Ivan Chajda; Jiří Rachůnek

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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 1, 111–120

Czechoslovak Mathematical Journal, 51 (126) (2001), 111–120

ANNIHILATORS IN NORMAL AUTOMETRIZED ALGEBRAS

Ivan Chajda and Jiří Rachůnek, Olomouc

(Received November 14, 1997)

Abstract. The concepts of an annihilator and a relative annihilator in an autometrized l-algebra are introduced. It is shown that every relative annihilator in a normal autometrized l-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$ and every principal ideal of $\mathcal{A}$ is an annihilator of $\mathcal{A}$. The set of all annihilators of $\mathcal{A}$ forms a complete lattice. The concept of an I-polar is introduced for every ideal I of $\mathcal{A}$. The set of all I-polars is a complete lattice which becomes a two-element chain provided I is prime. The I-polars are characterized as pseudocomplements in the lattice of all ideals of $\mathcal{A}$ containing I.

Keywords: autometrized algebra, annihilator, relative annihilator, ideal, polar

MSC 2000: 06F05

1. Autometrized l-algebras, basic concepts

The concept of an annihilator was introduced for lattices by M. Mandelker [5] as a generalization of the concept of a pseudocomplement. Since the set of all annihilators of a lattice $\mathcal{L}$ need not form a lattice with respect to inclusion, the first author introduced in [2] the concept of the so called indexed annihilator; the set of indexed annihilators in $\mathcal{L}$ does form a lattice. Both the annihilators and the indexed annihilators characterize distributive and modular lattices. Recall the that for a lattice $\mathcal{L} = (\mathcal{L}; \vee, \wedge)$ and elements $a, b \in L$ the annihilator $\langle a, b\rangle$ is the set $\langle a, b\rangle = \{x \in L;\ a \wedge x \leqslant b\}$; an indexed annihilator in $\mathcal{L}$ is every subset of L which is the intersection of a system of annihilators of $\mathcal{L}$.

This work was supported by The Ministry of Education of Czech Republic (Project PG 98439).

Autometrized algebras were introduced by K. L. N. Swamy [8] as a common generalization of Brouwerian algebras and commutative lattice ordered groups (l-groups, for short). Let us recall this basic concept:

Definition. An algebraic system $\mathcal{A} = (\mathcal{A}; +, \prime, \leqslant, *)$ is called an *autometrized algebra* if

(1) $(A; +, 0)$ is a commutative monoid;
(2) $(A; +, \leqslant)$ is an ordered semigroup, i.e. $\leqslant$ is an order on A and $a \leqslant b \Longrightarrow a + c \leqslant b + c$ for all $a, b, c \in A$;
(3) $*$ is a binary operation on A satisfying

$$\begin{aligned}
&a * b \geqslant 0, \\
&a * b = 0 \quad \text{if and only if} \quad a = b, \\
&a * b = b * a, \\
&a * c \leqslant (a * b) + (b * c)
\end{aligned}$$

for all a, b, c in A; $*$ is called an *autometric* on A.

If, moreover, $(A, \leqslant)$ is a lattice whose operations are denoted by $\vee$ and $\wedge$ and

$$\begin{aligned}
a + (b \vee c) &= (a + b) \vee (a + c), \\
a + (b \wedge c) &= (a + b) \wedge (a + c)
\end{aligned}$$

for every $a, b, c \in A$, then $\mathcal{A}$ is called an *autometrized lattice algebra*, briefly an *Al-algebra*.

In this case $\mathcal{A}$ is considered to be also equipped by the lattice operations and this fact is expressed by the notation $\mathcal{A} = (\mathcal{A}; +, 0, \vee, \wedge, *)$.

However, the concept of an Al-algebra can be too general for our purpose, so we use the following specification (which was introduced by Swamy [8]):

Definition. An Al-algebra $\mathcal{A} = (\mathcal{A}; +, 0, \vee, \wedge, *)$ is called *normal* (briefly an *NAl-algebra* if

$$\begin{gathered}
a \leqslant a * 0, \\
(a + c) * (b + d) \leqslant (a * b) + (c * d), \\
(a * c) * (b * d) \leqslant (a * b) + (c * d), \\
a \leqslant b \Longrightarrow \exists x \geqslant 0 \ \text{such that} \ a + x = b
\end{gathered}$$

for all $a, b, c, d \in A$.

Remark.

(a) Having an abelian l-group $\mathcal{G} = (\mathcal{G}; +, 0, -, \vee, \wedge)$ we can set

$$a * b = |a - b| = (a - b) \vee (b - a)$$

for $a, b \in G$. Then $(G; +, 0, \vee, \wedge, *)$ is an *NAl*-algebra.

(b) Having a Brouwerian algebra $\mathcal{B} = (B; \vee, \wedge)$, i.e. a dually relative pseudocomplemented lattice with the greatest element (it means that for each $a, b \in B$ there is a least $x \in B$ with $b \vee x \geqslant a$), denote by $a - b$ this relative pseudocomplement x of b with respect to a and set $a * b = (a - b) \vee (b - a)$. Thus also $(B; +, 0, \vee, \wedge, *)$ is an *NAl*-algebra where $+$ denotes the lattice join $\vee$.

The concept of an ideal of an *NAl*-algebra was introduced in [9]:

Definition. Let $\mathcal{A} = (\mathcal{A}; +, 0, \vee, \wedge, *)$ be an *NAl*-algebra and $\emptyset \neq I \subseteq A$. The set I is called an *ideal* of $\mathcal{A}$ if it satisfies

$$a, b \in I \Longrightarrow a + b \in I,$$
$$a \in I, \ \ x \in A, \ \ x * 0 \leqslant a * 0 \Longrightarrow x \in I$$

for all $a, b, x \in A$.

Denote by $\mathcal{I}(\mathcal{A})$ the set of all ideals of an *NAl*-algebra $\mathcal{A}$. Following Theorem 1 in [9], $\mathcal{I}(\mathcal{A})$ is an algebraic lattice with respect to set inclusion where $\inf M = \bigcap M$ for every subset $M \subseteq \mathcal{I}(\mathcal{A})$. If $B \subseteq A$, denote by $I(B)$ the ideal of A generated by B, i.e. the least ideal of A containing B; if B is a singleton, say $\{b\}$, we will write briefly $I(b)$. Then $I(b)$ is called a *principal ideal* of $\mathcal{A}$ generated by b.

It is easy to verify that

$$I(B) = \{x \in A; \ \ x * 0 \leqslant (b_1 * 0) + \ldots + (b_n * 0); \ \ b_1, \ldots, b_n \in B\},$$
$$I(b) = \{x \in A; \ \ x * 0 \leqslant m(b * 0), \ \text{ for } \ m \in \mathbb{N}\}.$$

Two elements a, b in any *NAl*-algebra $\mathcal{A}$ are said to be *orthogonal* (denoted by $a \perp b$) if

$$(a * 0) \wedge (b * 0) = 0.$$

For a subset B of A we denote by $B^{\perp}$ the set of all elements of A which are orthogonal to every element of B, i.e.

$$B^{\perp} = \{x \in A; \ \ x \perp b \ \text{ for each } \ b \in B\}.$$

The set $B^{\perp}$ is called the *polar of* B. For $B = \{b\}$ we will write briefly $b^{\perp}$ instead of $\{b\}^{\perp}$. A subset C of A is called a *polar in* $\mathcal{A}$ if $C = B^{\perp}$ for some subset B of A.

Now, we specify some kinds of *NAl*-algebras: An *NAl*-algebra $\mathcal{A}$ is called

(a) *semiregular* if for every $a \in A$

$$a \geqslant 0 \Longrightarrow a * 0 = a;$$

(b) *interpolation* if for all $a, b, c \in A$, $0 \leqslant a, b, c$ and $a \leqslant b + c$ imply the existence of $b_1, c_1 \in A$ such that $0 \leqslant b_1 \leqslant b$, $0 \leqslant c_1 \leqslant c$ and $a = b_1 + c_1$.

Denote by $\mathcal{P}(\mathcal{A})$ the set of all polars of an *NAl*-algebra $\mathcal{A}$. It was proved in [9], Theorem 7, that for a semiregular $\mathcal{A}$ the set $\mathcal{P}(\mathcal{A})$ ordered by inclusion is a complete Boolean algebra. The properties of $\mathcal{P}(\mathcal{A})$ for an interpolation semiregular *NAl*-algebra $\mathcal{A}$ were investigated in [7].

On the other hand, the assumption "to be interpolation" can be omitted by virtue of Lemma 1.2 in [3]. Further, Lemma 5 in [9] enables us to omit the assumption of semiregularity in the most cases as it was done in [4], where some results on lattices $\mathcal{I}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ are generalized to arbitrary *NAl*-algebras. This way will be used also here for an investigation of the above introduced concepts in a general setting.

2. Annihilators and relative annihilators

Definition. Let a, b be elements in an *NAl*-algebra $\mathcal{A}$. A subset

$$\langle a, b\rangle = \{x \in A;\ \ (a * 0) \wedge (x * 0) \leqslant n(b * 0)\ \text{ for some }\ n \in \mathbb{N}\}$$

will be called the *relative annihilator of a with respect to b.*

A subset B of $\mathcal{A}$ is a *relative annihilator in* $\mathcal{A}$ if $B = \langle a, b\rangle$ for some elements $a, b \in A$.

Theorem 1. *Every relative annihilator of an NAl-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$.*

P r o o f. Let $a, b, x, y \in A$ and suppose $x, y \in \langle a, b\rangle$. Then there are $n_1, n_2 \in \mathbb{N}$ such that

$$(a * 0) \wedge (x * 0) \leqslant n_1(b * 0),$$
$$(a * 0) \wedge (y * 0) \leqslant n_2(b * 0).$$

On account of normality of $\mathcal{A}$ we have

$$(a * 0) \wedge \big((x + y) * 0\big) \leqslant (a * 0) \wedge \big((x * 0) + (y * 0)\big).$$

By Lemma 1.2 in [3], this yields

$$\begin{aligned}(a*0)\wedge\big((x*0)+(y*0)\big)&\leqslant\big((a*0)\wedge(x*0)\big)+\big((a*0)\wedge(y*0)\big)\\&\leqslant n_1(b*0)+n_2(b*0)\\&=(n_1+n_2)(b*0),\end{aligned}$$

whence $x+y\in\langle a,b\rangle$.

It is obvious that for $z\in A$ we have $z*0\leqslant x*0\Longrightarrow z\in\langle a,b\rangle$. □

Remark.

(a) Of course, $\langle a,a\rangle=A$ for each $a\in A$, thus A is a relative annihilator of $\mathcal{A}$ for each NAl-algebra $\mathcal{A}$.

(b) If $a\in A$ then $\langle a,0\rangle=a^{\perp}$, the polar of a.

(c) The set of all relative annihilators of $\mathcal{A}$ need not be a complete lattice with respect to set inclusion. We can illuminate this fact by the following example:

Let G be an abelian l-group. For $a\in G$ we denote $|a|=a\vee -a$. Then $a*b=|a-b|$ is an autometric on G with $a*0=|a|$, thus

$$\langle a,b\rangle=\{x\in G;\ \ |a|\wedge|x|\leqslant n|b|,\ \ n\in\mathbb{N}\},$$

and hence $a^{\perp}=\{x\in G;\ |x|\wedge|a|=0\}$. Therefore polars in the autometrized algebra $\mathcal{G}$ coincide with polars in the l-group G. Recall that an element b in an l-group G is a weak unit of G if $b^{\perp}=\{0\}$.

Suppose now that the l-group G contains no weak units and let $a,b\in G$ be elements with $\langle a,b\rangle=\{0\}$. Since $|a|\wedge|b|\leqslant n|b|$ for each $n\in\mathbb{N}$, we have $n|b|=0$. Since G is torsion free, this yields $b=0$. Then $\langle a,b\rangle=\langle a,0\rangle=a^{\perp}$, i.e. $a^{\perp}=\{0\}$, a contradiction. Hence there are no elements $a,b\in G$ with $\langle a,b\rangle=\{0\}$, i.e. $\{0\}$ is not a relative annihilator of G.

On the other hand, $\{0\}=I(0)$, and, as will be shown in Theorem 4 later, every ideal generated by a singleton is the intersection of a set of relative annihilators. Altogether, $\{0\}$ is the intersection of all relative annihilators of G but it is not a relative annihilator of G.

The foregoing Remark (c) motivates us to introduce the following concept:

Definition. A subset B of an NAl-algebra $\mathcal{A}$ is called an *annihilator of* $\mathcal{A}$ if $B=\bigcap\{B_\gamma;\ \gamma\in\Gamma\}$ for a system of relative annihilators in $\mathcal{A}$.

Let us note that for lattices a different terminology was used, see [2] and [5], namely, relative annihilators in our sense are annihilators in [5] and annihilators in our sense are called indexed annihilators in [2].

Corollary 2. *Every annihilator of an NAl-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$.*

Proof. It follows from Theorem 1 and the fact that $\mathcal{I}(\mathcal{A})$ forms a lattice where meets are intersections. □

Corollary 3. *The set $Ann(\mathcal{A})$ of all annihilators of an NAl-algebra $\mathcal{A}$ forms a complete lattice with respect to set inclusion. For $B_\gamma \in Ann(\mathcal{A})$, $\gamma \in \Gamma$, we have*

$$\inf\{B_\gamma;\ \gamma \in \Gamma\} = \bigcap\{B_\gamma;\ \gamma \in \Gamma\}.$$

Applying Corollary 3, we conclude that for every NAl-algebra $\mathcal{A}$ and each subset M of A there exists the least annihilator of $\mathcal{A}$ containing M. We denote it by $A(M)$ and call it the *annihilator generated by M*.

For principal ideals of $\mathcal{A}$, we can prove

Theorem 4. *Every principal ideal of an NAl-algebra $\mathcal{A}$ is an annihilator of $\mathcal{A}$.*

Proof. Let $c \in A$ and $A(c) = A(\{c\})$. For the principal ideal $I(c)$ we clearly have $I(c) \subseteq A(c)$. Let us prove the converse inclusion. Let $z \in A(c)$. Then for every $a, b \in A$ we obviously have $c \in \langle a, b\rangle \Rightarrow z \in \langle a, b\rangle$. Since $(z * 0) \wedge (c * 0) \leqslant c * 0$, there must exist $s \in \mathbb{N}$ with $(z * 0) \wedge (z * 0) \leqslant s(c * 0)$, i.e. $z * 0 \leqslant s(c * 0)$. Then, of course, $z \in I(c)$. □

Remark.

(a) By Theorem 4, $I(0) = \{0\}$ is the least element of the lattice $Ann(\mathcal{A})$; of course, A is the greatest element of $Ann(\mathcal{A})$ by Remark after Theorem 1.
(b) By the proof of Theorem 4, $I(c) = A(c) = A(I(c))$ for each element $c \in A$.

The concept of a relative annihilator can be also generalized to subsets:

Definition. Let B, C be non-void subsets of an NAl-algebra $\mathcal{A}$. The set $\langle B, C\rangle = \bigcap\{\langle b, c\rangle;\ b \in B,\ c \in C\}$ is called the *generalized relative annihilator of B with respect to C*. A subset D of A is a *generalized relative annihilator of* $\mathcal{A}$ if $D = \langle B, C\rangle$ for some non-void subsets B, C of A.

Remark.

(a) Every relative annihilator of $\mathcal{A}$ is a generalized annihilator since $\langle a, b\rangle = \langle \{a\}, \{b\}\rangle$.
(b) Every generalized annihilator is an annihilator of $\mathcal{A}$.
(c) For every subset B of A we have $B^\perp = \langle B, \{0\}\rangle$, thus each polar of $\mathcal{A}$ is a generalized relative annihilator of $\mathcal{A}$.

It can be of some interest to study the set of generalized relative annihilators with a fixed second component:

Theorem 5. *Let B be a non-void subset of an NAl-algebra $\mathcal{A}$. The set of all generalized relative annihilators $\langle X, B\rangle$ where X runs over all non-void subsets of A forms a complete lattice with respect to set inclusion where infima coincide with intersections and A is the greatest element.*

Proof. Of course, $\langle\{0\}, B\rangle = A$, thus A is the greatest generalized annihilator of $\mathcal{A}$. It is an easy computation that for any non-void subsets C_γ of A we have $\bigcap\{\langle C_\gamma, B\rangle;\ \gamma\in\Gamma\} = \bigcap\{\langle c, b\rangle;\ c\in C_\gamma,\ b\in B;\ \gamma\in\Gamma\} = \bigcap\{\langle c, b\rangle;\ b\in B,\ c\in \bigcup\{C_\gamma;\ \gamma\in\Gamma\}\} = \langle\bigcup\{C_\gamma;\ \gamma\in\Gamma\}, B\rangle$. □

3. I-polars

Let $\mathcal{A}$ be an NAl-algebra and a, b elements of $\mathcal{A}$. Using the concept of a principal ideal, we have

$$\langle a, b\rangle = \{x\in A;\ (a*0)\wedge(x*0)\in I(b)\}.$$

Since $I(0)=\{0\}$, the polar of a can be expressed by

$$a^{\perp} = \{x\in A;\ (a*0)\wedge(x*0)\in I(0)\}.$$

From this point of view, it is natural to substitute $I(0)$ by an arbitrary ideal I of $\mathcal{A}$ to obtain the following concept:

Definition. Let I be an ideal of an NAl-algebra $\mathcal{A}$ and let $a\in A$. By the *I-polar of a* we mean the set

$$a(I)^{\perp} = \{x\in A;\ (a*0)\wedge(x*0)\in I\}.$$

By the *I-polar of a non-void subset* B of $\mathcal{A}$ we mean the set

$$B(I)^{\perp} = \bigcap\{a(I)^{\perp};\ a\in B\}.$$

A subset C is called an *I-polar of* $\mathcal{A}$ if $C = B(I)^{\perp}$ for some non-void subset B of A.

Remark.

(a) Of course, if $I = I(0)$ then $a(I(0))^{\perp} = a^{\perp}$ and $B(I(0))^{\perp} = B^{\perp}$ for each $a\in A$ and every $\emptyset\neq B\subseteq \dot{A}$. Moreover, a subset C of A is an $I(0)$-polar of $\mathcal{A}$ if and only if C is a polar of $\mathcal{A}$.

(b) For every two elements $a, b\in A$ we have $a(I(b))^{\perp} = \langle a, b\rangle$ and for each subset $\emptyset\neq C\subseteq A$ we have $C(I(b))^{\perp} = \langle C, \{b\}\rangle$.

We are able to prove the following theorem.

Theorem 6. *Let I be an ideal of an NAl-algebra $\mathcal{A}$. The set $\mathcal{P}(I)$ of all I-polars of $\mathcal{A}$ forms a complete lattice with respect to set inclusion where infima coincide with intersections, the least element is I and the greatest one is A. Moreover, every I-polar of $\mathcal{A}$ is an ideal of $\mathcal{A}$ and for each non-void subset B of A we have $B(I)^{\perp} = \{x \in A;\ I(x) \cap I(B) \subseteq I\}$.*

P r o o f. Let I be an ideal of $\mathcal{A}$ and $B \subseteq A$. Denote

$$C = \{x \in A;\ I(x) \cap I(B) \subseteq I\}.$$

(a) Suppose $x \in B(I)^{\perp}$ and $z \in I(x) \cap I(B)$. Then there exist $m \in \mathbb{N}$ and elements $b_1, \ldots, b_n \in B$ such that

$$\begin{aligned} &z * 0 \leqslant m(x * 0),\\ &z * 0 \leqslant (b_1 * 0) + \ldots + (b_n * 0). \end{aligned}$$

Hence

$$\begin{aligned} 0 \leqslant z * 0 &\leqslant m(x * 0) \wedge \big((b_1 * 0) + \ldots + (b_n * 0)\big)\\ &\leqslant m\big((x * 0) \wedge (b_1 * 0)\big) + \ldots + m\big((x * 0) \wedge (b_n * 0)\big) \in I, \end{aligned}$$

thus $z * 0 \in I$ and also $z \in I$. We have $I(x) \cap I(B) \subseteq I$, i.e. $B(I)^{\perp} \subseteq C$.

(b) Let $x \in A$ be an element satisfying $I(x) \cap I(B) \subseteq I$, let $b \in B$ and put $c = (x * 0) \wedge (b * 0)$. Then $0 \leqslant c \leqslant x * 0$ and, by Lemma 2 and Theorem 5 in [4], $I(x) = I(x * 0)$ and every ideal of $\mathcal{A}$ is a convex subset of A, i.e. $c \in I(x)$.

Analogously, $c \in I(b)$, which implies $c \in I$. Thus $x \in B(I)^{\perp}$ proving $C \subseteq B(I)^{\perp}$.

We conclude $B(I)^{\perp} = \{x \in A;\ I(x) \cap I(B) \subseteq I\}$. Suppose now $x \notin I$. Then $(x * 0) \wedge (x * 0) \notin I$ whence $x \notin A(I)^{\perp}$. Conversely, if $x \notin A(I)^{\perp}$ then there exists $a \in A$ with $(a * 0) \wedge (x * 0) \notin I$. Suppose $x \in I$. Then $x * 0 \in I$ and, on account of convexity of I, also $(a * 0) \wedge (x * 0) \in I$, a contradiction. Hence $x \notin I$. We have shown $A(I)^{\perp} = I$, i.e. $I \in \mathcal{P}(I)$. Since $B \subseteq C \subseteq A$ implies $C(I)^{\perp} \subseteq B(I)^{\perp}$, I is clearly the least element of $\mathcal{P}(I)$. Of course, A is the greatest element of $\mathcal{P}(I)$ because $\{0\}(I)^{\perp} = A$.

Let us prove that every I-polar is an ideal of $\mathcal{A}$. To this end, let $a \in A$ and $x, y \in a(I)^{\perp}$. Then $(a * 0) \wedge (x * 0) \in I$ and $(a * 0) \wedge (y * 0) \in I$. Applying the normality of $\mathcal{A}$ we have

$$\begin{aligned} 0 &\leqslant \big((x + y) * 0\big) \wedge (a * 0)\\ &\leqslant \big((x * 0) + (y * 0)\big) \wedge (a * 0)\\ &\leqslant \big((x * 0) \wedge (a * 0)\big) + \big((y * 0) \wedge (a * 0)\big) \in I. \end{aligned}$$

Since I is convex, we obtain $\big((x+y)*0\big)\wedge(a*0)\in I$, whence $x+y\in a(I)^{\perp}$.

Suppose now $x\in a(I)^{\perp}$, $z\in A$, $z*0\leqslant x*0$. Then $0\leqslant(a*0)\wedge(z*0)\leqslant(a*0)\wedge(x*0)\in I$, i.e. also $(a*0)\wedge(z*0)\in I$, which implies $z\in a(I)^{\perp}$.

Hence $a(I)^{\perp}$ is an ideal of $\mathcal{A}$ and, moreover, for any non-void subset B of A we have $B(I)^{\perp}=\bigcap\{a(I)^{\perp};\ a\in B\}$, thus also $B(I)^{\perp}$ is an ideal of $\mathcal{A}$. This yields the fact that infima in $\mathcal{P}(I)$ coincide with intersections. □

Corollary 7. *Let I be an ideal of an NAl-algebra $\mathcal{A}$ and let $C\in\mathcal{P}(I)$. Then there exists an ideal J of $\mathcal{A}$ with $C=J(I)^{\perp}$.*

P r o o f. Of course, if $C=B(I)^{\perp}$ then $C=J(I)^{\perp}$ for $J=I(B)$. □

An ideal I of an *NAl*-algebra $\mathcal{A}$ is called a *prime ideal* if for each ideals J and K of $\mathcal{A}$ the implication $J\cap K=I\Longrightarrow J=I$ or $K=I$ holds. This concept was introduced by the second author in [6] where it was also shown that for $\mathcal{A}$ semiregular, I is a prime ideal of $\mathcal{A}$ if and only if $0\leqslant a\wedge b\in I\Longrightarrow a\in I$ or $b\in I$ for every a,b in A. On account of Theorem 9 in [4], this equivalent condition holds in every *NAl*-algebra. Hence we have

Corollary 8. *If I is a prime ideal of an NAl-algebra $\mathcal{A}$ then $\mathcal{P}(I)$ is the two-element chain $\{I,A\}$.*

P r o o f. Let I be a prime ideal of $\mathcal{A}$ and let $a\notin I$, $x\in A$. If $(a*0)\wedge(x*0)\in I$ then $x*0\in I$ and also $x\in I$. Hence $a(I)^{\perp}=I$. If $a\in I$ then $a(I)^{\perp}=A$. This yields that for $\emptyset\neq B\subseteq A$ we have only two possibilities:

$$B\nsubseteq I\Longrightarrow B(I)^{\perp}=I\quad\text{and}$$
$$B\subseteq I\Longrightarrow B(I)^{\perp}=A.$$

□

Remark. Applying Corollary 7, we can restrict ourselves to I-polars of ideals when investigating properties of arbitrary I-polars.

Let $\mathcal{A}$ be an *NAl*-algebra and I an ideal of $\mathcal{A}$. Denote

$$\mathcal{I}(\mathcal{A})_I=\{J\in\mathcal{I}(\mathcal{A});\ \ I\subseteq J\},$$

i.e. $\mathcal{I}(\mathcal{A})_I$ is the principal filter of the lattice $\mathcal{I}(\mathcal{A})$ generated by I. This fact together with Theorem 6 in [9] (stating that $\mathcal{I}(\mathcal{A})$ is a complete and Brouwerian lattice, i.e. $K\cap\bigvee_{\gamma\in\Gamma}J_{\gamma}=\bigvee_{\gamma\in\Gamma}(K\cap J_{\gamma})$ for every $K,J_{\gamma}\in\mathcal{I}(\mathcal{A}),\ \ \gamma\in\Gamma$) immediately imply

Corollary 9. *For every ideal I of an NAl-algebra $\mathcal{A}$, $\mathcal{I}(\mathcal{A})_I$ is a complete Brouwerian lattice.*

Hence, we can ask about pseudocomplements in the lattice $\mathcal{I}(\mathcal{A})_I$.

Theorem 10. *Let I be an ideal of an NAl-algebra $\mathcal{A}$ and $J \in \mathcal{I}(\mathcal{A})_I$. Then the pseudocomplement of J in the lattice $\mathcal{I}(\mathcal{A})_I$ is $J(I)^\perp$.*

Proof. Since $J(I)^\perp \in \mathcal{P}(\mathcal{A})$, we have $I \subseteq J(I)^\perp$, i.e. $J(I)^\perp \in \mathcal{I}(\mathcal{A})_I$. Suppose $x \in J \cap J(I)^\perp$. Then $x \in J$ and $x \in J(I)^\perp$, thus $x * 0 = (x * 0) \wedge (x * 0) \in I$ whence $x \in I$. We have $J \cap J(I)^\perp = I$.

Let $K \in \mathcal{I}(\mathcal{A})_I$ with $J \cap K = I$. Let $x \in K$ and $a \in J$. Then $0 \leqslant (x*0) \wedge (a*0) \leqslant x * 0$. Since K is convex, this yields $(x * 0) \wedge (a * 0) \in K$. Analogously we obtain $(x * 0) \wedge (a * 0) \in J$, thus also $(x * 0) \wedge (a * 0) \in K \cap J = I$. However, this means $x \in J(I)^\perp$, i.e. $K \subseteq J(I)^\perp$. We have shown that $J(I)^\perp$ is the pseudocomplement of J in $\mathcal{I}(\mathcal{A})_I$. □

Applying Theorem 10 together with Glivenko's Theorem (see e.g. Theorem VIII. 4.3 in [1]), we immediately conclude

Corollary 11. *For every NAl-algebra $\mathcal{A}$ and $I \in \mathcal{I}(\mathcal{A})$, the mapping $J \longmapsto J(I)^{\perp\perp}$ is a closure operator on $\mathcal{I}(\mathcal{A})_I$. The closed subsets are just all I-polars of $\mathcal{A}$. The set $\mathcal{P}(\mathcal{A})_I$ of all I-polars of $\mathcal{A}$ is a complete Boolean algebra with respect to set inclusion.*

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Authors' address: Palacký University, Dept. of Algebra and Geometry, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mails: `chajda@risc.upol.cz`, `rachunek@risc.upol.cz`.