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ON THE TOUGHNESS OF CYCLE PERMUTATION GRAPHS

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Abstract. Motivated by the conjectures in [11], we introduce the maximal chains of a cycle permutation graph, and we use the properties of maximal chains to establish the upper bounds for the toughness of cycle permutation graphs. Our results confirm two conjectures in [11].

Keywords: cycle permutation graph, toughness, maximal chain

MSC 2000: 05C58

1. Introduction

Chartrand and Wilson, in [5], introduced a series of properties for the Petersen graph. They used a conjecture of Tutte to explain why so much attention had been paid to the Petersen graph and its various generalizations. (That is, every known bridgeless 3-regular graph whose edges cannot be colored with three colors contains a subgraph isomorphic to the Petersen graph. Tutte conjectured that this is always the case (see [12]).

Two classes of generalization of Petersen graphs are generalized Petersen graphs and cycle permutation graphs. Let n and k be integers with $n \geqslant 5$ and $k \geqslant 1$. A generalized Petersen graph, $G(n,k)$, is the graph with vertex set $V(G(n,k)) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$, and edge set $E(G(n,k)) = \{[x_i, x_{i+1}], [y_i, y_{i+k}], [x_i, y_i]$, $i = 1, 2, \ldots, n$ where the subscripts are taken modulo $n\}$. The subgraph induced from $x_1, x_2, \ldots, x_n$ is denoted by C_x and the subgraph induced from $y_1, y_2, \ldots, y_n$ is denoted by C_y. When $n = 5$ and $k = 2$, $G(5,2)$ is the Petersen graph.

Let α be a permutation of the symmetric group, S_n, acting on the set $\{1, 2, \ldots, n\}$. A cycle permutation graph $P_n(\alpha)$ is the graph with $2n$ vertices, $V(P_n(\alpha)) = V_1 \cup V_2$ where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ for $i = 1, 2$, $V_1 \cap V_2 = \emptyset$, and $E(P_n(\alpha)) = E_1 \cup E_2 \cup E_{12}$

where $E_i = \{[v_{ij}, v_{i(j+1)}]$ for $j = 1, 2, \ldots, n\}$ for $i = 1, 2$ and $E_{1,2} = \{[v_{1t}v_{2\alpha(t)}];$ $t = 1, 2, \ldots, n\}$. (See [6], [7].) When $n = 5$ and $\alpha = (1)(2453)$, $P_5(\alpha)$ is the Petersen graph.

The toughness $t(G)$ of a graph G was defined by Chvátal [6]. If G is not a complete graph,

$$t(G) = \min_S \left\{ \frac{|S|}{\omega(G-S)} \right\}$$

where S is taken over all disconnecting subsets of the vertex set of G, $|S|$ is the cardinality of S and $\omega(G-S)$ is the number of components in the subgraph induced from $G-S$.

Recently, the toughness of graphs has received a lot of attention. Much work has been done concerning the toughness, which is considered to be more sensitive to the structure of the graph than the connectivity of the graph (see [1] and [2]).

In [6], Chvátal first considered the toughness of the cross product of two complete graphs. Guichard, Piazza, and Stueckle, in [10], proved that for $\alpha \in S_{m+n}$ and $m \leqslant n$, the toughness of a cycle permutation graph is given by

$$t(P_\alpha(K_{m,n})) = \begin{cases} \dfrac{2m}{n+m-q} & \text{if } q < \dfrac{n^2+m^2}{n+3m}, \\[2ex] \dfrac{n+m}{n+q} & \text{if } q \geqslant \dfrac{n^2+m^2}{n+3m} \end{cases}$$

where $K_{m,n}$ is the complete bipartite graph of mn vertices.

Some results and conjectures were given by Piazza, Ringeisen, and Stueckle in [11]; the authors proved that the toughness of $G(n,k)$ is more than $n/(n-1)$, if n is an positive odd integer with n and k being relatively prime, and $k \notin \{1, n-1\}$. An upper bound for the toughness of cycle permutation graphs was obtained as follows:

$$t(P_n(\alpha)) \leqslant \frac{(k+2)}{(k+1)}, \quad \text{if } \alpha(i) = i \text{ for all } 1 \leqslant i \leqslant k \leqslant n-2.$$

Based on the set of permutations which generate all nonisomorphic cycle permutation graphs of C_n, $n \leqslant 8$ in [11], the authors found that for all $\sigma \in S_n$, the toughness of $P_3(\sigma)$ is equal to $3/2$ and the toughness of $P_n(\sigma)$ is less than or equal to $4/3$ for $4 \leqslant n \leqslant 8$. Three conjectures were stated as follows.

Conjecture 1. *For $n \geqslant 4$ and $\alpha \in S_n$, $t(P_n(\alpha)) \leqslant 4/3$.*

If this upper bound cannot be obtained, perhaps, the following looser upper bound can be obtained.

Conjecture 2. *For $n \geqslant 4$ and $\alpha \in S_n$, $t(P_n(\alpha)) < 3/2$.*

Since $G(5,2)$ and $G(9,2)$ have their toughness equal to $4/3$, could such a class be the generalized Petersen graphs when $n \equiv 1 \pmod 4$ and $k = 2$?

Conjecture 3. *If* $n \geqslant 5$ *and* $n \equiv 1 \pmod 4$, *then* $t(G(n,2)) = 4/3$.

In [3] and [8], the authors proved that the upper bound for toughness of generalized Petersen graph is $4/3$. Here, we shall study the structure of cycle permutation graphs with some maximal chains and establish the upper bound for the toughness of cycle permutation graphs. Some of these results confirm Conjectures 1 and 2. Throughout this paper, all integers and subscripts are taken modulo n.

2. Maximal chains

Let $P_n(\alpha)$ be a cycle permutation graph consisting of two n-cycles C_n and C'_n with a connecting set of edges, i.e., $V(P_n(\alpha)) = V(C_n) \cup V(C'_N)$ where $V(C_n) = \{1, 2, \ldots, n\}$, $V(C'_n) = \{y_1, y_2, \ldots, y_n\}$ such that $V(C_n) \cap V(C'_n) = \emptyset$, and $E(P_n(\alpha)) = E(C_n) \cup E(C'_n) \cup E_{1,2}$ where $E(C_n) = \{[i, i+1] \text{ for } i = 1, 2, \ldots, n\}$, $E(C'_n) = \{[y_i, y_{i+1}] \text{ for } i = 1, 2, \ldots, n\}$ and $E_{12} = \{[i, y_{\alpha(i)}] \text{ for } i = 1, 2, \ldots, n\}$.

Let B be a nonempty (proper) subset of $V(C_n)$. On C'_n, a chain of edges $[y_i, y_{i+1}][y_{i+1} y_{i+2}] \ldots [y_{t-1} y_t]$ is said to be related to B, if $\alpha^{-1}(i), \alpha^{-1}(i+1), \ldots, \alpha^{-1}(t)$ belong to B. For simplicity, this chain of edges on C'_n will be written as $y_i y_{i+1} \ldots y_t$. A chain of edges, $y_i y_{i+1} \ldots y_t$ is said to be maximal, if $\alpha^{-1}(i-1) \notin B$ and $\alpha^{-1}(t+1) \notin B$. A maximal chain related to B will be denoted by $M(B)$. Similarly, replacing B by $C_n - B$, we may define chains related to $C_n - B$, and maximal chains related to $C_n - B$. Also, a maximal chain related to $C_n - B$ will be denoted by $M(C_n - B)$. Two chains on $C'_n, y_i y_{i+1} \ldots y_t$ and $y_j y_{j+1} \ldots y_s$ are said to be related, denoted by $y_i y_{i+1} \ldots y_t \backsim y_j y_{j+1} \ldots y_s$ or $y_j y_{j+1} \ldots y_s \backsim y_i y_{i+1} \ldots y_t$, if $[y_t, y_j] \in E(C'_n)$ or $[y_s, y_i] \in E(C'_n)$.

For a nonempty independent subset B of $V(C_n)$, C'_n is partitioned into disjoint maximal chains $M_1(B), M_2(B), \ldots, M_k(B)$ related to B, and disjoint maximal chains $M_1(C_n - B), M_2(C_n - B), \ldots, M_k(C_n - B)$ related to $C_n - B$. This partition of maximal chains will be denoted by $p(n, \alpha, B)$. We note that since C'_n is a cycle, the number of maximal chains related to B is equal to the number of maximal chains related to $(C_n - B)$.

Example 1. Let $n = 12$, and

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 2 & 4 & 5 & 7 & 9 & 6 & 12 & 1 & 8 & 10 & 11 \end{pmatrix}.$$

In $P_{12}(\alpha)$, let $B = \{1, 3, 5, 7, 10\}$. Then we have $\alpha^{-1}(3) = 1$, $\alpha^{-1}(4) = 3$, $\alpha^{-1}(7) = 5$, $\alpha^{-1}(6) = 7$, $\alpha^{-1}(8) = 10$, $M_1(B) = y_3 y_4$, $M_2(B) = y_6 y_7 y_8$, $M_1(C_{12} -$

$B) = y_5$, $M_2(C_{12} - B) = y_9 y_{10} y_{11} y_{12} y_1 y_2$, and $p(12, \alpha, B)$ is $M_1(B) \backsim (C_{12} - B) \backsim M_2(B) \backsim M_2(C_{12} - B) \backsim M_1(B)$. Also, $y_6 y_7$ is a chain related to B, but not a maximal chain related to B. Similarly, $y_{12} y_1 y_2$ is a chain related to $(C_{12} - B)$, but not a maximal chain related to $(C_{12} - B)$.

Each of $M_1(B)$ and $M_2(C_{12} - B)$ is said to have an even cardinality (i.e., each contains an even number of vertices), and each of $M_2(B)$ and $M_1(C_{12} - B)$ has an odd cardinality.

Theorem 1. *Let n be an integer $\geqslant 4$, $\alpha \in S_n$, $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices, B be a nonempty independent subset of vertices of C_n in $P_n(\alpha)$, and $p(n, \alpha, B)$ be the partition of maximal chains related to B. Then*

$$t(P_n(\alpha)) \leqslant \frac{2|B| + n + e_1(B) - e_2(B)}{3|B| + e_1(B)} \tag{1}$$

where $|B|$ is the cardinality of B, $e_1(B)$ is the number of maximal chains related to B with odd cardinality, and $e_2(B)$ is the number of maximal chains related to $C_n - B$ with odd cardinality.

Proof. For some positive integer q, $p(n, \alpha, B)$ is $M_1(B) \backsim M_1(C_n - B) \backsim M_2(B) \backsim M_2(C_n - B) \backsim \ldots \backsim M_q(B) \backsim M_q(C_n - B) \backsim M_1(B)$. □

We construct a disconnecting subset S of $P_n(\alpha)$ as follows:

$$S = B \cup B_y \quad \text{with } B_y = \bigcup_{j=1}^{q} [K_j(B) \cup K_j(C_n - B)] \tag{2}$$

where for $M_j(B) = y_{t_j+1} y_{t_j+2} \ldots y_{t_j+m_j}$, $1 \leqslant j \leqslant q$,

$$K_j(B) = \begin{cases} \varphi & \text{if } m_j = 1 \text{ or } 2, \\ \{y_{t_j+2}, y_{t_j+4}, \ldots, y_{t_j+m_j-1}\} & \text{if } m_j \text{ is odd } > 1, \\ \{y_{t_j+2}, y_{t_j+4}, \ldots, y_{t_j+m_j-2}\} & \text{if } m_j \text{ is even } > 2. \end{cases}$$

For $M_j(C_n - B) = y_{s_j+1} y_{s_j+2} \ldots y_{s_j+m'_j}$, $1 \leqslant j \leqslant q$,

$$K_j(C_n - B) = \begin{cases} \{y_{s_j+1}, y_{s_j+3}, \ldots, y_{s_j+m'_j-2}, y_{s_j+m'_j}\} & \text{if } m'_j \text{ is odd }, \\ \{y_{s_j+1}, y_{s_j+3}, \ldots, y_{s_j+m'_j-1}, y_{s_j+m'_j}\} & \text{if } m'_j \text{ is even.} \end{cases}$$

Thus, we have, for $j = 1, 2, \ldots, q$,

$$|M_j(B)| = m_j, \quad |K_j(B)| = \left[\frac{m_j - 1}{2}\right],$$

and

$$|M_j(C_n - B)| = m'_j, \quad |K_j(C_n - B)| = \left[\frac{m'_j + 2}{2}\right]$$

where $[x]$ is the largest integer $\leqslant x$.

For $1 \leqslant j \leqslant q$, the components of the induced graph $M_j(B) - B_y$ are:

$$\{y_{t_j+1}\}, \{y_{t_j+3}\}, \ldots, \{y_{t_j+2k+1}\}, \ldots, \{y_{t_j+m_j}\} \quad \text{if } m_j \text{ is odd },$$

and

$$\{y_{t_j+1}\}, \{y_{t_j+3}\}, \ldots, \{y_{t_j+2k+1}\}, \ldots, \{y_{t_j+m_j-1}, y_{t_j+m_j}\} \quad \text{if } m_j \text{ is even.}$$

Since each vertex $v \in V(M_j(C_n - B) - B_y)$ is incident with an edge $[v, i]$ in $E(P_n(\alpha))$ for some $i \in C_n - S$, the number of components, $\omega(M_j(B) - B_y)$, of the induced subgraph $M_j(B) - B_y$ is equal to $\left[\frac{m_j+1}{2}\right]$ for $j = 1, 2, \ldots, q$, and

(2)
$$\begin{aligned}\omega\big(P_n(\alpha) - 5\big) &= \omega(C_n - B) + \sum_{j=1}^{q} \omega\big(M_j(B) - B_y\big)\\ &= |B| + \sum_{j=1}^{q} \left[\frac{m_j + 1}{2}\right]\\ &= |B| + \frac{|B|}{2} + \frac{e_1(B)}{2}\end{aligned}$$

where $|B| = \sum_{j=1}^{q} m_j$ is used.

By using

$$\left[\frac{m_j - 1}{2}\right] = \begin{cases} \dfrac{m_j - 1}{2} & \text{if } m_j \text{ is odd },\\[2ex] \dfrac{m_j + 2}{2} & \text{if } m_j \text{ is even },\end{cases}$$

$$\left[\frac{m'_j + 2}{2}\right] = \begin{cases} \dfrac{m'_j + 1}{2} & \text{if } m'_j \text{ is odd},\\[2ex] \dfrac{m_j + 2}{2} & \text{if } m'_j \text{ is even},\end{cases}$$

and $n = \sum_{j=1}^{q} m_j + \sum_{j=1}^{q} m'_j$, we have

$$
\begin{aligned}
(3) \qquad |S| &= |B| + |B_y| \\
&= |B| + \sum_{j=1}^{q} \left[\frac{m_j - 1}{2}\right] + \sum_{j=1}^{q} \left[\frac{m'_j + 2}{2}\right] \\
&= |B| + \frac{1}{2}\Bigg(\Bigg(\sum_{j=1}^{q} m_j\Bigg) - e_1(B) - 2\big(q - e_1(B)\big)\Bigg) \\
&\quad + \frac{1}{2}\Bigg(\Bigg(\sum_{j=1}^{q} m'_j\Bigg) + e_2(B) + 2\big(q - e_2(B)\big)\Bigg) \\
&= |B| + \frac{n}{2} + \frac{e_1(B) - e_2(B)}{2}.
\end{aligned}
$$

By using (2) and (3), we have

$$t(P_n(\alpha)) \leqslant \frac{|S|}{\omega(P_n(\alpha) - S)} = \frac{2|B| + n + e_1(B) - e_2(B)}{3|B| + e_1(B)}.$$

Example 2. Let $n, \alpha, P_n(\alpha)$ and B be the same as in our Example 1. We have

$$M_1(B) = y_3y_4, \quad M_2(B) = y_6y_7y_8, \quad M_1(C_{12} - B) = y_5,$$
$$M_2(C_{12} - B) = y_9y_{10}y_{11}y_{12}y_1y_2.$$

Thus,

$$K_1(B) = \emptyset,\ K_2(B) = \{y_7\},\ K_1(C_{12} - B) = \{y_5\},\ K_2(C_{12} - B) = \{y_9, y_{11}, y_1, y_2\}.$$

$S = B \cup B_y = \{1, 3, 5, 7, 10\} \cup \{y_7\} \cup \{y_5\} \cup \{y_9, y_{11}, y_1, y_2\}$, and the components in $P_{12}(\alpha) - S$ are: $\langle 2\rangle$, $\langle 4\rangle$, $\langle 6\rangle$, $\langle 8, 9, y_{12}\rangle$, $\langle 11, 12, y_{10}\rangle$, $\langle y_3, y_4\rangle$, $\langle y_6\rangle$ and $\langle y_8\rangle$. (We note that $[y_{12}, 8]$ and $[y_{10}, 11]$ do belong to $E(P_{12}(\alpha))$. Thus, $|S| = 11$, $\omega(P_{12}(\alpha) - S) = 8$, and $t(P_{12}(\alpha)) \leqslant \frac{|S|}{\omega(P_{12}(\alpha) - S)} = \frac{11}{8}$.

Using our (1), with $|B| = 5$, $n = 12$, $e_1(B) = 1$ and $e_2(B) = 1$, we have

$$t(P_{12}(\alpha)) \leqslant \frac{2(5) + 12 + 1 - 1}{3(5) + 1} = \frac{22}{16} = \frac{11}{8}.$$

Corollary 1.1. *Let n be an integer $\geqslant 4$ and $n \neq 4k + 3$, $\alpha \in S_n$ and $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices. Then*

$$t(P_n(\alpha)) \leqslant \frac{4}{3}.$$

P r o o f. Let

$$B = \begin{cases} \{1,3,\ldots,n-1\} & \text{if } n \text{ is even,} \\ \{1,3,\ldots,n-2\} & \text{if } n \text{ is odd.} \end{cases}$$

Then $|B| = [n/2]$.

For $n = 4k$, $|B| = 2k$, and by using (1) in Theorem 1, we have

$$t(P_k(\alpha)) \leqslant \frac{2(2k) + 4k + e_1(B) - e_2(B)}{3(2k) + e_1(B)} \leqslant \frac{8k + e_1(B)}{6k + e_1(B)} \leqslant \frac{4}{3}.$$

For $n = 4k+1$ and $|B| = 2k$. We claim that $e_1(B) \geqslant 1$. Let $p(4k+1, \alpha, B) = \left(\bigcup_{j=1}^{q} M_j(B)\right) \cup \left(\bigcup_{j=1}^{q} M_j(C_{4k+1} - B)\right)$ for some positive integer q. Since $\sum_{j=1}^{q} |M_j(B)| = |B| = 2k$ and $\sum_{j=1}^{q} |M_j(B)| + \sum_{j=1}^{q} M_j(C_{4k+1} - B)| = 4k+1$, $\sum_{j=1}^{q} |M_j(C_{4k+1} - B)| = 2k+1$. Consequently, $e_2(B) \neq 0$, i.e., $e_2(B) \geqslant 1$.

By using (1) in Theorem 1 with $e_2(B) \geqslant 1$, we have

$$t(P_{4k+1}(\alpha)) \leqslant \frac{2(2k) + 4k + 1 + e_1(B) - e_2(B)}{3(2k) + e_1(B)} = \frac{8k + 1 + e_1(B) - e_2(B)}{6k + e_1(B)} \leqslant \frac{4}{3}$$

where $3 \leqslant e_1(B) + 3e_2(B)$ is used.

For $n = 4k+2$, $|B| = 2k+1$. By using (1) in Theorem 1, we have

$$t(P_{4k+2}(\alpha)) \leqslant \frac{2(2k+1) + 4k + 2 + e_1(B) - e_2(B)}{3(2k+1) + e_1(B)} \leqslant \frac{8k + 4 + e_1(B)}{6k + 3 + e_1(B)} \leqslant \frac{4}{3}.$$

□

Corollary 1.2. *Let n be an integer $\geqslant 4$, $\alpha \in S_n$ and $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices. Then*

$$t(P_n(\alpha)) < \frac{3}{2}.$$

P r o o f. For $n = 4k, 4k+1$ and $4k+2$, by Corollary 1.1, we have $t(P_n(\alpha)) \leqslant \frac{4}{3} < \frac{3}{2}$. For $n = 4k+3$, let $B = \{1, 3, \ldots, 4k+1\}$. Then $|B| = 2k+1$. By using (1) in Theorem 1, we have

$$\begin{aligned} t(P_{4k+3}(\alpha)) &\leqslant \frac{2(2k+1) + (4k+3) + e_1(B) - e_2(B)}{3(2k+1) + e_1(B)} \\ &\leqslant \frac{8k + 5 + e_1(B)}{6k + 3 + e_1(B)} < \frac{3}{2} \end{aligned}$$

where $k \geqslant 1$ is used. □

Our Corollary 1.2 confirms the conjecture 2 in [11].

We shall show that, for $n = 4k + 3$ with $k \geqslant 1$, a certain $P_n(\alpha)$ is isomorphic to a generalized Petersen graph. We shall also show that the toughness of this generalized Petersen graph is $\leqslant \frac{4}{3}$, and we use the results to prove $t(P_n(\alpha)) \leqslant \frac{4}{3}$.

For $n = 4k + 3$ and $|B| = 2k + 1$, by using (1) in Theorem 1, we have

$$\begin{aligned} t(P_{4k+3}(\alpha)) &\leqslant \frac{2(2k+1) + (4k+3) + e_1(B) - e_2(B)}{3(2k+1) + e_1(B)} \\ &= \frac{8k + 5 + e_1(B) - e_2(B)}{6k + 3 + e_1(B)}. \end{aligned}$$

In order to have $t(P_{4k+3}(\alpha)) \leqslant \frac{4}{3}$, we need

$$3 \leqslant e_1(B) + 3e_2(B). \tag{4}$$

If $e_2(B_i) \geqslant 1$, then (4) holds. If $e_2(B) = 0$ and $E_1(B_i) \geqslant 3$, then (4) holds. Since $n = 4k + 3$ and $|B| = 2k + 1$ are odd integers, we cannot have the case of $e_2(B) = 0$ and $e_1(B_i) = 2$. The only case which we need to consider is $e_2(B) = 0$ and $e_1(B_i) = 1$.

Let $n = 4k + 3$ with $k \geqslant 1$, and $B_i = \{1 + i, 3 + i, \ldots, (4k + 1) + i\}$ for $i = 0, 1, \ldots, 4k + 2$. (The integers are taken modulo n.) Then each B_i is an independent set of vertices in C_n. Each of $p(n, \alpha, B_i)$, $e_1(B_i)$ and $e_2(B_i)$ are defined in the same way as $p(n, \alpha, B_0) = p(n, \alpha, B)$, $e_1(B_0) = e_1(B)$ and $e_2(B_0) = e_2(B)$ respectively.

Example 3. We give two cycle permutation graphs. One has $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for some i. The other one has $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i.

Let $P_{11}(\beta)$ be the cycle permutation graph with

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 7 & 5 & 9 & 3 & 8 & 4 & 2 & 11 & 1 & 6 \end{pmatrix}$$

and $B = B_0 = \{1, 3, 5, 7, 9\}$. Then $p(11, \beta, B_0)$ is $M_1(B_0) = y_3y_4y_5 \backsim M_1(C_{11} - B_0) = y_6y_7y_8y_9 \backsim M_2(B_0) = y_{10}y_{11} \backsim M_2(C_{11} - B_0) = y_1y_2 \backsim M_1(B_0)$. Thus, $e_1(B_0) = 1$ and $e_2(B_0) = 0$. For $B_3 = \{4, 6, 8, 10, 1\}$, $p(11, \beta, B_3)$ is $M_1(B_3) = y_1y_2 \backsim M_1(C_{11} - B_3) = y_3y_4y_5y_6y_7 \backsim M_2(B_3) = y_8y_9y_{10} \backsim M_2(C_{11} - B_3) = y_{11} \backsim M_1(B_3)$. Thus, $e_1(B_3) = 1$ and $e_2(B_3) = 2$.

Let $P_{11}(\alpha)$ be the cycle permutation graph with

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

with $B = B_0 = \{1,3,5,7,9\}$. Then $p(11,\alpha,B_0)$ is

$$M_1(B_0) = y_1y_2 \backsim M_1(C_{11} - B_0) = y_3y_4 \backsim M_2(B_0) = y_5y_6$$
$$\backsim M_2(C_{11} - B_0) = y_7y_8 \backsim M_3(B_0) = y_9 \backsim M_3(C_{11} - B_0) = y_{10}y_{11} \backsim M_1(B_0).$$

Thus, $e_1(B_0) = 1$ and $e_2(B_0) = 0$.

For $B_i = \{1+i, 3+i, 5+i, 7+i, 9+i\}$, $i = 1,2,\ldots,10$, $p(11,\alpha,B_i)$ is

$$M_1(B_i) = y_{1+2i}\,y_{2+2i} \backsim M_1(C_{11} - B_i) = y_{3+2i}y_{4+2i} \backsim M_2(B_i)$$
$$= y_{5+2i}\,y_{6+2i} \backsim M_2(C_{11} - B_i) = y_{7+2i}\,y_{8+2i} \backsim M_3(B_i) = y_{9+2i} \backsim M_3(C_{11} - B_i)$$
$$= y_{10+2i}\,y_{11+2i} \backsim M_1(B_i).$$

Thus, $e_1(B_i) = 1$ and $e_2(B_i) = 0$, i.e., $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i.

Theorem 2. *Let $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices where $n = 4k+3$ and $k \geqslant 1$, and B_i be the same as above with $e_1(B_i) \geqslant 3$ or $e_2(B_i) \geqslant 1$ for some integer i. Then*

$$t(P_n(\alpha)) \leqslant \frac{4}{3}.$$

Proof. Replacing B in Theorem 1 by B_i, we have the inequality (1). By using the inequality (4), we have $t(P_n(\alpha)) \leqslant \frac{4}{3}$. □

Theorem 3. *Let $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices with $n = 4K+3$ and $k \geqslant 1$, and B_i be the same as above for $i = 0,1,\ldots,4k+2$. If $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all i, then $P_n(\alpha)$ is isomorphic to $G(n,2r)$ for some positive integer r such that r divides $2(k+1)$.*

In order to prove our Theorem 3, we need the following lemmas.

Lemma 3.1. *Let $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices where $n = 4k+3$ and $k \geqslant 1$, B_i be the same as above with $e_1(B_i) = 1$, and $e_2(B_i) = 0$ for all integers i, and for some integer t and some integer l $M_1(B_t) = y_{l+1}y_{l+2}\ldots y_{l+m}$ where m is a positive odd integer. Then*

$$\alpha\{t-1,t\} = \{l, l+m+1\}$$

where

$$\alpha\{t-1,t\} = \{\alpha(t-1), \alpha(t)\}.$$

P r o o f. We claim that if $M_1(B_0) = y_{l+1}y_{l+2}\dots y_{l+m}$, then $\alpha\{n-1,n\} = \{l, l+m+1\}$. Suppose the contrary, i.e., $\alpha\{n-1,n\} \neq \{l, l+m+1\}$. Then there are two cases to be considered:

Case 1. $\alpha^{-1}(l+m+1) \notin \{n-1,n\}$.

(i) $\alpha^{-1}(l) \notin \{n-1,n\}$.

Let $B_0 = \{1,3,\dots,4k+1\}$, $M_1(B_0) = y_{l+1}y_{l+2}\dots y_{l+m}$ with m being a positive odd integer, and $B_1 = \{2,4,\dots,4k+2\}$. Then $C_n - B_1$ consists of all odd integers in $\{1,2,\dots,n\}$ and $\{\alpha^{-1}(i);\ y_i \in M_1(B_0)\} \subseteq C_n - B_1$, i.e., $M_1(B_0)$ is a chain related to $C_n - B_1$. Since $\alpha^{-1}(l)$ and $\alpha^{-1}(l+m+1) \notin \{n-1,n\}$, neither the vertex y_l nor the vertex y_{l+m+1} is in the chain $M_1(B_0)$. Thus, $M_1(B_0)$ is a maximal chain related to $C_n - B_1$. Since $|M_1(B_0)| = m$ is an odd integer, $e_2(B_1) \geqslant 1$ which is a contradiction to $e_2(B_i) = 0$ for all integers i. Hence, $\alpha^{-1}(l) \in \{n-1,n\}$.

(ii) $\alpha^{-1}(l) = n-1$.

Similar to the proof of (i) in the Case 1. Since $n-1 \notin C_n - B_1$, $M_1(B_0)$ is a maximal chain related to $C_n - B_1$, and $e_2(B_1) \geqslant 1$ which is a contradiction.

(iii) $\alpha^{-1}(l) = n$.

Similar to the proof of (i) in the Case 1. Replacing B_1 by $B_{-1} = B_{4k+2}$, $M_1(B_0)$ is a maximal chain related to $C_n - B_{-1}$, and $e_2(B_{-1}) \geqslant 1$ which is a contradiction. Consequently, we have $\alpha^{-1}\{l, l+m+1\} \in \{n-1,n\}$.

Case 2. $\alpha^{-1}(l) \notin \{n-1,n\}$ and $\alpha^{-1}(l+m+1) \in \{n-1,n\}$.

(i) $\alpha^{-1}(l+m+1) = n-1$.

Similar to the proof of (ii) in the Case 1, $e_2(B_1) \geqslant 1$ which is a contradiction.

(ii) $\alpha^{-1}(l+m+1) = n$.

Similar to the proof of (iii) in the Case 1, $e_2(B_{-1}) \geqslant 1$ which is a contradiction. Hence, we have $\alpha^{-1}\{l, l+m+1\} = \{n-1,n\}$, i.e., $\alpha\{n-1,n\} = \{l, l+m+1\}$.

We claim that if, for some nonzero t,

$$M_1(B_t) = y_{l+1}y_{l+2}\dots y_{l+m},$$

then $\alpha\{t-1,t\} = \{l, l+m+1\}$. Relabeling k by $t+k$ for all $k \in V(C_n)$, we obtain a new cycle permutation graph $P_n(\beta)$ for a permutation $\beta \in S_n$ such that $\beta(i) = \alpha(t+i)$ for $i = 1,2,\dots,n$.

Obviously, $P_n(\beta) \cong P_n(\alpha)$ and, in $P_n(\beta)$, $e_1(B_i) = 1$ and $e_2(\beta_i) = 0$ for all integers i. Since $M_1(B_0) = y_{l+1}y_{l+2}\dots y_{l+m}$ in $P_n(\beta)$, $\beta\{n-1,n\} = \{l, l+m+1\}$. Thus,

$$\alpha\{(n-1)+t, n+t\} = \alpha\{t-1,t\} = \{l, l+m+1\},$$

i.e.,

$$\alpha\{t-1,t\} = \{l, l+m+1\}.$$

$\square$

Lemma 3.2. *Let $n = 4k+3$ with $k \geqslant 1$, $P_n(\alpha)$ be a cycle permutation graph with $B_0 = \{1, 3, \ldots, 4k+1\}$ and $p(n, \alpha, B_0)$ given by*

$$M_1 \backsim M_1' \backsim M_2 \backsim M_2' \backsim \ldots \backsim M_r \backsim M_r' \backsim M_1$$

where

$$M_j = M_j(B_0) = v_{j,1} v_{j,2} \ldots v_{j,m_j} \tag{5}$$

and

$$M_j' = M_j(C_n - B_0) = w_{j,1} w_{j,2} \ldots w_{j,m_j'} \tag{6}$$

for $j = 1, 2, \ldots, r$. If $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i, and $|M_i| = m$ is a positive odd integer, then

$$n \backsim w_{1,1} \quad \text{and} \quad (n-1) \backsim w_r, m_r' \tag{7}$$

or

$$n \backsim w_{r,m_r'} \quad \text{and} \quad (n-1) \backsim w_{1,1} \tag{8}$$

where $w_{1,1}$ and $w_{r,m_r'}$ are the initial vertex of M_1' and the terminal vertex of M_r', respectively, and $q \backsim w$ means $[q, w] \in E(P_n(\alpha))$ where $q \in C_n$ and $w \in C_n'$.

P r o o f. This follows from Lemma 3.1. $\square$

We shall repeatedly use (5), (6), (7) and (8) in the following

Lemma 3.3. *Let $n = 4k+3$ and $k \geqslant 1$, $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices, and M_j and M_j' for $j = 1, 2, \ldots, r$ be the same as in (5) and (6). If $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i, $|M_1| = m_1$ is an odd positive integer, and $n \backsim w_{1,1}$ and $(n-1) \backsim w_{r,m_r'}$, then we have*

(1) $(2dr-1) \backsim v_{1,d}$ and $2dr \backsim w_{1,d+1}$ *for* $1 \leqslant d \leqslant \min\limits_{1 \leqslant j \leqslant r} \{|M_j|, |M_j'|\}$, *and*

(2) $(2sr+2t-1) \backsim v_{t+1,s+1}$, *and* $(2sr+2t) \backsim w_{t+1,s+1}$ *for* $0 \leqslant s \leqslant m$ *and* $1 \leqslant t \leqslant r-1$.

Since the proof of Lemma 3.3. is very lengthy, we shall first consider the following examples which demonstrate Lemma 3.3.

Example 4. Let $n = 4(3) + 3 = 15$, $P_n(\alpha)$ be the cycle permutation graph with

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 5 & 9 & 13 & 2 & 6 & 10 & 14 & 3 & 7 & 11 & 15 & 4 & 8 & 12 \end{pmatrix}$$

and $B_0 = \{1, 3, 5, 7, 9, 11, 13\}$. Then we have $M_1 = y_9y_{10}y_{11} \backsim M_1' = y_{12}y_{13}y_{14}y_{15} \backsim M_2 = y_1y_2y_3y_4 \backsim M_2' = y_5y_6y_7y_8 \backsim M_1$. Thus, $e_1(B_0) = 1$ and $e_2(B_0) = 0$. In fact, for $B_i = \{1+i, 3+i, 5+i, 7+i, 9+i, 11+i, 13+i\}$, we have $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i. We see that, in this case, $k = 3$, $r = 2$, $\min_{1 \leqslant j \leqslant r} \{|M_j|, |M_j'|\} = 3$, and $(15 = n) \backsim (w_{1,1} = y_{12} = y_{\alpha(15)})$, and $(14 = n - 1) \backsim (w_{2,m_2'} = y_8 = y_{\alpha(14)})$.

(1) $$(2dr - 1) \backsim v_{1,d} \text{ and } 2dr \backsim w_{1,d+1} \text{ for } d = 1, 2, 3.$$

That is,

$$\begin{aligned}
((2(1)(2) - 1) = 3) &\backsim (v_{1,1} = y_9 = y_{\alpha(3)}),\\
((2(2)(2) - 1) = 7) &\backsim (v_{1,2} = y_{10} = y_{\alpha(7)}),\\
((2(3)(2) - 1) = 11) &\backsim (v_{1,3} = y_{11} = y_{\alpha(11)}),\\
((2(1)(2)) = 4) &\backsim (w_{1,2} = y_{13} = y_{\alpha(4)}),\\
((2(2)(2)) = 8) &\backsim (w_{1,3} = y_{14} = y_{\alpha(8)}),\\
((2(3)(2)) = 12) &\backsim (w_{1,4} = y_{15} = y_{\alpha(12)}).
\end{aligned}$$

(2) $$(2sr + 2t - 1) \backsim v_{t+1,s+1} \text{ and } (2sr + 2t) \backsim w_{t+1,s+1}$$
$$\text{for } s = 0, 1, 2, 3 \text{ and } t = 1.$$

$$\begin{aligned}
((2(0)(2) + 2(1) - 1) = 1) &\backsim (v_{2,1} = y_1 = y_{\alpha(1)}),\\
((2(1)(2) + 2(1) - 1) = 5) &\backsim (v_{2,2} = y_2 = y_{\alpha(5)}),\\
((2(2)(2) + 2(1) - 1) = 9) &\backsim (v_{2,3} = y_3 = y_{\alpha(9)}),\\
((2(3)(2) + 2(1) - 1) = 13) &\backsim (v_{2,4} = y_4 = y_{\alpha(13)}),\\
((2(0)(2) + 2(1)) = 2) &\backsim (w_{2,1} = y_5 = y_{\alpha(2)}),\\
((2(1)(2) + 2(1)) = 6) &\backsim (w_{2,2} = y_6 = y_{\alpha(6)}),\\
((2(2)(2) + 2(1)) = 10) &\backsim (w_{2,3} = y_7 = y_{\alpha(10)}),\\
((2(3)(2) + 2(1) = 14) &\backsim (w_{2,4} = y_8 = y_{\alpha(14)}).
\end{aligned}$$

We recall that the generalized Petersen graph $G(15, 4)$ has

$$V(G(15, 4)) = \{1', 2', \ldots, 15', y_1', y_2' \ldots y_{15}'\}$$

and $E(G(15,4)) = \{[i', i'+1], [y'_i, y'_{i+4}], [i', y'_i]$ for $i = 1, 2, \ldots, 15\}$. Then $P_{15}(\alpha) \backsimeq G(15,4)$ where the isomorphic map $\theta\colon V(P_{15}(\alpha)) \longrightarrow V(G(15,4))$ is defined by $\theta(i) = i'$ and $\theta(y_i) = y'_{\alpha^{-1}(i)}$ for $i = 1, 2, \ldots, 15$.

Example 5. The following two cycle permutation graphs are isomorphic. But one has the property (7) and the other has the property (8).

Let $P_7(\alpha)$ be the cycle permutation graph with 14 vertices,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 \end{pmatrix},$$

and $B_i = \{1+i, 3+i, 5+i\}$ for $i = 0, 1, \ldots, 6$. Then $M_1 = M_1(B_0) = y_6y_7y_1 \backsim M'_1 = M_1(C_7 - B_0) = y_2y_3y_4y_5 \backsim M_1$, and $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for $i = 0, 1, \ldots, 6$. $(n = 7) \backsim (w_{1,4} = y_5 = y_{\alpha(7)})$ and $((n-1) = 6) \backsim (w_{1,1} = y_2 = y_{\alpha(6)})$ which is the case of (8).

Let $P_7(\beta)$ be the cycle permutation graph with 14 vertices,

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 \end{pmatrix},$$

and $B_i = \{1+i, 3+i, 5+i\}$ for $i = 0, 1, \ldots, 6$. Then $M_1 = M_1(B_0) = y_1y_2y_3 \backsim M'_1 = M_1(C_7 - B_0) = y_4y_5y_6y_7 \backsim M_1$, and $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for $i = 0, 1, \ldots, 6$. $(n = 7) \backsim (w_{1,1} = y_4 = y_{\beta(7)})$ and $((n-1) = 6) \backsim (w_{1,4} = y_7 = y_{\beta(6)})$ which is the case of (7):

$$P_7(\alpha) \backsimeq G(7,3) \backsimeq G(7,4) \backsimeq P_7(\beta).$$

We also note that $r = 1$ for both $P_7(\alpha)$ and $P_7(\beta)$. Also, in $P_7(\alpha)$, if we use $B_3 = \{4, 6, 1\}$ instead of $B_0 = \{1, 3, 5\}$, then we have $M_1 = y_1y_2y_3$ and $M'_1 = y_4y_5y_6y_7$.

P r o o f. The proof of Lemma 3.2 goes as follows.

(1) We show that $(2dr - 1) \backsim v_{1,d}$ and $2dr \backsim w_{1,d+1}$ for $1 \leqslant d \leqslant m = \min\limits_{1 \leqslant j \leqslant r} \{|M_j|, |M'_j|\}$.

There are two cases to be considered:

Case 1. $r = 1$.

Let $B_0 = \{1, 3, \ldots, 4k+1\}$, and let the maximal chains be

$$M_1(B_0) = M_1 = v_{1,1} \ldots v_{1,m_1},$$

and

$$M_1(C_n - B_0) = M'_1 = w_{1,1}w_{1,2} \ldots w_{1,m'_1}$$

where m_1 is a positive odd integer, and

$$n \backsim w_{1,1} \quad \text{and} \quad (n-1) \backsim (w_{r,m'_r} = w_{1,m'_1}).$$

Since $r = 1$ and $n = 4k + 3$, $m_1 = 2k + 1$ and $m'_1 = 2k + 2$, i.e., $m_1 = m = \min\{|M_1|, |M'_1|\}$. Let $B_2 = \{3, 5, \ldots, 4k + 1, 4k + 3\}$. Then

$$\{\alpha^{-1}(i);\ y_i \in M'_1 - w_{1,1}\} \subseteq C_n - B_2.$$

We claim that $1 \backsim v_{1,1}$. If not, then $M'_1 - w_{1,1}$ is a maximal chain related to $C_n - B_2$, and $|M' - w_{1,1}|$ is an odd integer. Thus, $e_2(B_2) \geqslant 1$ which is a contradiction to $e_2(B_i) = 0$ for all integers i. Hence, $1 \backsim v_{1,1}$. Since $M_1 + w_{1,1} - v_{1,1}$ is a maximal chain related to B_2 and $M'_1 - w_{1,1} + v_{1,1}$ is a maximal chain related to $C_n - B_2$, by Lemma 3.1, $1 \backsim v_{1,1}$ (the terminal vertex of $M'_1 - w_{1,1} + v_{1,1}$), and $2 \backsim w_{1,2}$ (the initial vertex of $M'_1 - w_{1,1} + v_{1,1}$).

Repeatedly applying the same argument to $B_{2dr} = B_{2d}$ for $1 \leqslant d \leqslant m (= 2k + 1)$, we obtain a maximal chain related to B_{2d}:

$$M_1 + w_{1,1}w_{1,2} \ldots w_{1,d} - v_{1,1}v_{1,2} \ldots v_{1,d},$$

a maximal chain related to $C_n - B_{2d}$, $M'_1 - w_{1,1}w_{1,2} \ldots w_{1,d} + v_{1,1}v_{1,2} \ldots v_{1,d}$, and $(2d - 1) \backsim v_{1,d}$ and $2d \backsim w_{1,d+1}$ for $d = 1, 2 \ldots, (2k + 1)$. (We already know that $n \backsim w_{1,1}$.)

Case 2. $r > 1$.

(i) We show that $2t - 1 \backsim v_{t+1,1}$ and $2t \backsim w_{t+1,1}$ for $t = 1, 2, \ldots, r - 1$.

Let $B_0 = \{1, 3, \ldots, 4k + 1\}$, let the maximal chains be, for $j = 2, \ldots, r$,

$$M_j = M_j(B_0) = v_{j,1}v_{j,2}, \ldots, v_{j,m_j},$$

and

$$M'_j = M_j(C_n - B_0) = w_{j,1}w_{j,2} \ldots w_{j,m'_j}$$

where m_1 is a positive odd integer, and

$$n \backsim w_{1,1} \quad \text{and} \quad (n - 1) \backsim w_{r,m'_r}.$$

Similar to the proof of Case 1, let $B_2 = \{3, 5, \ldots, 4k + 1, 4k + 3\}$. Then

$$\{\alpha^{-1}(i);\ y_i \in M'_1 - w_{1,1}\} \subseteq C_n - B_2.$$

We claim that $1 \backsim v_{2,1}$. If not, then $M'_1 - W_{1,1}$ is a maximal chain related to $C_n - B_2$, and $|M'_1 - w_{1,1}|$ is an odd integer. Thus, $e_2(B_2) \geqslant 1$ which is a contradiction to $e_2(B_i) = 0$ for all integers i. Hence, $1 \backsim v_{2,1}$. Consequently,

$$M_1 + w_{1,1}, M_2 - v_{2,1}, M_3, \ldots, M_r$$

are maximal chains related to B_2, and

$$M'_1 - w_{1,1} + v_{2,1}, M'_2, M'_3, \dots, M'_r$$

are maximal chains related to $C_n - B_2$. Since $|M_2 - v_{2,1}| = m_2 - 1$ is odd and the others are even, by Lemma 2.1, we have $1 \backsim v_{2,1}$ (the terminal vertex of $M'_1 - w_{1,1} + v_{2,1}$), and $2 \backsim w_{2,1}$ (the initial vertex of M'_2).

Repeatedly applying the same argument to B_{2t} for $1 \leqslant t \leqslant r-1$, we obtain the maximal chains related to B_{2t}:

$$M_1 + w_{1,1}, M_2 - v_{2,1} + w_{2,1}, M_3 - v_{3,1} + w_{3,1}, \dots, M_{t+1} - v_{t+1,1}, M_{t+2} \dots, M_r,$$

the maximal chains related to $C_n - B_{2t}$:

$$M'_1 - w_{1,1} + v_{2,1}, M'_2 - w_{2,1} + v_{3,1}, M'_3 - w_{3,1} + v_{4,1}, \dots, M'_t - w_{t,1} + v_{t+1,1},$$
$$M'_{t+1}, M'_{t+2}, \dots, M'_r,$$

and $(2t-1) \backsim v_{t+1,1}$ and $2t \backsim v_{t+1,1}$ and $2t \backsim w_{t+1,1}$ for $t = 1, 2, \dots, r-1$.

(ii) We show that $(2dr-1) \backsim v_{1,d}$ and $2dr \backsim w_{1,d+1}$ for $d = 1, 2, \dots, m$ (where $m = \min_{1 \leqslant j \leqslant r} \{|M_j|, |M'_j|\}$). For $t = r-1$, by the Case 2 (i), we obtain the maximal chains related to $B_{2(r-1)}$:

$$M_1 + w_{1,1}, M_2 - v_{2,1} + w_{2,1}, M_3 - v_{3,1} + w_{3,1}, \dots, M_r - v_{r,1},$$

the maximal chains related to $C_n - B_{2(r-1)}$:

$$M'_1 - w_{1,1} + v_{2,1}, M'_2 - w_{2,1} + v_{3,1}, M'_3 - w_{3,1} + v_{4,1}, \dots, M'_{r-1} - w_{r-1,1} + v_{r,1}, M'_r$$

and $(2r-3) \backsim v_{r,1}$ and $(2r-2) \backsim w_{r,1}$.

Let $B_{2r} = \{1+2r, 3+2r, \dots, 4k+1+2r\}$. Since $4k+1+2r$ is congruent to $2r-2$ modulo n, $\{\alpha^{-1}(i);\ y_i \in M'_r - w_{r,1}\} \subseteq C_n - B_{2r}$.

We claim that $(2r-1) \backsim v_{1,1}$. If not, then $M'_r - w_{r,1}$ is a maximal chain related to $C_n - B_{2r}$, and $|M'_r - w_{r,1}| = m'_r - 1$ is an odd integer. Thus, $e_2(B_{2r}) \geqslant 1$ which is a contradiction to $e_2(B_i) = 0$ for all integers i. Hence, $(2r-1) \backsim v_{1,1}$.

The maximal chains related to B_{2r} are:

$$M_1 - v_{1,1} + w_{1,1} M_2 - v_{2,1} + w_{2,1}, M_3 - v_{3,1} + w_{3,1}, \dots, M_r - v_{r,1} + w_{r,1},$$

and the maximal chains related to $C_n - B_{2r}$ are:

$$M'_1 - w_{1,1} + v_{2,1}, M'_2 - w_{2,1} + v_{3,1}, M'_3 - w_{3,1} + v_{4,1}, \dots, M'_r - w_{r,1} + v_{1,1}.$$

Since $|M_1 - v_{1,1} + w_{1,1}|$ is an odd integer, by Lemma 3.1, $(2r-1) \backsim v_{1,1}$ (the terminal vertex of $M'_r - w_{1,1} + v_{1,1}$), and $2r \backsim w_{2,1}$ (the initial vertex of $M'_1 - w_{1,1} + v_{2,1}$).

Repeatedly applying the same argument to B_{2dr} for $d = 1, 2, \ldots, m$, we obtain the maximal chains related to B_{2dr}:

$$\begin{gathered} M_1 - v_{1,1}v_{1,2}\ldots v_{1,d} + w_{1,1}w_{1,2}\ldots w_{1,d} \\ M_2 - v_{2,1}v_{2,2}\ldots v_{2,d} + w_{2,1}w_{2,2}\ldots w_{2,d} \\ \vdots \\ M_r - v_{r,1}v_{r,2}\ldots v_{r,d} + w_{r,1}w_{r,2}\ldots w_{r,d}, \end{gathered}$$

the maximal chains related to $C_n - B_{2dr}$:

$$\begin{gathered} M'_1 - w_{1,1}w_{1,2}\ldots w_{1,d} + v_{2,1}v_{2,2}\ldots v_{2,d}, \\ M'_2 - w_{2,1}w_{2,2}\ldots w_{2,d} + v_{3,1}v_{3,2}\ldots v_{3,d}, \\ \vdots \\ M'_r - w_{r,1}w_{r,2}\ldots w_{r,d} + v_{1,1}v_{1,2}\ldots v_{1,d}, \end{gathered}$$

and $(2dr-1) \backsim v_{1,d}$ and $2dr \backsim w_{1,d+1}$ for $d = 1, 2, \ldots, m$. Thus, the proof of (1) is completed.

(2) Repeatedly applying the same argument as in the proof of Case 2 (ii) to B_{2kr} for $k = 1, 2, \ldots, m-1$, then applying the same argument as in the proof of Case 2 (i) to B_{2kr+2t} for $t = 1, 2, \ldots, r-1$, we obtain the maximal chains related to B_{2kr+2t}:

$$\begin{gathered} M_1 - v_{1,1}v_{1,2}\ldots v_{1,k} + w_{1,1}, w_{1,2}\ldots w_{1,k}w_{1,k+1}, \\ M_2 - v_{2,1}v_{2,2}\ldots v_{2,k}v_{2,k+1} + w_{2,1}w_{2,2}\ldots w_{2,k}w_{2,k+1}, \\ \vdots \\ M_t - v_{t,1}v_{t,2}\ldots v_{t,k}v_{t,k+1} + w_{t,1}w_{t,2}\ldots w_{t,k}w_{t,k+1}, \\ M_{t+1} - v_{t+1,1}v_{t+1,2}\ldots v_{t+1,k}v_{t+1,k+1} + w_{t+1,1}w_{t+1,2}\ldots w_{t+1,k}, \\ M_{t+2} - v_{t+2,1}v_{t+2,2}\ldots v_{t+2,k} + w_{t+2,1}w_{t+2,2}\ldots w_{t+2,k}, \\ \vdots \\ M_r - v_{r,1}v_{r,2}\ldots v_{r,k} + w_{r,1}w_{r,2}\ldots w_{r,k}, \end{gathered}$$

the maximal chains related to $C_n - B_{2kr+2t}$:

$$\begin{gathered} M'_1 - w_{1,1}w_{1,2}\dots w_{1,k}w_{1,k+1} + v_{2,1}v_{2,2}\dots v_{2,k}v_{2,k+1}, \\ M'_2 - w_{2,1}w_{2,2}\dots w_{2,k}w_{2,k+1} + v_{3,1}v_{3,2}\dots v_{3,k+1}, \\ \vdots \\ M'_t - w_{t,1}w_{t,2}\dots w_{t,k}w_{t,k+1} + v_{t+1,1}v_{t+1,2}\dots v_{t+1,k}v_{t+1,k+1}, \\ M'_{t+1} - w_{t+1,1}w_{t+1,2}\dots w_{t+1,k} + v_{t+2,1}v_{t+2,2}\dots v_{t+2,k}, \\ \vdots \\ M'_r - w_{r,1}w_{r,2}\dots w_{r,k} + v_{1,1}v_{1,2}\dots v_{1,k}, \end{gathered}$$

and

$$(2kr+2t-1) \backsim v_{t+1,k+1} \quad \text{and} \quad (2kr+2t) \backsim w_{t+1,k+1}$$

for $k = 1, 2, \dots, m-1$ and $t = 1, 2, \dots, r-1$. □

Lemma 3.4. *Let $n = 4k+3$ with $k \geqslant 1$, $P_n(\alpha)$, B_i, M_j and M'_j for $j = 1, 2, \dots, r$ be the same as in Lemma 3.3. If $|M_1| = m$ is an odd positive integer, $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i, and $n \backsim w_{1,1}$ and $(n-1) \backsim w_{r,m'_r}$, then $|M_j| = m+1$ for $j = 2, 3, \dots, r$, and $|M'_j| = m+1$ for $j = 1, 2, \dots, r$ where $r(m+1) = 2(k+1)$.*

Proof. Let $|M_1| = m = \min\limits_{1\leqslant j\leqslant r}\{|M_j|, |M'_j|\}$. We claim that $|M_j| = m+1$ for $j = 2, 3, \dots, r$ and $|M'_j| = m+1$ for $j = 1, 2, \dots, r$. First, we show that $|M'_1| > m$. Suppose $|M'_1| = m$. Then by Lemma 3.3, Case 2 (ii), $2mr \backsim w_{1,m+1}$. Since $|M'_1| = m$, $w_{1,m+1} = v_{2,1}$. By Lemma 3.3, Case 2 (i), $1 \backsim v_{2,1}$. Since $P_n(\alpha)$ is a cycle permutation graph, each vertex in C_n is incident with exactly one vertex in C'_n. Thus, $2mr \equiv 1 \pmod n$, i.e., $2mr - 1 = n = 4k+3$ which is a contradiction to $2mr \leqslant 4k+3$. Hence, $|M'_1| > m$.

Suppose $|M_t| = m$ for some $t = 2, 3, \dots, r$. Then

$$M_t(B_{2mr}) = M_t - v_{t,1}v_{t,2}\dots v_{t,m} + w_{t,1}w_{t,2}\dots w_{t,m},$$

i.e., $M_t(B_{2mr}) = w_{t,1}w_{t,2}\dots w_{t,m}$ is a maximal chain related to B_{2mr}. Repeatedly using the procedure in Lemma 3.3, Case 2 (i) with $B_{2mr+2(t-1)}$, we have $(2mr + 2(t-1) - 1) \backsim v_{t,m+1}$. Since $|M_t| = m$, $v_{t,m+1} = w_{t,1}$. By Lemma 3.3, Case 2 (i), $w_{t,1} \backsim 2t-2$. Thus, $2mr + 2t - 3 \equiv 2t - 2 \pmod n$, $2mr - 1 = n = 4k+3$ which is a contradiction to $2mr \leqslant 4k+3$. Hence, $|M_t| > m$ for $t = 2, 3, \dots, r$. By using a similar reasoning, we have $|M'_t| > m$ for $t = 1, 2, \dots, r$.

We want to show that

$$|M_j| = m + 1 \quad \text{for} \quad j = 2, 3, \ldots, r,$$

and

$$|M'_j| = m + 1 \quad \text{for} \quad j = 1, 2, \ldots, r.$$

We know that $|M_j| \geqslant m+1$ for $j = 2, 3, \ldots, r$, and $|M'_j| \geqslant m+1$ for $j = 1, 2, \ldots, r$. By using the procedure in Lemma 3.3, Case 2 (i) with $B_{2mr+2(r-1)}$ repeatedly, we have $(2mr + 2(r-1) - 1) \backsim w_{(r-1)+1,m+1}$ and $(2mr + 2(r-1)) \backsim w_{(r-1)+1,m+1} = w_{r,m+1}$. If $w_{r,m+1}$ is the terminal vertex of M'_r, then $2mr+2(r-1) \equiv 4k+2 \pmod{n}$, i.e., $2r(m+1) = 4k+4$. Hence, $|M_j| = m+1$ for $j = 2, 3, \ldots, r$, $|M'_j| = m+1$ for $j = 1, 2, \ldots, r$, and r divides $2(k+1)$. If $w_{r,m+1}$ is not the teerminal vertex of M'_r, then, by using the procedure in Lemma 3.3, Case 2 (ii) with B_{2mr+2r}, we have $(2(m+1)r-1) \backsim v_{1,m+1}$. Since $v_{1,m+1} = w_{1,1}$ and $4k+3 \backsim w_{1,1}$, $2(m+1)r-1 \equiv 4k+3 \pmod{n}$, i.e., $2r(m+1) = 4k+4$. Hence, $|M_j| = m+1$ for $j = 2, 3, \ldots, r$, $|M'_j| = m+1$ for $j = 1, 2, \ldots, r$ and r divides $2(k+1)$.

Proof of Theorem 3 goes as follows. We want to show that, for $n = 4k+3$ with $k \geqslant 1$, $P_n(\alpha)$ is isomorphic to $G(n, 2r)$ for some r which divides $2(k+1)$. Let $V(G(n,2r)) = \{1', 2', \ldots, (4k+3)', y'_1, y'_2, \ldots, y'_{4k+3}\}$ and $E(G(n,2r)) = \{[i', (i+1)'], [y'_i, y'_{i+2r}], [i', y'_i]$ for $i = 1, 2, \ldots, 4k+3\}$. Also, let $V(P_n(\alpha)) = \{1, 2, \ldots, 4k+3, y_1, y_2, \ldots, y_{4k+3}\}$ and $E(P_n(\alpha)) = \{[i, i+1], [y_i, y_{i+1}][i, y_{\alpha(i)}]$ for $i = 1, 2, \ldots, 4k+3\}$. In $P_n(\alpha)$, let $B_i = \{1+i, 3+i, \ldots, (4k+1)+i\}$, with $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for $i = 0, 1, \ldots, 4k+2$, the maximal chains $M_j = M_j(B_0) = v_{j,1}v_{j,2}\ldots v_{j,m_j}$ and $M'_j = M_j(C_n - B_0) = w_{j,1}w_{j,2}\ldots w_{j,m'_j}$, for $j = 1, 2, \ldots, r$, where r divides $2(k+1)$.

Case 1. $n \backsim w_{1,1}$ and $(n-1) \backsim w_{r,m'_r}$. We define a map $\theta\colon V(P_n(\alpha)) \longrightarrow V(G(n,2r))$ by $\theta(i) = i'$ and $\theta(y_i) = y'_{\alpha^{-1}(i)}$ for $i = 1, 2, \ldots, 4k+3$. Then θ is a well defined map between the vertices of these two cubic graphs. We show that θ preserves the edges: Since $[i,j] = E(P_n(\alpha))$ if and only if $j = i+1$ and $[\theta(i), \theta(j)] = [i', j'] \in G(n,2r)$ if and only $j' = i'+1$, $\theta[i,j] = [\theta(i), \theta(j)]$ for $i, j = 1, 2, \ldots, 4k+3$ and $i \neq j$. Since $[i, y_j] \in E(P_n(\alpha))$ if and only if $j = \alpha(i)$ and $[\theta(i), \theta(y_j)] = [i', y'_{\alpha^{-1}(j)}] \in E(G(n,2r))$ if and only if $j = \alpha(i)$, $\theta[i, y_j] = [\theta(i), \theta(y_j)]$ for $i, j = 1, 2, \ldots, 4k+3$. We know that $[y_i, y_j] \in E(P_n(\alpha))$ if and only if $j = i+1$, and $[\theta(y_i), \theta(y_j)] = [y'_{\alpha^{-1}(i)}, y'_{\alpha^{-1}(j)}]$. Say, $\alpha^{-1}(i) = q$ for some q such that $1 \leqslant q \leqslant 4k+3$. Then $\alpha(q) = i$. By (1) and (2) of lemma 3.3, we know that $[y_i, y_{i+1}] \in E(P_n(\alpha))$ if and only if $i+1 = \alpha(q) + (2r)$. By Lemma 3.4, this holds for all $i = 1, 2, \ldots, 4l+3$. Since $2r(m+1) = 4k+4 \equiv 1 \pmod{n}$, $[\theta(y_i), \theta(y_{i+1})] = [y'_{\alpha^{-1}}(i), y'_{\alpha^{-1}(i+1)}] = [y'_q, y'_{q+2r}] \in E(G(n,2r))$. Thus, $\theta[y_i, y_j] = [\theta(y_i), \theta(y_j)]$ for all $i, j = 1, 2, \ldots, 4k+3$ and $i \neq j$. Hence, $P_n(\alpha) \backsimeq G(n, 2r)$.

Case 2. $n \backsim w_{r,M'_r}$ and $(n-1) \backsim w_{1,1}$. Relabeling the vertices on C'_n by $v_{1,m_1} \longrightarrow z_1, v_{1,m_1-1} \longrightarrow z_2, \ldots, v_{1,1} \longrightarrow z_{m_1}, w_{r,m'_r} \longrightarrow z_{m_1+1}, z_{r,m'_r-1} \longrightarrow z_{m_1+2}, \ldots, w_{1,1} \longrightarrow z_n$, we obtain a new cycle permutation graph $P_n(\beta)$ such that $P_n(\beta)$ has a cycle with vertex set $\{z_1, z_2, \ldots, z_n\}$ and $n \backsim z_{m_1+1}$ (the initial vertex of $M_1(C'_n - B_0)$ and $(n-1) \backsim w_{r,m'_r}$ (the end vertex related to $M_r(C'_n - B_0)$). Clearly, by Case 1 above, we have $P_n(\beta) = P_n(\alpha) \backsimeq G(n, 2r)$. □

We know that $G(n,t)$ is isomorphic to a cycle permutation graph $P_n(\alpha)$ for some $\alpha \in S_n$ if and only if t and n are relatively prime. The following example shows that the converse of Theorem 3 does not hold.

Example 6. Consider $G(11,4)$ where

$$V(G(11,4)) = \{1, 2, \ldots, 11, y_1, y_2, \ldots, y_{11}\},$$

and $E(G(11,4)) = \{[i, i+1], [i, y_i], [y_i, y_{i+4}]$ for $i = 1, 2, \ldots, 11\}$, i.e., the outer cycle of $G(11,4)$, C_x, is $1 - 2 - \ldots - 11 - 1$, and the inner cycle of $G(11,4)$, C_y, is $y_1 - y_5 - y_9 - y_2 - y_6 - y_{10} - y_3 - y_7 - y_{11} - y_4 - y_8 - y_1$. Let

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 11 & 4 & 8 \end{pmatrix}$$

and

$$\alpha = \gamma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 4 & 7 & 10 & 2 & 5 & 8 & 11 & 3 & 6 & 9 \end{pmatrix}.$$

Clearly, $G(11,4) \backsimeq P_{11}(\alpha)$. Let $B_0 = \{1,3,5,7,9\}$. Then the maximal chains in $G(11,4)$ are:

$$M_1 = y_3y_7 \backsim M'_1 = y_{11}y_4y_8 \backsim M_2 = y_1y_5y_9 \backsim M'_2 = y_2y_6y_{10} \backsim M_1.$$

Thus, $r = 2$ and $m = 2$. But $e_1(B_0) = 1$ and $e_2(B_0) = 2$.

Theorem 4. *Let* $n = 4k+3$ *with* $k \geqslant 1$ *and* r *divide* $2(k+1)$. *Then,* $t(G(n,2r)) \leqslant 4/3$.

Proof. Let $V(G(n,2r)) = \{1, 2, \ldots, 4k+3, y_1, y_2, \ldots, y_{4k+3}\}$ and $E(G(n,2r)) = \{[i, i+1], [i, y_i], [y_i, y_{i+2r}]$ for $i = 1, 2, \ldots, 4k+3\}$.

Case 1. $r = 2(k+1)$, $G(n,2r) = G(n,1)$. Take the disconnecting set $S = \{1, 3, 5, \ldots, 4k+1, y_2, y_4, y_6, \ldots, y_{4k+2}\}$. Then, $\omega(G(n,1) - S)$, the set of components induced from $G(n,1) - S$ is $\{\{2\}, \{4\}, \{6\}, \ldots, \{4k\}, \{y_3\}, \{y_5\}, \{y_7\}, \ldots, \{y_{4k+1}\}, \{4k+2, 4k+3, y_{4k+3}, y_1\}\}$. Thus, $|S| = (2k+1) + (2k+1) = 4k+2$, $\omega(G(n,1) - S) = 2k + 2k + 1 = 4k+1$ and

$$t(G(n,1)) \leqslant \frac{4k+2}{4k+1} < \frac{4}{3}.$$

Case 2. $r = 1$. $G(n, 2r) = G(n, 2)$. It was proved in [3] and [8] that $t(G(n, 2)) \leqslant \frac{4}{3}$.

Case 3. $r = k + 1$. $G(n, 2r) = G(n, 2k + 2)$. Let $B_0 = \{1, 3, 5, \ldots, 4k + 1\}$. Then the partition of maximal chains is:

$$y_{2k+1} \backsim y_{4k+3}y_{2k+2} \backsim y_1 y_{2k+3} \backsim y_2 y_{2k+4}$$
$$\backsim y_3 y_{2k+5} \backsim y_4 y_{2k+6} \backsim \cdots \backsim y_{2k-1}y_{4k+1} \backsim y_{2k}y_{4k+2},$$

and the maximal chains are:

$$\begin{aligned} M_1(B_0) &= y_{2k+1}, & M_1(C_n - B_0) &= y_{4k+3}y_{2k+2}, \\ M_2(B_0) &= y_1 y_{2k+3}, & M_2(C_n - B_0) &= y_2 y_{2k+4}, \\ &\vdots \\ M_{k+1}(B_0) &= y_{2k-1}y_{4k-1}, & M_{k+1}(C_n - B_0) &= y_{2k}y_{4k+2}. \end{aligned}$$

Let $B_y = \{y_{2k+2}, y_2, y_{2k+4}, y_4, y_{2k+6}, \ldots, y_{2t}, y_{2k+2t+2}, \ldots, y_{2k-2}, y_{4k}, y_{2k}\}$ and $S = B_0 \cup B_y$. Then the components of $G(n, 2k + 2) - S$ are the following sets:

$$\{2\}, \{4\}, \ldots, \{2t\}, \ldots, \{4k\},$$
$$\{y_1, y_{2k+3}\}, \{y_3, y_{2k+5}\}, \ldots, \{y_{2t-1}, y_{2k+2t+1}\}, \ldots,$$
$$\{y_{2k-1}, y_{4k+1}\}, \{y_{4k+3}, 4k + 3, 4k + 2, y_{4k_{+2}}, y_{2k+1}\}.$$

Thus,

$$|S| = (2k + 1) + 2(k - 1) + 2 = 4k + 1,$$
$$\omega(G(n, 2k + 2) - S) = 2k + k + 1 = 3k + 1,$$
$$\text{and } t(G(n, 2k + 2) \leqslant \frac{|S|}{\omega(G(n, 2k + 2) - S)} = \frac{4k + 1}{3k + 1} < \frac{4}{3}.$$

Case 4. Let $1 < r < k + 1$ and $r(m + 1) = 2k + 2$. Let $B_0 = \{1, 3, 5, \ldots, 4k + 1\}$. Then the partition of maximal chains is:

$$y_{2r-1}y_{4r-1} \cdots y_{2mr-1} \backsim y_{4k+3}y_{2r}y_{4r} \cdots y_{2mr}$$
$$\backsim y_1 y_{1+2r} y_{1+4r} \cdots y_{1+2mr} \backsim y_2 y_{2+2r} y_{2+4r} \cdots y_{2+2mr}$$
$$\backsim y_{1+2(3-2)} y_{1+2(3-2)+2r} y_{1+2(3-2)+4r} \cdots y_{1+2(3-2)+2mr}$$
$$\backsim y_{2+2(3-2)} y_{2+2(3-2)+2r} y_{2+2(3-2)+4r} \cdots y_{2+2(3-2)+2mr}$$
$$\vdots$$
$$\backsim y_{1+2(r-2)} y_{1+2(r-2)+2r} y_{1+2(r-2)+4r} \cdots y_{1+2(r-2)+2mr}$$
$$\backsim y_{2+2(r-2)} y_{2+2(r-2)+2r} y_{2+2(r-2)+4r} \cdots y_{2+2(r-2)+2mr}.$$

Since $1 < r < k+1$ and $r(m+1) = 2k+2$, $m+1 \geqslant 3$ and $m \geqslant 2$. If $m = 2$, then

$$|M_1(C_n - B_0)| = |y_{4k+3}y_{2r}y_{4r}\ldots y_{2mr}| = m+1$$

is a positive odd integer. Thus, $e_2(B_0) \geqslant 1$ and by (1) in Theorem 1 or by (4), $t(G(n,2r)) \leqslant 4/3$, since $G(n,2r)$ is isomorphic to a cycle permutation graph. If $m \geqslant 3$, then we define an independent set of vertices in C_n by

$$B = (B_0 \cup \{4r\}) \setminus \{4r-1, 4r+1\}.$$

Then, since $m \geqslant 3$ and $4r \neq 2mr$, $y_{4r-1}, y_{6r}, \ldots, y_{2mr}$ and y_{4r+1} are three maximal chains, related to $C_n - B$, of odd cardinalities. Thus, $e_2(B) \geqslant 3$ and by (1) in Theorem 1 or by (4), $t(G(n,2r)) \leqslant \frac{4}{3}$ since $G(n,2r)$ is isomorphic to a cycle permutation graph. □

Corollary 4.1. *For $n \geqslant 4$ and $\alpha \in S_n$,*

$$t(P_n(\alpha)) \leqslant \frac{4}{3}.$$

P r o o f. If $n \neq 4k+3$, then, by Corollary 1.1, $t(P_n(\alpha)) \leqslant \frac{4}{3}$ for every $\alpha \in S_n$. By Theorem 2, if $e_1(B_i) \geqslant 3$ or $e_2(B_i) \geqslant 1$, then $t(P_{4k+3}(\alpha)) \leqslant \frac{4}{3}$ for every $\alpha \in S_n$ and any positive integer k. We know that the case of $n = 4k+3$ with $k \geqslant 1$, $e_1(B_i) = 2$ and $e_2(B_i) = 0$ does not exist, because thus means that there are exactly 2 maximal chains with odd cardinalities and the rest are of even cardinalities. Then the total number of vertices in C'_n is even which contradicts $n = 4k+3$. Thus, the remaining case which we have to consider is $n = 4k+3$ with $k \geqslant 1$, $e_1(B_i) = 1$ and $e_2(B_i) = 0$ for all integers i. By Theorem 3, $P_{4k+3}(\alpha)$, for every $\alpha \in S_n$, is isomorphic to $G(4k+3, 2r)$ for some positive integer r which divides $2k+2$. By Theorem 4, $t(G(4k+3,2r)) \leqslant \frac{4}{3}$. Hence, $t(P_n(\alpha)) \leqslant \frac{4}{3}$ for every integer $n \geqslant 4$ and every $\alpha \in S_n$. □

Our Corollary 4.1 confirms the conjecture 1 in [11].

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