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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 2, 231–238

Persistent URL: http://dml.cz/dmlcz/127644

Czechoslovak Mathematical Journal, 51 (126) (2001), 231–238

AN UPPER BOUND ON THE BASIS NUMBER OF THE POWERS OF THE COMPLETE GRAPHS

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(Received June 10, 1997)

Abstract. The basis number of a graph G is defined by Schmeichel to be the least integer h such that G has an h-fold basis for its cycle space. MacLane showed that a graph is planar if and only if its basis number is $\leqslant 2$. Schmeichel proved that the basis number of the complete graph K_n is at most 3. We generalize the result of Schmeichel by showing that the basis number of the d-th power of K_n is at most $2d+1$.

1. Introduction

Throughout this paper, we assume that graphs are finite, undirected, and simple. Our terminology and notations will be as in [8]. Let G be a graph, and let $e_1, e_2, \ldots, e_q$ be an ordering of its edges. Then, any subset S of $E(G)$ corresponds to a $(0,1)$-vector $(a_1, a_2, \ldots, a_q)$ with $a_i = 1$ if $e_i \in S$ and $a_i = 0$ if $e_i \notin S$. These vectors form a q-dimensional vector space over $\mathcal{Z}_2$ denoted by $(\mathcal{Z}_2)^q$. Let $\mathcal{C}(G)$, called the *cycle space* of G, be the subspace of $(\mathcal{Z}_2)^q$ generated by the vectors corresponding to the cycles in G. We shall say, however, that the cycles themselves, rather than the vectors corresponding to the cycles, generate $\mathcal{C}(G)$. It is well known that if G is connected, then the dimension of $\mathcal{C}(G)$ is $q-p+1$, where p and q denote, respectively, the number of vertices and edges in G. In fact, given any spanning tree T in G, every graph $T+e$, $e \notin T$, contains exactly one cycle C_e, and the collection of cycles $\{C_e\colon\ e \notin T\}$ forms a basis of $\mathcal{C}(G)$, called the *fundamental basis corresponding to* T. While each edge outside of T occurs in exactly one cycle of this basis, an edge of T itself may occur in many cycles of the basis. This observation suggests the following definition.

Definition. Let h be a positive integer. A basis of $\mathcal{C}(G)$ is called *h-fold* if each edge of G occurs in at most h of the cycles in the basis. The *basis number* of G (denoted by $b(G)$) is the smallest integer h such that $\mathcal{C}(G)$ has an h-fold basis.

The first important result concerning the basis number was given by MacLane [9]. He proved the following Theorem:

Theorem 1. *A graph G is planar if and only if $b(G) \leqslant 2$.*

Schmeichel [10] proved the following theorem:

Theorem 2. *For every integer $n \geqslant 5$, $b(K_n) = 3$.*

Also in [10] he proved that for $m, n \geqslant 5$, the basis number $b(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ is equal 4 except for $K_{6,10}$, $K_{5,n}$, $K_{6,n}$, with $n = 5, 6, 7, 8$. Moreover, Alsardary and Ali [6] established that $b(K_{5,n}) = b(K_{6,n}) = 3$ for $n = 5, 6, 7, 8$. Banks and Schmeichel [7] proved that for $n \geqslant 7$, $b(Q_n) = 4$, where Q_n is the n-cube. Ali [2], [3] and [4] investigated the basis number of the join of graphs, the complete multipartite graphs, the direct product of paths and cycles. Finally, Ali and Marougi [5] found the basis number of the cartesian product of some graphs.

In this paper we investigate the basis number of the d-th power K_n^d of the complete graph K_n. We show that $b(K_n^d) \leqslant 2d + 1$ which is a generalization of Theorem 2.

2. An upper bound for the basis number of K_n^d

If G and H are graphs, then the *product* of G and H is the graph $G \times H$ with $V(G) \times V(H)$ as the vertex set and (g_1, h_1) adjacent to (g_2, h_2) if either $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or else $g_1 = g_2$ and $h_1 h_2 \in E(H)$. Let K_n^d be the product of d copies of the complete graph K_n, $n \geqslant 2$, $d \geqslant 1$. It will be convenient to think of the vertices of K_n^d, as d-tuples of n-ary digits, i.e. the elements of the set $\{0, 1, \ldots, n-1\}$, with edges between two d-tuples differing at exactly one coordinate.

We will say that two vertices $v = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $v' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_d)$ in K_n^d *match* if and only if $\alpha_i = \alpha'_i$, for $i = 1, 2, \ldots, d-1$ but $\alpha_d \neq \alpha'_d$. Let X_i denote the set of vertices of K_n^d having $\alpha_d = i$, $i = 0, 1, \ldots, n-1$. Then $X_0, X_1, \ldots, X_{n-1}$ induce subgraphs $H_0, H_1, \ldots, H_{n-1}$ of K_n^d, respectively, which are isomorphic to K_n^{d-1}.

It is easy to construct a Hamiltonian path in K_n^d for any $n \geqslant 2, d \geqslant 1$ (see for example Wojciechowski [11]). Let $P_0 = v_1^{(0)}, v_2^{(0)}, \ldots, v_{n^{d-1}}^{(0)}$ be a Hamiltonian path in H_0. Let $v_j^{(i)} \in X_i$ be the vertex that matches $v_j^{(0)}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n^{d-1}$. Then

$$P_i = v_1^{(i)}, v_2^{(i)}, \ldots, v_{n^{d-1}}^{(i)}$$

is a Hamiltonian path in H_i, $i = 1, 2, \ldots, n-1$. Moreover, the edges of K_n^d joining a vertex in H_j to a vertex in H_k are precisely the edges $v_i^{(j)} v_i^{(k)}$, $0 \leqslant j < k \leqslant n-1$, $i = 1, 2, \ldots, n^{d-1}$. Let J_i be the subgraph of K_n^d induced by the set of vertices $Y_i = \{v_i^{(j)} \colon j = 0, 1, \ldots, n-1\}$, $i = 1, 2, \ldots, n^{d-1}$. Clearly, J_i is isomorphic to K_n, for every $i = 1, 2, \ldots, n^{d-1}$.

By Theorem 1 and Theorem 2, $b(K_n) \leqslant 3$. Let D_i be a 3-fold basis of $J_i, i = 1, 2, \ldots, n^{d-1}$. Let $C_i^{(j,k)}$ be the 4-cycle $v_i^{(j)} v_{i+1}^{(j)} v_{i+1}^{(k)} v_i^{(k)}$ for every $i = 1, 2, \ldots, n^{d-1} - 1$, and $0 \leqslant j < k \leqslant n-1$. Let

$$E_i = \{C_i^{(j,k)} \colon 0 \leqslant j < k \leqslant n-1\},$$

$i = 1, 2, \ldots, n^{d-1} - 1$.

Define a collection $T_n^{(d)}$ of cycles in K_n^d by taking:

$$T_n^{(d)} = \bigcup_{i=1}^{n^{d-1}-1} E_i \cup \{D_1\}.$$

We say that

$$\mathcal{B} = \{B_0, B_1, \ldots, B_{n-1}\}$$

is a *foundation* of K_n^d if B_i is a basis of H_i, $i = 0, 1, \ldots, n-1$.

Lemma 3. *If $\mathcal{B}$ is a foundation of K_n^d, then the collection*

$$\bigcup_{B \in \mathcal{B}} B \cup T_n^{(d)}$$

is a basis of $\mathcal{C}(K_n^d)$.

Proof. Let

$$\mathcal{B} = \{B_i \colon i = 0, 1, \ldots, n-1\}$$

be any foundation of K_n^d and let

$$B_n^{(d)} = \bigcup_{B \in \mathcal{B}} B \cup T_n^{(d)}.$$

Since K_n^d is $(n-1)d$-regular, it has $\frac{n^d(n-1)d}{2}$ edges and thus

$$\dim \mathcal{C}(K_n^d) = \frac{n^d(n-1)d}{2} - n^d + 1 = n^d \left(\frac{(n-1)d}{2} - 1 \right) + 1.$$

Thus

$$|B_i| = \dim \mathcal{C}(K_n^{d-1}) = n^{d-1}\left(\frac{(n-1)(d-1)}{2} - 1\right) + 1,$$

$i = 0, 1, \ldots, n-1$. Moreover, we have

$$|E_i| = \frac{n(n-1)}{2},$$

and

$$|D_i| = \dim \mathcal{C}(K_n) = \frac{n(n-1)}{2} - n + 1,$$

$i = 1, 2, \ldots, n^{d-1}$. Therefore, it follows from the definition of $B_n^{(d)}$ that

$$\begin{aligned} |B_n^{(d)}| &= n\left(n^{d-1}\left(\frac{(n-1)(d-1)}{2} - 1\right) + 1\right) \\ &\quad + \left(n^{d-1} - 1\right)\left(\frac{n(n-1)}{2}\right) + \left(\frac{n(n-1)}{2} - n + 1\right) \\ &= n^d\left(\frac{(n-1)d}{2} - 1\right) + 1 \\ &= \dim \mathcal{C}(K_n^d). \end{aligned}$$

Thus to prove that $B_n^{(d)}$ is a basis of $\mathcal{C}(K_n^d)$, it suffices to show that the cycles of $B_n^{(d)}$ are independent.

Indeed, suppose that some collection S of cycles in $B_n^{(d)}$ satisfies a nontrivial relation modulo 2 (that is, $\sum_{C \in S} C = 0 \pmod 2$). Since the graphs $H_0, H_1, \ldots, H_{n-1}$ are mutually vertex disjoint, and B_i is a basis of $H_i, i = 0, 1, \ldots, n-1$, it follows that S must include at least one cycle C

$$C \in B_n^{(d)} \setminus \left(\bigcup_{i=1}^{n-1} B_i\right).$$

Because of symmetry we may assume without loss of generality that $C = C_i^{(0,1)}$ for some $i \in \{1, 2, \ldots, n^{d-1} - 1\}$. We claim that $C_1^{(0,1)} \in S$.

Indeed, if $i = 1$, then we are done. If $i > 1$, then since $C_i^{(0,1)}$ contains the edge $v_i^{(0)}v_i^{(1)}$ and the only other cycle in $B_n^{(d)}$ containing the edge $v_i^{(0)}v_i^{(1)}$ is $C_{i-1}^{(0,1)}$, we conclude that $C_{i-1}^{(0,1)} \in S$. Continuing by induction we get $C_1^{(0,1)} \in S$. But the cycle $C_1^{(0,1)}$ contains the edge $v_1^{(0)}v_1^{(1)}$ which occurs in no other cycle of $B_n^{(d)}$, and in particular in no other cycle of S. This means that $\sum_{C \in S} C$ could not be 0 modulo 2, a contradiction. Thus a nontrivial relation among the cycles of $B_n^{(d)}$ is impossible, and so $B_n^{(d)}$ is an independent collection of cycles and hence a basis of $\mathcal{C}(K_n^d)$, and the proof of this lemma is complete. $\square$

Theorem 4. *For every $n \geqslant 2$ and $d \geqslant 1$, we have $b(K_n^d) \leqslant 2d+1$.*

P r o o f. By Theorem 2, the result is true for $d = 1$. We will proceed by induction on d. Assume that $d \geqslant 2$ and that the theorem is true for smaller values of d. By the inductive hypothesis, since H_i is isomorphic to K_n^{d-1}, we can find a $(2d-1)$-fold basis B_i for $\mathcal{C}(H_i), i = 0, 1, \ldots, n-1$.

Let $C_i^{(j)} = C_i^{(j,j+1)}$, i.e. let $C_i^{(j)}$ be the 4-cycle $v_i^{(j)} v_{i+1}^{(j)} v_{i+1}^{(j+1)} v_i^{(j+1)}$ for every $i = 1, 2, \ldots, n^{d-1}-1$ and $j = 0, 1, \ldots, n-2$.

Set

$$F_i = \{C_i^{(j)} \colon \; j = 0, 1, \ldots, n-2\},$$

$i = 1, 2, \ldots, n^{d-1}-1$. Define the collection B of cycles in K_n^d by taking:

$$B = \bigcup_{i=0}^{n-1} B_i \cup \bigcup_{i=1}^{n^{d-1}} D_i \cup \bigcup_{i=1}^{n^{d-1}-1} F_i,$$

where D_i's are defined as before, $i = 1, 2, \ldots, n^{d-1}$. We have:

$$|B_i| = \dim \mathcal{C}(K_n^{d-1}) = n^{d-1}\left(\frac{(n-1)(d-1)}{2} - 1\right) + 1,$$

$i = 0, 1, \ldots, n-1$,

$$\text{(1)} \qquad |D_i| = \dim \mathcal{C}(K_n) = \frac{n(n-1)}{2} - n + 1,$$

$i = 1, 2, \ldots, n^{d-1}$, and

$$\text{(2)} \qquad |F_i| = n - 1,$$

where $i = 1, 2, \ldots, n^{d-1}-1$. Therefore,

$$\begin{aligned} |B| &= n\left(n^{d-1}\left(\frac{(n-1)(d-1)}{2} - 1\right) + 1\right) \\ &\quad + n^{d-1}\left(\frac{n(n-1)}{2} - n + 1\right) + (n^{d-1}-1)(n-1) \\ &= n^d\left(\frac{(n-1)d}{2} - 1\right) + 1 \\ &= \dim \mathcal{C}(K_n^d). \end{aligned}$$

Thus to prove that B is a basis of $\mathcal{C}(K_n^d)$, it is enough to show that B generates all of $\mathcal{C}(K_n^d)$. Since

$$\mathcal{B} = \{B_0, B_1, \ldots, B_{n-1}\}$$

is a foundation of K_n^d, the collection

$$B_n^{(d)} = \bigcup_{i=0}^{n-1} B_i \cup T_n^{(d)}$$

is a basis of $\mathcal{C}(K_n^d)$ by Lemma 3. Therefore, it is enough to show that B generates $B_n^{(d)}$, and since $\bigcup_{i=0}^{n-1} B_i \subseteq B$, it is enough to prove that B generates $T_n^{(d)}$.

Let $G_n^{(d)}$ be the spanning subgraph of K_n^d such that

$$E(G_n^{(d)}) = \bigcup_{i=0}^{n-1} E(P_i) \cup \bigcup_{i=1}^{n^{d-1}} E(J_i).$$

Clearly $G_n^{(d)}$ is isomorphic to $P \times K_n$, where P is a path of length n^{d-1}. Define a collection B' of cycles in $G_n^{(d)}$ as follows:

$$B' = \bigcup_{i=1}^{n^{d-1}} D_i \cup \bigcup_{i=1}^{n^{d-1}-1} F_i.$$

We claim that B' is a basis of $G_n^{(d)}$.

Since J_i has $\frac{n(n-1)}{2}$ edges and P_j has n^{d-1} edges, $i = 1, 2, \ldots, n^{d-1}$, and $j = 0, 1, \ldots, n-1$. We get

$$\begin{aligned}\dim \mathcal{C}(G_n^{(d)}) &= \left(\frac{n(n-1)n^{d-1}}{2} + n(n^{d-1}-1)\right) - n^d + 1 \\ &= \left(\frac{n-1}{2}\right) n^d - n + 1.\end{aligned}$$

Therefore, by (1) and (2) we get

$$\begin{aligned}|B'| &= n^{d-1}\left(\frac{n(n-1)}{2} - n + 1\right) + (n^{d-1}-1)(n-1) \\ &= \left(\frac{n-1}{2}\right) n^d - n + 1 \\ &= \dim \mathcal{C}(G_n^{(d)}).\end{aligned}$$

Thus to show that B' is a basis of $\mathcal{C}(G_n^{(d)})$ it suffices to show that the cycles of B' are independent. Suppose that some collection R of cycles in B' satisfies a nontrivial relation modulo 2 (that is, $\sum_{C \in R} C = 0 \pmod 2$). Since the graphs $J_1, J_2, \ldots, J_{n^{d-1}}$

are mutually vertex disjoint and D_i is a basis of $J_i, i = 1, 2, \ldots, n^{d-1}$, it follows that R must include at least one cycle C in $\bigcup_{i=1}^{n^{d-1}-1} F_i$. Let

$$C = (v_i^{(j)} v_{i+1}^{(j)} v_{i+1}^{(j+1)} v_i^{(j+1)}).$$

Suppose that $j > 0$. Since the cycle $C' = (v_i^{(j-1)} v_{i+1}^{(j-1)} v_{i+1}^{(j)} v_i^{(j)})$ is the only other cycle of B' containing the edge $v_i^{(j)} v_{i+1}^{(j)}$, we conclude that $C' \in R$. Continuing by induction, we see that R must contain the cycle $(v_i^{(0)} v_{i+1}^{(0)} v_{i+1}^{(1)} v_i^{(1)})$ which is the only cycle of B' containing the edge $v_i^{(0)} v_{i+1}^{(0)}$ and in particular is the only cycle of R containing the edge $v_i^{(0)} v_{i+1}^{(0)}$. This means that $\sum_{C \in R} C$ could not be 0 modulo 2, which is a contradiction. Thus a nontrivial relation among the cycles of B' is impossible, and so B' is an independent collection of cycles and hence a basis of $\mathcal{C}(G_n^{(d)})$.

Since $B' \subseteq B$, and each cycle in $T_n^{(d)}$ is a cycle in the graph $G_n^{(d)}$, it follows that B generates $T_n^{(d)}$ and hence is a basis of K_n^d.

To complete the proof, it remains to show that B is $(2d+1)$-fold.

Assume first that

$$e \in \bigcup_{j=0}^{n-1} E(H_j).$$

Then by the induction hypothesis, e occurs in at most $2d-1$ cycles of $\bigcup_{i=0}^{n-1} B_i$, in at most 2 cycles of $\bigcup_{i=1}^{n^{d-1}-1} F_i$ and in no cycles of $\bigcup_{i=1}^{n^{d-1}} D_i$. Thus e occurs in at most $2d+1$ cycles of B.

Now assume that

$$e \in \bigcup_{j=1}^{n^{d-1}} E(J_j).$$

Then e occurs in at most 3 cycles of $\bigcup_{i=1}^{n^{d-1}} D_i$, in at most 2 cycles of $\bigcup_{i=1}^{n^{d-1}-1} F_i$, and in no cycles of $\bigcup_{i=0}^{n-1} B_i$. Thus e occurs in at most 5 cycles of B. Since $d \geqslant 2$, e occurs in at most $2d+1$ cycles of B and the proof is complete. $\square$

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