{f_1, \dots, f_k\}$ to a set $\{f_1, \dots, f_k, g_{k+1}, \dots, g_n\}$ of orthogonal idempotents of $E(A)$ with $\bar{e}_j = g_j + tE(A)$. Moreover, the g_j 's can be chosen in such a way that $A_j = g_j(A) \oplus C_j$ for some bounded C_j . Setting $A'_j = g_j(A)$ and $f = f_1 + \dots + f_k + g_{k+1} + \dots + g_n$ gives a decomposition $A = B_1 \oplus \dots \oplus B_k \oplus A'_{k+1} \oplus \dots \oplus A'_n \oplus (1 - f)(A)$. Since $1 - f \in t(E(A))$, we obtain that $(1 - f)(A)$ is bounded, as desired. \square

Combining the last result with Proposition 3.3, we immediately obtain

Corollary 5.3. *Let $A \in \mathcal{G}$. Every \mathcal{G}_A -presented group is a direct sum of essentially indecomposable \mathcal{G}_A -presented groups.*

The proof of the next result is based on that of [4, Lemma 26.4], which is used in the proof of the Crawley-Johnson-Warfield-Theorem, but several modifications to the arguments used in [4] are necessary due to the fact that [4, Lemma 26.4] deals with direct sums of modules whose endomorphism ring is local, while we consider direct sums of groups A for which $\overline{E(A)}$, but not $E(A)$ is local.

Lemma 5.4. *Let $G = B \oplus C = N \oplus H$ be a \mathcal{G} -decomposable group. If $N \in \mathcal{G}$ is not torsion, then $G = N \oplus T \oplus B' \oplus C'$ for subgroups $B' \subseteq B$ and $C' \subseteq C$ and a sigma-cyclic torsion group T such that $T_p \neq 0$ for only finitely many primes p .*

Proof. As in the proof of [4, Lemma 26.4], it suffices to consider the case that $G = N \oplus H \oplus H' = H \oplus K \oplus L$. We show that $G = N \oplus H \oplus T \oplus K' \oplus L'$ for some $K' \subseteq K$, $L' \subseteq L$, and a sigma-cyclic torsion group T with $T_p \neq 0$ for only finitely many primes p .

Once this has been shown, we use Theorem 5.2 to write $N = \bigoplus_{i=1}^k N_i$ where N_1, \dots, N_k are essentially indecomposable groups which are not torsion, and prove the lemma by induction on k . The case $k = 1$ is trivial since it corresponds to $H = 0$. Setting $H = N_1 \oplus \dots \oplus N_k$, we can find T_1 , $K_1 \subseteq B$, and $L_1 \subseteq C$ as desired with $G = H \oplus T_1 \oplus K_1 \oplus L_1$. By the result in the first paragraph, there are $K_2 \subseteq K_1$, $L_2 \subseteq L_1 \oplus T_1$, and a suitable torsion group T_2 with $G = N_{k+1} \oplus H \oplus T_2 \oplus K_2 \oplus L_2$. Let p_1, \dots, p_t be the primes for which $T_1[p_i] \neq 0$. Since G is \mathcal{G} -decomposable, we can write $G = V \oplus U$ such that $V = G_{p_1} \oplus \dots \oplus G_{p_t}$ is a direct sum of cyclics and U is a fully invariant subgroup of G for which $U[p_i] = 0$ and $U = p_i U$ for $i = 1, \dots, t$. Therefore, $L_1 \oplus T_1 = T_1 \oplus (L_1 \cap V) \oplus (L_1 \cap U) = L_2 \oplus S$ for some subgroup $S \subseteq L_1 \oplus T_1$. Observe that $L_2 \cap V, S \cap V \subseteq T_1 \oplus (L_1 \cap V)$. Since $L_1 \cap U$ is a direct summand of U , it is fully invariant in $L_1 \oplus T_1$. Therefore, $T_1 \oplus (L_1 \cap V) = (L_2 \cap V) \oplus (S \cap V)$ and $(L_1 \cap U) = (L_2 \cap U) \oplus (S \cap U)$. We set $T = (L_2 \cap V) \oplus T_2$ and obtain $G = N \oplus T \oplus K_2 \oplus (L_2 \cap U)$ with $L_2 \cap U \subseteq L_1 \subseteq C$ as desired.

Following [4], we choose idempotents $e, e', f \in E(G)$ such that $ee' = e'e = 0$, $K = e(G)$, $L = e'(G)$, $H = (1 - e - e')(G)$, $N = f(G)$, and $H \oplus H' = (1 - f)(G)$. Because of $(1 - e - e')(G) \subseteq (1 - f)(G)$, we have $f = fef + fe'f$. Since f is the identity in $fE(G)f \cong E(N)$, it is impossible that \overline{fef} and $\overline{fe'f}$ are both in the Jacobson-radical of $\overline{fE(G)f}$. Observe that $\overline{E(N)}$ is non-zero since N is not torsion. But $\overline{E(N)} \cong \overline{fE(G)f}$ is a local ring, and one of these two elements is a unit in $\overline{fE(G)f}$. Without loss of generality, this is the case for \overline{fef} . There is $r \in E(G)$

such that $s = frf$ satisfies $\overline{sfe\bar{f}} = \overline{fef\bar{s}} = \bar{f}$. We can find $t_1, t_2 \in t(fe(G)f)$ with $sfe\bar{f} = f + t_1$ and $fefs = f + t_2$. Choose a non-zero integer m with $mt_1 = mt_2 = 0$, and let p_1, \dots, p_n be the primes dividing m . As before, $T = G_{p_1} \oplus \dots \oplus G_{p_n}$ is sigma-cyclic, and $G = T \oplus U$ for some fully invariant subgroup U of G for which multiplication by p_i is an automorphism. In contrast to [4], ese need not be an idempotent of $E(G)$. Nevertheless, we are able to show $U = (ese)(U) \oplus (1 - ese)(U)$:

Since $s = fs = fs$, we have $(ese)^2 = efsfe.fsfef = efs(f + t_1)fe = ese + efst_1fe$ with $mefst_1fe = 0$. Hence, $(ese)^2 = ese + t'$ for some $t' \in tE(G)$ with $t'(G) \subseteq T$. Since U is fully invariant in G , it remains to show $ese(U) \cap (1 - ese)(U) = 0$. If $w = ese(u) = (1 - ese)(v)$ for some $u, v \in U$, then $ese(w) = [ese - (ese)^2](u) = (-t')(u) \in T \cap U$. Thus, $0 = (ese)^2(v) = ese(v) + t'(v) = ese(v) = w$.

Moreover, $(1 - e)(G) = L \oplus H$, and hence $(1 - e)(U) = (1 - e)(G) \cap U = (L \cap U) \oplus (H \cap U)$ is contained in $(1 - ese)(U)$. Since $U = (1 - e)(U) \oplus (K \cap U)$ by the full invariance of U in G , we have $(1 - ese)(U) = (L \cap U) \oplus (H \cap U) \oplus (K \cap (1 - ese)(U))$. On the other hand, $ese(U) \subseteq K \cap U$. Hence, $K \cap U = ese(U) \oplus (K \cap (1 - ese)(U))$. We set $K' = K \cap (1 - ese)(U)$ and obtain $U = ese(U) \oplus K' \oplus (L \cap U) \oplus (H \cap U)$. We show that $N \cap U$ can replace $ese(U)$ in this decomposition. For this, define $\varphi: U \rightarrow ese(U)$ by $\varphi(u) = ese(u)$. Since $K' \oplus (L \cap U) \oplus (H \cap U) = \ker \varphi$, it suffices to show that $\varphi|_{(N \cap U)}$ is an isomorphism. For $u \in U$, we have $fse(u) \in N \cap U$ and $ese(u) = (ese)^2(u) - t'(u) = ese(se)(u) = ese(fse)(u) \in \varphi(N \cap U)$. Furthermore, if $ese(x) = 0$ for some $x \in N \cap U$, then $x = f(x)$ yields $0 = esef(x) = efsfef(x) = ef(f + t_2)(x) = ef(x)$ since $mef t_2 = 0$ yields $eft_2(x) \in T \cap U$. Thus, $0 = sfe\bar{f}(x) = (f + t_1)(x) = f(x) = x$.

Therefore, $U = (N \cap U) \oplus K' \oplus (L \cap U) \oplus (H \cap U)$ and $T = (N \cap T) \oplus (H \cap T) \oplus (H' \cap T)$. Consequently, $G = N \oplus H \oplus (H' \cap V) \oplus K' \oplus L'$ with $L' = L \cap U$. Observe that $H' \cap T$ is sigma-cyclic with only finitely many non-zero p -primary components. \square

Lemma 5.5. *Let $G = N \oplus H = B \oplus C$ be \mathcal{G} -decomposable where $N \in \mathcal{G}$ or N is finite. If X is a finite subset of $N \cap B$, then $B = B_0 \oplus B^0$ and $C = C_0 \oplus C^0$ such that $X \subseteq B_0$ and $N \oplus D \cong B_0 \oplus C_0 \oplus E$ for finite groups D and E .*

Proof. We first consider the case that N is finite. Since G is \mathcal{G} -decomposable, we can write $G = T \oplus U$ where U and T are fully invariant and T is a sigma-cyclic group containing N with $T_p \neq 0$ for only finitely many primes p . We have $T = (T \cap B) \oplus (T \cap C)$. There are finite direct summands B_0 of $T \cap B$ and C_0 of $T \cap C$ with $N \subseteq B_0 \oplus C_0$.

In the case that $N \in \mathcal{G}$, we use Lemma 5.4 to write $G = N \oplus T \oplus B'' \oplus C''$ where $B'' \subseteq B$, $C'' \subseteq C$ and T is a direct sum of cyclics with $T_p \neq 0$ for only finitely many primes p . Set $B_* = B \cap (N \oplus T \oplus C'')$ and $C_* = C \cap (N \oplus T \oplus B'')$. We have $X \subseteq B_*$, $B = B_* \oplus B''$, and $C = C_* \oplus C''$. Moreover, $B_* \oplus C_* \cong N \oplus T$. Let p_1, \dots, p_n be the

primes with $T_p \neq 0$, and set $V = G_{p_1} \oplus \dots \oplus G_{p_n}$. Write $G = V \oplus U$ for some fully invariant subgroup U of G . Since $T \subseteq V$, we have $(N \cap V) \oplus T \cong (B_* \cap V) \oplus (C_* \cap V)$ and $N \cap U \cong (B_* \cap U) \oplus (C_* \cap U)$. Observe that $N \cap V$ is finite since it is a torsion direct summand of a group in \mathcal{G} . Moreover, $B_* \cap V$ is a direct sum of cyclics, and we can write $B_* \cap V = W \oplus E$ where W is finite and $B_0 = B_* \oplus W$ contains X . We set $C_0 = C_* \cap U$, $B' = E \oplus B''$, and $C' = (C_* \cap V) \oplus C''$. Since $(N \cap V) \oplus B_0 \oplus C_0 = (B_* \cap U) \oplus (C_* \cap U) \oplus W \oplus (N \cap V) \cong (N \cap U) \oplus (N \cap V) \oplus W = N \oplus W$, we have obtained the desired decomposition of G . \square

We are now able to show that the class of \mathcal{G} -decomposable groups is closed with respect to direct summands. By Theorem 5.2, every \mathcal{G} -decomposable group G has the form $G = \bigoplus_{i \in I} G_i$ such that G_i is either a cyclic p -group or an essentially indecomposable honest mixed group.

Theorem 5.6. *Let $G = B \oplus C = \bigoplus_{i \in I} G_i$ be \mathcal{G} -decomposable where each G_i is either an essentially indecomposable group in \mathcal{G} or a cyclic p -group. Then, $B \cong \bigoplus_{j \in J} H_j$ where, for each $j \in J$, we can find $i \in I$ such that H_j is a direct summand of G_i .*

Proof. By Kaplansky's Theorem, we may assume that G is countable. We write $B = \{b_n \mid n < \omega\}$ with $b_0 = 0$. We construct an ascending chain $0 = B_0 \subseteq \dots \subseteq B_n \subseteq \dots$ of direct summands of B such that $B_n \in \mathcal{G}$ and contains b_n for all $n < \omega$.

We write $B = B_n \oplus D_n$ and write $b_{n+1} = x + y$ with $x \in B_n$ and $y \in D_n$. We can find a finite subset I_n of I such that $y \in N = \bigoplus_{i \in I_n} G_i$. Observe that $N \in \mathcal{G}$. We apply Lemma 5.5 to the decomposition $G = D_n \oplus [B_n \oplus C]$ to obtain $D_n = K_n \oplus K'$ and $B_n \oplus C = L_n \oplus L'$ such that $y \in K_n$ and $K_n \oplus L_n \oplus S_n \cong N \oplus T_n$ for some finite groups S_n and T_n . We set $B_{n+1} = B_n \oplus K_n$. Since $B = \bigcup_{n < \omega} B_n$, we have $B \cong \bigoplus_{n < \omega} K_n$. Every p -primary cyclic direct summand of K_n is isomorphic a direct summand of $\bigoplus_{i \in I_n} (G_i)_p$ as desired. If K_n has an essentially indecomposable direct summand W , then there is $i \in I_n$ such that $W \oplus U_i \cong G_i \oplus V_i$ for some finite groups U_i and V_i . But then, $W = W' \oplus W''$ where W' is isomorphic to a direct summand of G_i and W'' is finite. Hence, $K_n = (\bigoplus_{i \in I_n} H_i) \oplus T_n$ where each H_i is isomorphic to a direct summand of G_i and T is finite. This proves the theorem. \square

Corollary 5.7. *Let $A \in \mathcal{G}$ and $G = B \oplus C = \bigoplus_{i \in I} G_i$.*

- a) *If each G_i is a reduced A -generated group in \mathcal{G} , then $B \cong T \oplus \bigoplus_{j \in J} B_j$, where each B_j is a reduced A -generated group in \mathcal{G} , and T is an A -solvable torsion group.*
- b) *If each G_i is \mathcal{G}_A -presented, then B is a direct sum of an A -solvable torsion group and a direct sum of \mathcal{G}_A -presented groups.*

- c) Every A -projective group P is of the form $P = \bigoplus_I P_i$ where each P_i is isomorphic to a cyclic or essentially indecomposable direct summand of A .

PROOF. By Theorem 4.6, we have $B \cong \bigoplus_J B_j$ where each B_j is either torsion or isomorphic to a direct summand of G_i . Let $J_1 = \{j \in J \mid B_j \text{ is torsion}\}$. Since $\bigoplus_{J_1} B_j$ is an A -generated reduced torsion group, it is A -solvable by Lemma 2.1. This proves the corollary. \square

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