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GEOMETRY OF HOLOMORPHIC DISTRIBUTIONS
OF REAL HYPERSURFACES
IN A COMPLEX PROJECTIVE SPACE

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Abstract. We characterize homogeneous real hypersurfaces M 's of type (A_1) , (A_2) and (B) of a complex projective space in the class of real hypersurfaces by studying the holomorphic distribution T^0M of M .

Keywords: complex projective space, real hypersurfaces, holomorphic distribution

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0. INTRODUCTION

Let $P_n(\mathbb{C})$ be an n -dimensional complex projective space with Fubini-Study metric G of constant holomorphic sectional curvature 4, and let M^{2n-1} be a real hypersurface of $P_n(\mathbb{C})$. Then M has an almost contact metric structure (φ, ξ, η, g) induced by the complex structure J of $P_n(\mathbb{C})$. This structure is a useful tool in the study of real hypersurfaces M 's in $P_n(\mathbb{C})$ (for examples, see [1], [4], [7]). In this paper we study the holomorphic distribution T^0M which is defined by $(T^0M)_p = \{X \in T_p(M) \mid X \perp \xi\}$ for $p \in M$.

It is known that if the structure vector ξ of a real hypersurface M is a principal curvature vector, the holomorphic distribution T^0M is not integrable (see [3]). This implies that the holomorphic distribution of any homogeneous real hypersurfaces in

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$P_n(\mathbb{C})$, that is any of the real hypersurfaces given as orbits under subgroups of the projective unitary group $PU(n+1)$, is not integrable.

Takagi ([7]) classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$. By virtue of his work, we find that a homogeneous real hypersurface in $P_n(\mathbb{C})$ is locally congruent to one of the six model spaces of type A_1, A_2, B, C, D and E . They are realized as tubes of constant radii over compact Hermitian symmetric spaces of rank 1 or rank 2 (see Theorem A). A homogeneous real hypersurface of type A_1 is usually called a *geodesic hypersphere*. In the study of real hypersurfaces in $P_n(\mathbb{C})$, many differential geometers have considered the following two problems:

- (I) Give a characterization of homogeneous real hypersurfaces in $P_n(\mathbb{C})$.
- (II) Construct non-homogeneous *nice* real hypersurfaces in $P_n(\mathbb{C})$ and characterize such examples.

We first investigate in detail the distribution T^0M of any homogeneous real hypersurface M in $P_n(\mathbb{C})$. From the viewpoint of Problem (I) we establish the following two theorems.

Theorem 1. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then M is locally congruent to a homogeneous real hypersurface of type A_1 or type A_2 if and only if the holomorphic distribution T^0M satisfies the following two conditions:*

- (1) T^0M is decomposed as the direct sum of principal foliations V_{λ_i} 's of M in $P_n(\mathbb{C})$.
- (2) For each principal foliation V_{λ_i} in condition (1), the distribution $V_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}}$ is integrable.

Theorem 2. *Let M be a real hypersurface of $P_n(\mathbb{C})$, $n \geq 3$. Then M is locally congruent to a homogeneous real hypersurface of type B if and only if the holomorphic distribution T^0M satisfies the following three conditions:*

- (1) T^0M is decomposed as the direct sum of principal foliations V_{λ_i} 's of M in $P_n(\mathbb{C})$ with $\dim V_{\lambda_i} \geq 2$.
- (2) Every principal foliation V_{λ_i} in condition (1) is integrable.
- (3) Every leaf of any principal foliation V_{λ_i} in condition (1) is a totally geodesic submanifold of the real hypersurface M .

We remark that if we omit the condition (3), Theorem 2 is not true. We will construct a certain class of non-homogeneous real hypersurfaces M 's (in $P_n(\mathbb{C})$) satisfying the conditions (1), (2) in Theorem 2. In this paper, a real hypersurface satisfying the conditions (1), (2) in Theorem 2 is called a *real hypersurface of Dupin type*. Needless to say, the characteristic vector ξ of any real hypersurface M of Dupin type is a principal curvature vector of M in $P_n(\mathbb{C})$.

From the viewpoint of Problem (II) it is interesting to study non-homogeneous real hypersurfaces of Dupin type in $P_n(\mathbb{C})$. We here review the definition of a Dupin

hypersurface M^n of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c (that is, $\widetilde{M}^{n+1}(c) = \mathbb{R}^{n+1}$, $S^{n+1}(c)$ or $H^{n+1}(c)$) as the curvature c is zero, positive or negative). A hypersurface M^n in $\widetilde{M}^{n+1}(c)$ is called a *Dupin hypersurface* if each of its principal curvatures has constant multiplicity and is constant along the leaves of its principal foliation. So every leaf of its principal foliation is totally umbilic in $\widetilde{M}^{n+1}(c)$, but generally it is not totally geodesic in the hypersurface M^n .

Finally, we will construct non-homogeneous real hypersurfaces of Dupin type in $P_n(\mathbb{C})$.

1. PRELIMINARIES

Let M be a real hypersurface of $P_n(\mathbb{C})$ and let N be a unit normal local vector field on M . The Riemannian connections $\widetilde{\nabla}$ of $P_n(\mathbb{C})$ and ∇ of M are related by

$$(1.1) \quad \widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$$

and

$$(1.2) \quad \widetilde{\nabla}_X N = -AX,$$

where g denotes the Riemannian metric of M induced by the Fubini-Study metric G of $P_n(\mathbb{C})$ and A is the shape operator of M in $P_n(\mathbb{C})$. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. In what follows, we denote by V_λ the eigenspace of A associated with the eigenvalue λ . It is known that M admits an almost contact metric structure (φ, ξ, η, g) induced by the complex structure of $P_n(\mathbb{C})$, which satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

It follows from (1.1) and (1.2) that

$$(1.3) \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi$$

and

$$(1.4) \quad \nabla_X \xi = \varphi AX.$$

Let \tilde{R} and R be the curvature tensors of $P_n(\mathbb{C})$ and M , respectively. Since the curvature tensor \tilde{R} has a nice form, we have the following Gauss and Codazzi equations:

$$(1.5) \quad \begin{aligned} g(R(X, Y)Z, W) &= g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &\quad + g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W) \\ &\quad - 2g(\varphi X, Y)g(\varphi Z, W) + g(AY, Z)g(AX, W) \\ &\quad - g(AX, Z)g(AY, W), \end{aligned}$$

$$(1.6) \quad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi.$$

In the following, we use the same terminology and notation as above unless otherwise stated. Now we present without proof the following results in order to prove our theorems:

Theorem A ([7]). *Let M be a homogeneous real hypersurface of $P_n(\mathbb{C})$. Then M is a tube of radius r over one of the following Kaehler submanifolds:*

- (A₁) hyperplane $P_{n-1}(\mathbb{C})$, where $0 < r < \frac{\pi}{2}$,
- (A₂) totally geodesic $P_k(\mathbb{C})$ ($1 \leq k \leq n - 2$), where $0 < r < \frac{\pi}{2}$,
- (B) complex quadric Q_{n-1} , where $0 < r < \frac{\pi}{4}$,
- (C) $P_1(\mathbb{C}) \times P_{\frac{n-1}{2}}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$ and n (≥ 5) is odd,
- (D) complex Grassmann $G_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,
- (E) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

The number of distinct principal curvatures of these homogeneous real hypersurfaces is 2, 3, 3, 5, 5, 5, respectively.

Theorem B ([2]). *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

Proposition A ([5]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α . Then α is locally constant. In addition, $A\varphi X = \frac{\alpha\lambda+2}{2\lambda-\alpha}\varphi X$ holds for any $X(\perp\xi) \in V_\lambda$.*

2. PROOF OF THEOREMS

P r o o f of Theorem 1. Let M be a real hypersurface satisfying the conditions (1), (2) in Theorem 1. We shall show that our manifold is of type A_1 or type A_2 . T^0M is decomposed as $T^0M = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_d}$, where d is the number of distinct principal curvatures λ_i corresponding to the principal curvature vectors $v_{\lambda_i}(\perp \xi)$. Then for any $X (= \sum_{i=1}^d X^i v_{\lambda_i}) \in T^0M$, $g(A\xi, X) = g(\xi, AX) = \sum_{i=1}^d g(\xi, X^i \lambda_i v_{\lambda_i}) = 0$, so that ξ is principal. By hypothesis, for any $X (\in V_{\lambda_i})$ we have $\nabla_X \xi - \nabla_\xi X \in V_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}}$ ($i = 1, \dots, d$). We note that $\nabla_X \xi - \nabla_\xi X$ is perpendicular to ξ for any $X (\in V_{\lambda_i})$, because ξ is a principal curvature (unit) vector, so that

$$A(\nabla_X \xi - \nabla_\xi X) = \lambda_i(\nabla_X \xi - \nabla_\xi X) \quad \text{for any } X \in V_{\lambda_i}.$$

This, together with (1.4) and Proposition A, shows

$$(2.1) \quad (A - \lambda_i I)\nabla_\xi X = \lambda_i \left(\frac{\alpha \lambda_i + 2}{2\lambda_i - \alpha} - \lambda_i \right) \varphi X.$$

It follows from (1.4), (2.1) and Proposition A that

$$\begin{aligned} (\nabla_X A)\xi - (\nabla_\xi A)X &= \nabla_X(\alpha\xi) - A\nabla_X \xi - \nabla_\xi(AX) + A\nabla_\xi X \\ &= \alpha\varphi AX - A\varphi AX - (\xi\lambda_i)X + (A - \lambda_i I)\nabla_\xi X \\ &= \lambda_i \left(\alpha - \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha} \right) \varphi X - (\xi\lambda_i)X + \lambda_i \left(\frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha} - \lambda_i \right) \varphi X \\ &= \lambda_i(\alpha - \lambda_i)\varphi X - (\xi\lambda_i)X. \end{aligned}$$

On the other hand, the Codazzi equation (1.6) implies

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\varphi X.$$

Hence, the principal curvature λ_i is a solution of the quadratic equation

$$(2.2) \quad \lambda_i^2 - \alpha\lambda_i - 1 = 0.$$

Then our manifold M is locally congruent to one of the homogeneous real hypersurfaces in $P_n(\mathbb{C})$ (see Theorem B). Moreover, again by using (2.2) we find that $\varphi V_{\lambda_i} = V_{\lambda_i}$ ($i = 1, \dots, d$), which yields that M is of type A_1 or type A_2 (cf. [5]).

Our theorem is obvious for type A_1 . So, let M be of type A_2 (which is a tube of radius r). Let $x = \cot r$ ($0 < r < \frac{\pi}{2}$). Then at any point p of M , $T_p(M)$ is decomposed as $T_p(M) = \{\xi\}_{\mathbb{R}} \oplus V_{\lambda_1} \oplus V_{\lambda_2}$, where $\lambda_1 = x$, $\lambda_2 = -\frac{1}{x}$, $\alpha = x - \frac{1}{x}$. Note

that $\varphi V_{\lambda_i} = V_{\lambda_i}$ ($i = 1, 2$) (for details, see [8]). We remark that neither V_{λ_1} nor V_{λ_2} is integrable. Our aim here is to prove that $V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$ ($V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$) is integrable and moreover, that any leaf of the distribution $V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$ (respectively, $V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$) is a totally geodesic submanifold of M . Let $\mathcal{T} = V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$. Then we can show the following:

$$\nabla_{\xi}\xi \in \mathcal{T}, \quad \nabla_X\xi \in \mathcal{T}, \quad \nabla_{\xi}X \in \mathcal{T} \quad \text{and} \quad \nabla_XY \in \mathcal{T} \quad \text{for any } X, Y \in V_{\lambda_1}.$$

In fact, (1.4) yields $\nabla_{\xi}\xi = 0 \in \mathcal{T}$ and $\nabla_X\xi = \varphi AX = \lambda_1\varphi X \in V_{\lambda_1} \subset \mathcal{T}$. Next,

$$\begin{aligned} (\nabla_{\xi}A)X - (\nabla_XA)\xi &= \nabla_{\xi}(AX) - A\nabla_{\xi}X - \nabla_X(A\xi) + A\nabla_X\xi \\ &= (\lambda_1I - A)\nabla_{\xi}X - \alpha\varphi AX + A\varphi AX \\ &= (\lambda_1I - A)\nabla_{\xi}X + \lambda_1(\lambda_1 - \alpha)\varphi X. \end{aligned}$$

On the other hand, it follows from (1.6) that

$$(\nabla_{\xi}A)X - (\nabla_XA)\xi = \varphi X \in V_{\lambda_1}.$$

Thus for any $Z \in V_{\lambda}$ ($\lambda = \lambda_2, \alpha$) we find $g((\lambda_1I - A)\nabla_{\xi}X, Z) = 0$, so that $\nabla_{\xi}X \in V_{\lambda_1} \subset \mathcal{T}$. Finally, for any $X, Y \in V_{\lambda_1}$ and for any $Z \in V_{\lambda_2}$ we get

$$\begin{aligned} g((\nabla_XA)Y, Z) &= g(\nabla_X(AY) - A\nabla_XY, Z) \\ &= g((\lambda_1I - A)\nabla_XY, Z) \\ &= (\lambda_1 - \lambda_2)g(\nabla_XY, Z). \end{aligned}$$

On the other hand, it follows from (1.6) that

$$\begin{aligned} g((\nabla_XA)Y, Z) &= g((\nabla_XA)Z, Y) \\ &= g((\nabla_ZA)X, Y) \\ &= g(\nabla_Z(AX) - A\nabla_ZX, Y) \\ &= g((\lambda_1I - A)\nabla_ZX, Y) = 0. \end{aligned}$$

Hence, $\nabla_XY \in \mathcal{T}$. Thus we can see that every leaf L of the distribution \mathcal{T} is a totally geodesic submanifold of M . The manifold L is locally congruent to a homogeneous real hypersurface (with the unit vector $-N$) of type A_1 of radius $(\frac{\pi}{2} - r)$ in $P_{m+1}(\mathbb{C})$ which is a holomorphic totally geodesic submanifold of $P_n(\mathbb{C})$, where $2m = \dim V_{\lambda_1}$ (see Theorem 1 in [1]). The same discussion as above yields that the distribution $\mathcal{S} = V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$ is integrable and moreover, that every leaf K of the distribution \mathcal{S} is a totally geodesic submanifold of M . The manifold K is locally congruent to a homogeneous real hypersurface (with the unit normal vector N) of type A_1 of radius r in $P_{k+1}(\mathbb{C})$ which is a holomorphic totally geodesic submanifold of $P_n(\mathbb{C})$, where $2k = \dim V_{\lambda_2}$. \square

Proof of Theorem 2. Let M be a real hypersurface satisfying the conditions (1), (2), (3) in Theorem 2. We shall show that the manifold M is of type B . From the condition (1) we can set $T^0M = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_d}$. It follows from the conditions (2), (3) that $A\nabla_X Y = \lambda_i \nabla_X Y$ for any $X, Y \in V_{\lambda_i}$ ($i = 1, 2, \dots, d$). Hence, for any $X, Y \in V_{\lambda_i}$ we get $(\nabla_X A)Y = (X\lambda_i)Y$. On the other hand, it follows from the condition (3) and (1.1) that every leaf of V_{λ_i} is a totally umbilic submanifold of $P_n(\mathbb{C})$. Needless to say, the mean curvature of any totally umbilic submanifold whose dimension is greater than 1 in $P_n(\mathbb{C})$ is constant, which implies that $X\lambda_i = 0$ for any $X \in V_{\lambda_i}$. Hence,

$$(2.3) \quad (\nabla_X A)Y = 0 \quad \text{for any } X, Y \in V_{\lambda_i}.$$

Thus, for each unit $X \in V_{\lambda_i}$ and for any $Z \in TM$

$$\begin{aligned} 0 &= g((\nabla_X A)X, Z) && \text{(from (2.3))} \\ &= g((\nabla_X A)Z, X) \\ &= g((\nabla_Z A)X + \eta(X)\varphi Z - \eta(Z)\varphi X - 2 \cdot g(\varphi X, Z)\xi, X) && \text{(from (1.6))} \\ &= g((\nabla_Z A)X, X) = g(\nabla_Z(AX) - A\nabla_Z X, X) \\ &= g((Z\lambda_i)X + (\lambda_i I - A)\nabla_Z X, X) \\ &= Z\lambda_i. \end{aligned}$$

Then Theorem B tells us that the manifold M is homogeneous in $P_n(\mathbb{C})$. However, the principal foliation V_λ is not integrable in the case that $\varphi V_\lambda = V_\lambda$ (see (1.6)). Thus we can see that M is of type B (for details, see [8]).

Conversely, let M be of type B . Then at any point p of M , $T_p(M)$ is decomposed as $T_p(M) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$, where $\lambda_1 = \frac{1+x}{1-x}$, $\lambda_2 = \frac{x-1}{x+1}$, $\alpha = x - \frac{1}{x}$ and $x = \cot r$ ($0 < r < \frac{\pi}{4}$) (cf. [8]). We note that $\varphi V_{\lambda_1} = V_{\lambda_2}$ (see Proposition A). We shall prove that the principal foliation V_{λ_1} (resp. V_{λ_2}) on M is integrable, and moreover that every leaf of the distribution V_{λ_1} (resp. V_{λ_2}) is a totally geodesic submanifold of M . It suffices to verify that $\nabla_X Y \in V_{\lambda_1}$ for any $X, Y \in V_{\lambda_1}$. We first have

$$\begin{aligned} A\nabla_X Y &= \nabla_X(AY) - (\nabla_X A)Y \\ &= \lambda_1 \nabla_X Y - (\nabla_X A)Y. \end{aligned}$$

For any $Z \in TM$, since A is symmetric, from (1.6) we find

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g((\nabla_X A)Z, Y) \\ &= g((\nabla_Z A)X + \eta(X)\varphi Z - \eta(Z)\varphi X - 2 \cdot g(\varphi X, Z)\xi, Y) \\ &= g((\nabla_Z A)X, Y) = g(\nabla_Z(AX) - A\nabla_Z X, Y) \\ &= g((Z\lambda_1)X + (\lambda_1 I - A)\nabla_Z X, Y) = 0, \end{aligned}$$

so that $(\nabla_X A)Y = 0$ for any $X, Y \in V_{\lambda_1}$. This implies that every leaf L_{λ_1} of the principal foliation V_{λ_1} is a totally geodesic submanifold of the real hypersurface M . L_{λ_1} is locally congruent to a totally umbilic hypersurface of constant curvature c (with $\sqrt{c-1} = |\lambda_1|$) in $P^n(\mathbb{R})$ which is a totally real totally geodesic submanifold of $P_n(\mathbb{C})$. \square

Example. Let V_{n-1} be a complex hypersurface of $P_n(\mathbb{C})$, $n \geq 3$ such that

- (1) any principal curvature with respect to the shape operator A_ξ for any unit normal vector ξ of V_{n-1} is non zero, and
- (2) multiplicity of each principal curvature with respect to A_ξ for any unit normal vector ξ of V_{n-1} is constant.

Then by [1, Proposition 3.1, p. 487] we can see that a real hypersurface M which lies on the tube of radius $r > 0$ over V_{n-1} satisfies the conditions (1), (2) in Theorem 2. We remark that there exists such complex hypersurfaces V_{n-1} .

References

- [1] *T. Cecil and P. Ryan*: Focal sets and real hypersurfaces in complex projective space. *Trans. Amer. Math. Soc.* *269* (1982), 481–499.
- [2] *M. Kimura*: Real hypersurfaces and complex submanifolds in complex projective space. *Trans. Amer. Math. Soc.* *296* (1986), 137–149.
- [3] *M. Kimura and S. Maeda*: On real hypersurfaces of a complex projective space. *Math. Z.* *202* (1989), 299–311.
- [4] *M. Kimura and S. Maeda*: Lie derivatives on real hypersurfaces in a complex projective space. *Czechoslovak Math. J.* *45* (1995), 135–148.
- [5] *Y. Maeda*: On real hypersurfaces of a complex projective space. *J. Math. Soc. Japan* *28* (1976), 529–540.
- [6] *K. Ogiue*: Differential geometry of Kaehler submanifolds. *Adv. Math.* *13* (1974), 73–114.
- [7] *R. Takagi*: On homogeneous real hypersurfaces of a complex projective space. *Osaka J. Math.* *10* (1973), 495–506.
- [8] *R. Takagi*: Real hypersurfaces in a complex projective space with constant principal curvatures I, II. *J. Math. Soc. Japan* *27* (1975), 43–53, 507–516.

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