

Wiesław Aleksander Dudek; Zoran Stojaković
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ON RUSAKOV'S n -ARY rs -GROUPS

WIESŁAW A. DUDEK, Wrocław, and ZORAN STOJAKOVIĆ, Novi Sad

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Abstract. Properties of n -ary groups connected with the affine geometry are considered. Some conditions for an n -ary rs -group to be derived from a binary group are given. Necessary and sufficient conditions for an n -ary group (θ, b) -derived from an additive group of a field to be an rs -group are obtained. The existence of non-commutative n -ary rs -groups which are not derived from any group of arity $m < n$ for every $n \geq 3$, $r > 2$ is proved.

Keywords: n -ary group, symmetry

MSC 2000: 20N15, 51A25, 51D15

1. INTRODUCTION

If the standard (affine) geometry has a fixed point O , then any point P of this geometry is uniquely determined by the vector $\vec{p} = \overrightarrow{OP}$, and conversely, the vector $\vec{p} = \overrightarrow{OP}$ uniquely determines the point P . Moreover, any interval \overline{PQ} may be interpreted as the vector $\vec{q} - \vec{p}$ or as the vector $\vec{p} - \vec{q}$. In the latter case,

$$\overline{AB} = \overline{CD} \quad \text{if and only if} \quad \vec{a} - \vec{b} + \vec{d} = \vec{c},$$

or in other words

$$\overline{AB} = \overline{CD} \quad \text{if and only if} \quad f(a, b, d) = c,$$

where any vector \vec{v} is treated as an element v of a commutative group $(G, +)$. The operation f has the form $f(x, y, z) = x - y + z$. Groups (also non-commutative) with a ternary operation defined in this way were considered by J. Certainé (cf. [3]) as a special case of *ternary heaps* first described by H. Prüfer (cf. [11]). Ternary heaps have interesting applications to projective geometry (cf. [1]), affine geometry (cf. [2]), theory of nets (webs), theory of knots and even to the differential geometry.

Moreover, all affine geometries may be treated as geometries defined by some ternary relations (cf. for example [15]). Such geometries may be defined also by some n -ary ($n > 3$) relations (cf. [16]). The class of affine geometries defined by n -ary groups, which are a natural generalization of the notion of groups, was introduced by S. A. Rusakov (cf. [13], [14]) and in detail described by J. I. Kulachgenko (cf. [10]).

Note that n -ary structures are interesting also for their applications to problems of modern mathematical physics (cf. for example [17], [18]).

2. PRELIMINARIES

Traditionally in the theory of n -ary groups we use the following abbreviated notation: the sequence x_i, \dots, x_j is denoted by x_i^j (for $j < i$ this symbol is empty). If $x_{i+1} = \dots = x_{i+k} = x$, then instead of x_{i+1}^{i+k} we write $x^{(k)}$. Obviously $x^{(0)}$ is the empty symbol. In this notation the formula

$$f(x_1, \dots, x_i, x_{i+1}, \dots, x_{i+k}, x_{i+k+1}, \dots, x_n),$$

where $x_{i+1} = \dots = x_{i+k} = x$, will be written as $f(x_1^i, x^{(k)}, x_{i+k+1}^n)$.

By an n -ary group (G, f) we mean (cf. [4]) a non-empty set G together with one n -ary operation $f: G^n \rightarrow G$, which for all $i = 1, 2, \dots, n$ satisfies the following two conditions:

1° *the associative law*:

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}),$$

2° for all $a_1, a_2, \dots, a_n, b \in G$ there exists a unique $x_i \in G$ such that

$$f(a_1^{i-1}, x_i, a_{i+1}^n) = b.$$

Such an n -ary group may be considered also (see for example [5], [6] or [8]) as an algebra (G, f, g) with one associative n -ary operation f and one unary operation g satisfying some identities. In particular, an n -ary group may be treated as an algebra $(G, f, [^{-2}])$ with one associative n -ary operation f and one unary operation $[^{-2}]$: $x \mapsto x^{[-2]}$ in which the identities

$$(1) \quad f(x^{[-2]}, \binom{n-2}{x}, f(\binom{n-1}{x}, y)) = f(f(y, \binom{n-1}{x}), \binom{n-2}{x}, x^{[-2]}) = y,$$

hold (cf. [12]).

In the affine geometry defined on an n -ary group (G, f) (for details see [13]) four points $a, b, c, d \in G$ define a parallelogram if and only if

$$f(f(a, b^{[-2]}, \overset{(n-2)}{b}), \overset{(n-2)}{b}, c) = d.$$

Two points a and c are called *symmetric* if and only if there exists a uniquely determined point $x \in G$ such that

$$f(f(a, x^{[-2]}, \overset{(n-2)}{x}), \overset{(n-2)}{x}, c) = x.$$

3. n -ARY rs -GROUPS

For $n \geq 3$ the above definitions can be modified. Indeed, since (1) and the associativity of f give

$$f(f(x^{[-2]}, \overset{(n-1)}{x}), \overset{(n-2)}{x}, y) = f(y, \overset{(n-2)}{x}, f(\overset{(n-1)}{x}, x^{[-2]})) = y,$$

the results obtained in [8] and [5] imply

$$f(x^{[-2]}, \overset{(n-1)}{x}) = f(\overset{(n-1)}{x}, x^{[-2]}) = \bar{x},$$

where \bar{x} denotes the *skew element* to x (cf. [4], [5] or [8]). In general $\bar{x} \neq x$, but in the so-called *idempotent* n -ary groups, i.e. in n -ary groups (G, f) in which $f(\overset{(n)}{x}) = x$ for all $x \in G$, we have $\bar{x} = x$.

Thus for $n \geq 3$ the above two definitions can be formulated in the following equivalent form:

Definition 1. Elements a, b, c, d of an n -ary group (G, f) , where $n \geq 3$, define a parallelogram if and only if

$$f(a, \bar{b}, \overset{(n-3)}{b}, c) = d.$$

Definition 2. Two elements a and c of an n -ary group (G, f) are symmetric if and only if there exists one and only one $x \in G$ such that

$$(2) \quad f(a, \bar{x}, \overset{(n-3)}{x}, c) = x.$$

Thus for symmetric elements a and c there exist a uniquely determined $x \in G$ and a symmetry S_x such that $S_x(a) = c$. Since the definition of an n -ary group implies

This by the so-called *Dörnte's identity*

$$f\left(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y\right) = y,$$

which for $i = 2, 3, \dots, n$ holds in any n -ary group (for details see [4], [5], [6] or [8]), gives

$$x_3 = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_4).$$

In a similar way, replacing x_3 in the third identity by the above formula, we get

$$x_4 = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_5).$$

This together with the fourth identity gives

$$x_5 = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_6).$$

Continuing the above procedure we obtain

$$x_k = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_{k+1})$$

for $k = 2, 3, \dots, r$. Thus

$$x_r = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, c)$$

and

$$x_{r-1} = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_r) = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, f(a, \overline{x}_2, \overset{(n-3)}{x_2}, c)).$$

Analogously we obtain

$$(4) \quad x_k = f_{(r-k+1)}\left(\underbrace{a, \overline{x}_2, \overset{(n-3)}{x_2}, \dots, a, \overline{x}_2, \overset{(n-3)}{x_2}}_{r-k+1\text{-times}}, c\right)$$

for all $k = 2, 3, \dots, r$, where $f_{(r-k+1)}$ means that the n -ary operation f is used $(r - k + 1)$ -times.

However, by Hosszú's theorem every n -ary group (G, f) is $\langle \theta, b \rangle$ -derived from some binary group, i.e. every n -ary group has the form

$$f(x_1^n) = x_1 \cdot \theta x_2 \cdot \theta^2 x_3 \cdot \dots \cdot \theta^{n-1} x_n \cdot b,$$

where (G, \cdot) is a binary group (called *the creating group*), $b \in G$, θ is an automorphism of (G, \cdot) such that $\theta b = b$ and $\theta^{n-1} x \cdot b = b \cdot x$ for all $x \in G$.

Since this form may be written also as

$$(5) \quad f(x_1^n) = x_1 \cdot \theta x_2 \cdot \theta^2 x_3 \cdot \dots \cdot \theta^{n-2} x_{n-1} \cdot b \cdot x_n,$$

and $f(z, \bar{z}, \overset{(n-2)}{z}) = z$ for every $z \in G$, hence $\theta \bar{z} \cdot \theta^2 z \cdot \dots \cdot \theta^{n-2} z \cdot b \cdot z$ is the identity of (G, \cdot) and $\theta \bar{z} \cdot \theta^2 z \cdot \dots \cdot \theta^{n-2} z \cdot b$ is the inverse of z in (G, \cdot) . Thus the identity (2) may be written in (G, \cdot) as $a \cdot x^{-1} \cdot c = x$. Similarly (4), for $k = 2$ and $a = e$, where e is the identity of (G, \cdot) , may be written as $x_2 = (x_2^{-1})^{r-1} \cdot c$, which proves that in a creating group of an n -ary rs -group (G, \cdot) for any element c there exists only one x such that $c = x^r$. In particular, there exists only one x such that $e = x^r$.

Thus the following theorem is true.

Theorem 1. *An n -ary rs -group is $\langle \theta, b \rangle$ -derived from a binary group without elements of order r .*

Corollary 1. *An n -ary $2s$ -group is $\langle \theta, b \rangle$ -derived from a binary group without elements of order two.*

Corollary 2. *A finite n -ary $2s$ -group is $\langle \theta, b \rangle$ -derived from a binary group with an odd exponent.*

Corollary 3. *There are no n -ary $2s$ -groups of an even order.*

Since all binary retracts of a given n -ary group (G, f) $\langle \theta, b \rangle$ -derived from a group (G, \cdot) , i.e. groups $(G, *_a)$ where $x *_a y = f(x, \overset{(n-2)}{a}, y)$ and $a \in G$ is fixed are isomorphic to (G, \cdot) (see [9]), we have

Corollary 4. *If a binary retract of an n -ary group (G, f) has an element of order $d|r$, then (G, f) is not a rs -group.*

Corollary 5. *Commutative idempotent n -ary rs -groups do not exist for $r \equiv 0 \pmod{(n-1)}$.*

Proof. Let (G, f) be a commutative idempotent n -ary group. By Hosszú's theorem there exists a group (G, \cdot) with the identity e such that (5) holds. Since the operation f is commutative and $x = \bar{x}$ for every $x \in G$, we have $f(y, e, \dots, e) = f(e, y, e, \dots, e) = y$ for all $y \in G$ (cf. [4] or [8]). This together with (5) gives $\theta y \cdot b = y$. Hence $b = e$ and $\theta y = y$. Therefore the operation f has the form $f(x_1^n) = x_1 \cdot x_2 \cdot \dots \cdot x_n$. Thus for every x in (G, \cdot) we have $x^{n-1} = e$, which proves (by Corollary 4) that for $r \equiv 0 \pmod{(n-1)}$ this n -ary group is not an rs -group. \square

The assumption that $(K, +)$ is an additive group of a field is essential. Indeed, as was mentioned above, an n -ary group (Z, f) derived from the additive group of an integral domain $(Z, +, \cdot)$ is not an rs -group.

As is well known (cf. for example [4] and [9]) some n -ary groups may be reduced to groups of arity $m < n$, i.e. for some n -ary groups (G, f) there exists an m -ary group (G, g) , where $n = s(m - 1) + 1$, $s > 1$, such that the following identity holds:

$$f(x_1^n) = g(g(\dots g(g(x_1^m), x_{m+1}^{2m-1}), \dots), x_{(s-1)(m-1)+2}^{s(m-1)+1}).$$

Such n -ary groups are not interesting since all results on such groups immediately follow from the results on the corresponding m -ary groups. Obviously the affine geometry defined by such n -ary groups is identical with the geometry defined by the corresponding m -ary groups.

Theorem 3. *For every $n \geq 3$ and $r \geq 2$ there exists a non-commutative n -ary rs -group which is not derived from any group of arity $m < n$.*

P r o o f. Let C be the set of complex numbers and let ω be the primitive $(n-1)$ -st root of unity. Then $G = C^3$ with the operation

$$(x_1, x_2, x_3) \bullet (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2 + x_1 y_3, x_3 + y_3)$$

is a group and $\theta(x_1, x_2, x_3) = (\omega x_1, \omega^2 x_2, \omega x_3)$ is its automorphism.

It is not difficult to verify that an n -ary group $\langle \omega, 1 \rangle$ -derived from (G, \bullet) is an rs -group for every $r \geq 2$. Since it is isomorphic to the n -ary group from the proof of Theorem 3 in [7], it is not reduced to any group of arity $m < n$. \square

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Authors' addresses: W. A. Dudek, Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland, e-mail: dudek@im.pwr.wroc.pl; Z. Stojaković, Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21 000 Novi Sad, Yugoslavia, e-mail: stojakov@eunet.yu.