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ON ASYMPTOTIC PROPERTIES OF A STRONGLY NONLINEAR DIFFERENTIAL EQUATION

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Abstract. The paper describes asymptotic properties of a strongly nonlinear system $\dot{x} = f(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. The existence of an $\lfloor n/2 \rfloor$ parametric family of solutions tending to zero is proved. Conditions posed on the system try to be independent of its linear approximation.

Keywords: ordinary differential equations, asymptotic properties

MSC 2000: 34D99

1. Introduction

There is a vast amount of literature about solutions of weakly nonlinear systems of ordinary differential equations

$$\dot{x} = Ax + f(t, x), \quad f(t, 0) \equiv 0 \tag{1}$$

in the vicinity of the trivial solution $x(t) \equiv 0$. In the case of strongly nonlinear systems

$$\dot{x} = f(t, x), \quad f(t, 0) \equiv 0 \tag{2}$$

the situation is much more complicated mainly because of the lack of any possibility of usage of a linear approximation of (2) in the vicinity of the trivial solution. In this case it is even difficult to say what kind of reasonable conditions should be posed on (2). When studying oscillatory properties it is often supposed that

$$x_{i+1} f_i(t, x_1, \ldots, x_n) > 0 \qquad \text{for } x_{i+1} \neq 0,\ i = 1, 2, \ldots, n, \quad x_{n+1} = x_1 \tag{3}$$

and such conditions are supposed to be useful ([1], [3]). This paper is not interested in oscillatory properties, but in a description of solutions approaching the trivial solution of (2), therefore instead of (3) hypotheses like

$$x_i f_{n-i+1}(t, x_1, \dots, n_n) > 0 \qquad \text{for } x_i \neq 0,\ i = 1, \dots, n \tag{4}$$

will be used. It is interesting to note that (3) and (4) coincide for $n = 2$.

2. Main results

Consider the differential equation

$$\dot{x} = f(t, x), \tag{5}$$

in which $\dot{}= \mathrm{d}/\mathrm{d}t$ and $f(t, x)$ is a continuous function from $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$, where $n > 1$, such that

(H1) *all solutions of* (5) *are uniquely determined by initial conditions*

and where $x = [x_1, \dots, x_n]^T$. Our main hypotheses are

(H2) $x_i f_{n-i+1}(t, x_1, \dots, x_n) > 0 \qquad \text{for } x_i \neq 0,\ i = 1, \dots, n$

or

(H3) $x_i f_{n-i+1}(t, x_1, \dots, x_n) + x_{n-i+1} f_i(t, x_1, \dots, x_n) > 0$
$\text{for } |x_i| + |x_{n-i+1}| > 0,\ i = 1, \dots, n.$

In the following we shall use functions $\lfloor . \rfloor$, $\lceil . \rceil$ defined for $x \in \mathbb{R}$ by $\lfloor x \rfloor = n$ if $x \in [n, n+1)$, and $\lceil x \rceil = n$ if $x \in (n-1, n]$, where n is an integer. The right endpoint of the maximal interval of existence of a solution of (5) will be denoted by ω^+.

Theorem 2.1. *Suppose that* (5) *satisfies* (H1) *and* (H2). *Then the system* (5) *has an* $\lfloor \frac{n}{2} \rfloor$ *parametric family of solutions* $u(t) = [u_1(t), \dots, u_n(t)]$ *such that the function* $\|u(t)\|$ *is nonincreasing, the limit* $\lim\limits_{t \to \infty} u(t)$ *exists and* $u_i(t)$ *are monotonous functions.*

Proof. Under the change of variables

$$\left.\begin{aligned} x_i &= y_i + y_{n-i+1} \\ x_{n-i+1} &= -y_i + y_{n-i+1} \end{aligned}\right\} i = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \quad n \text{ even or odd}$$
$$x_{\lfloor \frac{n}{2} \rfloor + 1} = y_{\lfloor \frac{n}{2} \rfloor + 1}, \qquad n \text{ odd}$$

the equation (5) becomes

$$\dot{y} = g(t, y) \tag{6}$$

where

$$g_i(t,y) = \begin{cases} \tfrac{1}{2} f_i(t, y_1 + y_n, \dots, y_n - y_1) - \tfrac{1}{2} f_{n-i+1}(t, y_1 + y_n, \dots, y_n - y_1) \\ \quad \text{for } i = 1, \dots, \lfloor \tfrac{n}{2} \rfloor, \quad n \text{ even or odd} \\ \tfrac{1}{2} f_i(t, y_1 + y_n, \dots, y_n - y_1) + \tfrac{1}{2} f_{n-i+1}(t, y_1 + y_n, \dots, y_n - y_1) \\ \quad \text{for } i = \lceil \tfrac{n}{2} \rceil + 1, \dots, n, \quad n \text{ even or odd} \\ f_{\lfloor \frac{n}{2} \rfloor + 1}(t, y_1 + y_n, \dots, y_{\lfloor \frac{n}{2} \rfloor}, \dots, y_n - y_1) \\ \quad \text{for } i = \lfloor \tfrac{n}{2} \rfloor + 1, \quad n \text{ odd.} \end{cases}$$

In order to prove the theorem we will use the Ważewski topological principle (e.g. [2]). Let $\varepsilon > 0$ be an arbitrary small fixed number,

$$u_i^\varepsilon \colon \ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},$$

$$u_i^\varepsilon \colon \ (t,y) \mapsto \begin{cases} (y_{n-i+1})^2 - y_i^2 - \varepsilon & i = 1, \dots, \lfloor \tfrac{n}{2} \rfloor, \ n \text{ even or odd} \\ (y_{\lfloor \frac{n}{2} \rfloor + 1})^2 - \varepsilon & i = \lfloor \tfrac{n}{2} \rfloor + 1, \ n \text{ odd} \end{cases}$$

and

$$\Omega^\varepsilon := \Big\{ (t,y) \in \mathbb{R} \times \mathbb{R}^n : \ t > t_0, \ u_i^\varepsilon(t,y) < 0 \text{ for } i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil \Big\}.$$

It will be verified that Ω^ε is a (u,v) subset of $\mathbb{R} \times \mathbb{R}^n$ ([2] p. 281) determined by functions u_i^ε. The derivative of u_i^ε along a solution of (6) satisfies

$$\begin{aligned} \dot{u}_i^\varepsilon(t,y) &= 2 y_{n-i+1} \dot{y}_{n-i+1} - 2 y_i \dot{y}_i \\ &= y_{n-i+1} \left[f_i(t, y_1 + y_n, \dots) + f_{n-i+1}(t, y_1 + y_n, \dots) \right] \\ &\quad - y_i \left[f_i(t, y_1 + y_n, \dots) - f_{n-i+1}(t, y_1 + y_n, \dots) \right] \\ &= (y_{n-i+1} - y_i) f_i(t, y_1 + y_n, \dots) + (y_i + y_{n-i+1}) f_{n-i+1}(t, y_1 + y_n, \dots) \\ &= x_{n-i+1} f_i(t, x_1, \dots, x_n) + x_i f_{n-i+1}(t, x_1, \dots, x_n) \\ &> 0 \qquad \text{for } |x_i| + |x_{n-i+1}| > 0, \ i = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \ n \text{ even or odd,} \\ \dot{u}_i^\varepsilon(t,y) &= 2 y_{\lfloor \frac{n}{2} \rfloor + 1} \dot{y}_{\lfloor \frac{n}{2} \rfloor + 1} \\ &= 2 x_{\lfloor \frac{n}{2} \rfloor + 1} f_{\lfloor \frac{n}{2} \rfloor + 1}(t, x_1, \dots, x_n) \\ &> 0 \qquad \text{for } x_{\lfloor \frac{n}{2} \rfloor} \neq 0, \ i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \ \ n \text{ odd,} \end{aligned}$$

hence Ω^ε is a (u,v) subset of $\mathbb{R} \times \mathbb{R}^n$ and the set of strict egress points of Ω^ε is $\partial\Omega^\varepsilon$. Let S^ε be a subset of $\mathbb{R} \times \mathbb{R}^n$ defined by

$$\begin{aligned} S^\varepsilon := \Big\{ &(t_0, y_1^0, \dots, y_{\lfloor \frac{n}{2} \rfloor}^0, y_{\lfloor \frac{n}{2} \rfloor + 1}, \dots, y_n) : \ y_i^0 = \text{const}_i \in \mathbb{R} \quad \text{for } i = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \\ &|y_{n-i+1}| \leqslant \sqrt{\varepsilon + (y_i^0)^2}, \quad i = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \quad n \text{ even or odd} \\ &|y_{\lfloor \frac{n}{2} \rfloor + 1}| \leqslant \sqrt{\varepsilon}, \quad n \text{ odd} \Big\}. \end{aligned}$$

S^ε is essentially an $\lceil \frac{n}{2} \rceil$ dimensional ball and $S^\varepsilon \cap \partial\Omega^\varepsilon$ is its boundary, therefore $S^\varepsilon \cap \partial\Omega^\varepsilon$ is not a retract of S^ε. Since the mapping

$$\begin{aligned}
&\pi^\varepsilon\colon\ \partial\Omega^\varepsilon \to S^\varepsilon \cap \partial\Omega^\varepsilon,\\
&\pi^\varepsilon\colon\ (t,y) \mapsto (t_0, y_1^0, \dots, y^0_{\lfloor \frac{n}{2} \rfloor}, \tilde{y}_{\lfloor \frac{n}{2} \rfloor+1}, \dots, \tilde{y}_n),\\
&\tilde{y}_{n-i+1} := \begin{cases} y_{n-i+1}\dfrac{\sqrt{\varepsilon + (y_i^0)^2}}{\sqrt{\varepsilon + (y_i)^2}}, & i = 1, \dots, \left\lfloor \dfrac{n}{2} \right\rfloor,\ n \text{ even or odd}\\ y_{\lfloor \frac{n}{2} \rfloor+1}, & i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1,\ n \text{ odd} \end{cases}
\end{aligned}$$

is continuous and equal to identity on $S^\varepsilon \cap \partial\Omega^\varepsilon$, it is a retraction of $\partial\Omega^\varepsilon$ onto $S^\varepsilon \cap \Omega^\varepsilon$. The existence of (at least one) point $(t_0, y_0) \in S^\varepsilon$ such that the initial problem (5), $y(t_0) = y_0$ has a solution $y = u(t, t_0, y_0)$ satisfying $y(t) \in \Omega^\varepsilon$ on its right maximal interval $[t_0, \omega^+)$ follows now from the Ważewski topological principle.

This means that the system (5) has an $\lfloor \frac{n}{2} \rfloor$-parametrical system of solutions belonging to the set

$$\Theta^\varepsilon := \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n :\ t \geqslant t_0,\ x_i x_{n-i+1} < \varepsilon \text{ for } i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil \right\}.$$

Passing with ε to zero we obtain an $\lfloor \frac{n}{2} \rfloor$ parametric set of solutions $u(t) = [u_1(t), \dots, u_n(t)]$ of (5) such that

$$\begin{aligned}
&u_i(t)u_{n-i+1}(t) \leqslant 0 \quad \text{for } i = 1, \dots, \lfloor \tfrac{n}{2} \rfloor, && n \text{ even or odd},\\
&u_{\lfloor \frac{n}{2} \rfloor+1}(t) = 0, && n \text{ odd}
\end{aligned} \tag{7}$$

on the right maximal interval $[t_0, \omega^+)$.

From (H2) we have

$$\begin{aligned}
0 &\leqslant [u_i(t) f_{n+i-1}(t, u(t))]\,[u_{n-i+1}(t) f_i(t, u(t))]\\
&= [u_i(t) f_i(t, u(t))]\,[u_{n-i+1}(t) f_{n+i-1}(t, u(t))]
\end{aligned}$$

and together with (7) we obtain

$$u_i(t) f_i(t, u(t)) \leqslant 0 \qquad \text{for } i = 1, \dots, n \tag{8}$$

on $[t_0, \omega^+)$. Therefore $u_i(t)$ is monotonous and properly bounded by 0, hence the limit $\lim\limits_{t \to \omega^+} u(t)$ exists. It is clear that for such solutions

$$\|u(t)\| \frac{\mathrm{d}}{\mathrm{dt}} \|u(t)\| = \sum_{i=1}^n u_i(t) f_i(t, u(t)) \leqslant 0,$$

therefore $\|u(t)\|$ is bounded and $\omega^+ = \infty$. □

Theorem 2.1 does not claim that the limit of $u(t)$ is 0. Instead of a direct proof of this we shall first replace (H2) by a more general (H3).

From Theorem 2.1 we obtain immediately

Corollary 2.2. *Suppose that* (5) *satisfies* (H1) *and* (H3). *Then the system* (5) *has an* $\lfloor\frac{n}{2}\rfloor$ *parametric family of solutions* $u(t) = [u_1(t), \dots, u_n(t)]$ *such that the functions* $v_i(t) := u_i(t)u_{n-i+1}(t)$ *are nondecreasing and the limit* $\lim\limits_{t\to\omega^+} v_i(t)$ *exists* (*and is nonpositive*) *for* $i = 1, \dots, \lceil\frac{n}{2}\rceil$.

Proof. Exactly as in Theorem 2.1 we obtain the existence of an $\lfloor\frac{n}{2}\rfloor$ parametric set of functions $u(t) = [u_1(t), \dots, u_n(t)]$ of (5) fulfilling (7). Therefore $v_i \leqslant 0$ for $i = 1, \dots, \lceil\frac{n}{2}\rceil$. For any t_1, t_2 $t_0 < t_1 < t_2 < \omega^+$ we have

$$0 \leqslant \int_{t_1}^{t_2} [u_i(t)f_{n-i+1}(t, u(t)) + u_{n-i+1}(t)f_i(t, u(t))]\,\mathrm{d}t = v_i(t_2) - v_i(t_1)$$

and $v_i(t)$ is nondecreasing. □

Lemma 2.3. *Suppose that* (5) *is autonomous, satisfies* (H1), (H3) *and all its solutions are bounded. Then* (5) *has an* $\lfloor\frac{n}{2}\rfloor$ *parametric family of solutions* $u(t) = [u_1(t), \dots, u_n(t)]$ *such that* $\lim\limits_{t\to\infty} u_i(t) = 0$.

Proof. Let $u(t)$ be an element of the $\lfloor\frac{n}{2}\rfloor$ parametric family of solutions guaranteed by Corollary 2.2. Consider a function

$$\begin{aligned} V\colon\ & \mathbb{R}^n \to \mathbb{R}, \\ V\colon\ & x \mapsto \sum_{i=1}^{\lceil\frac{n}{2}\rceil} x_i x_{n-i+1}. \end{aligned}$$

Then $V(x) \leqslant 0$ and

$$\frac{\mathrm{d}}{\mathrm{dt}}(V \circ u)(t) = \sum_{i=1}^{\lceil\frac{n}{2}\rceil} [u_i(t)f_{n-i+1}(u(t)) + u_{n-i+1}(t)f_i(u(t))] \geqslant 0.$$

As $u(t)$ is bounded, its ω-limit set Ω is nonempty. Let $u^* \in \Omega$, then there is a sequence t_i, $t_i < t_{i+1} \to \omega^+$ as $i \to \infty$ such that $u(t_i) \to u^*$ as $i \to \infty$ and since the sequence $(V \circ u)(t_i)$ is bounded from above, nondecreasing and $V(x)$ is continuous, we conclude that

$$\lim_{i\to\infty} (V \circ u)(t_i) = V\big(\lim_{i\to\infty} u(t_i)\big) = \sum_{i=1}^{\lceil\frac{n}{2}\rceil} u_i^* u_{n-i+1}^* = \mathrm{Const} \leqslant 0,$$

where Const $= V(u^*)$. Since $(V \circ u)(t)$ is continuous, $\lim\limits_{t \to \omega+} (V \circ u)(t) =$ Const, hence for any sequence s_i, $s_i < s_{i+1} \to \omega^+$ as $i \to \infty$ also $(V \circ u)(s_i) \to$ Const as $i \to \infty$. Therefore $V(x)$ is a constant function on Ω, in particular, if $x^* \in \Omega$ then $\dot{V}(x^*) := \partial_x V(x^*) \circ f(x^*) = 0$. However, (H3) yields that $\dot{V}(x) = 0$ if and only if $x = 0$, hence $u(t) \to 0$ for $t \to \omega^+$ and $\omega^+ = \infty$. □

As a direct consequence of Theorem 2.1 and Lemma 2.3 we obtain

Theorem 2.4. *Suppose that* (5) *is autonomous and satisfies* (H1) *and* (H2). *Then* (5) *has an* $\lfloor \frac{n}{2} \rfloor$ *parametric family of solutions* $u(t)$ *such that* $\lim\limits_{t \to \infty} u(t) = 0$.

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