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OSCILLATORY PROPERTIES OF SECOND ORDER HALF-LINEAR DIFFERENCE EQUATIONS

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Abstract. We study oscillatory properties of the second order half-linear difference equation

$$\Delta(r_k|\Delta y_k|^{\alpha-2}\Delta y_k) - p_k|y_{k+1}|^{\alpha-2}y_{k+1} = 0, \quad \alpha > 1. \tag{HL}$$

It will be shown that the basic facts of oscillation theory for this equation are essentially the same as those for the linear equation

$$\Delta(r_k\Delta y_k) - p_k y_{k+1} = 0.$$

We present here the Picone type identity, Reid Roundabout Theorem and Sturmian theory for equation (HL). Some oscillation criteria are also given.

Keywords: half-linear difference equation, Picone identity, Reid Roundabout Theorem, oscillation criteria

MSC 2000: 39A10

1. Introduction

In this paper we establish the basic facts of oscillation theory for the second order half-linear difference equation

$$\Delta(r_k\Phi(\Delta y_k)) - p_k\Phi(y_{k+1}) = 0, \tag{1}$$

where p_k and r_k are real-valued sequences with $r_k \neq 0$ and $\Phi(y) := |y|^{\alpha-1}\operatorname{sgn} y = |y|^{\alpha-2}y$, $\Phi(0) = 0$, with $\alpha > 1$.

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This work was motivated by some recent papers [4], [9] dealing with the oscillation theory of the second order half-linear differential equation

$$(r(t)\Phi(y'))' - p(t)\Phi(y) = 0, \tag{2}$$

where r and p are real-valued continuous functions with $r(t) > 0$. The terminology *half-linear* equations is justified by the following fact. If a sequence y (a function y) is a solution of (1) (of (2)) then for any real constant c the sequence cy (the function cy, respectively) is also a solution of the same equation. Note that there are most frequently referred [7], [11] as basis papers concerning oscillation theory of (2).

We will show that the basic oscillatory properties of (1) are essentially the same as those of the linear difference equation

$$\Delta(r_k \Delta y_k) - p_k y_{k+1} = 0, \tag{3}$$

which is a special case of (1) with $\alpha = 2$; the oscillatory properties of this linear equation can be found e.g. in [1].

The objects of our examinations in the present paper are especially:

- the generalized Picone identity. We establish this identity in the general form, which involves two half-linear difference operators (for the precise statement see the next section). It is a very useful tool for proving the following result.
- the discrete half-linear version of Reid's Roundabout Theorem. This theorem provides, among other, the following equivalence: An "α-degree" functional

  $$\mathcal{F}(\xi; m, n) = \sum_{k=m}^{n} [r_k |\Delta \xi_k|^\alpha + p_k |\xi_{k+1}|^\alpha]$$

  is positive definite on $[m, n]$ in the class of the so called admissible sequences if and only if the equation (1) is disconjugate on $[m, n]$. These results are presented in Section 3. The Sturmian theory (comparison and separation theorems) is also included in this section.
- oscillation criteria as an application of the above results. In Section 4 we present Leighton-Wintner type and Hinton-Lewis type criteria. The proof of these statements is based specifically on the relationship between the positive definiteness of the above functional and the disconjugacy of (1).

Note that the last section is devoted to remarks and comments.

2. The Picone identity

Consider the second order difference operators of the form

$$l[y_k] \equiv \Delta(r_k \Phi(\Delta y_k)) - p_k \Phi(y_{k+1})$$

and

$$L[z_k] \equiv \Delta(R_k \Phi(\Delta z_k)) - P_k \Phi(z_{k+1}),$$

where $k \in [m,n] \equiv \{m, m+1, \ldots, n\}$, $m, n \in \mathbb{Z}$, $m \leqslant n$, and p_k, P_k are real-valued sequences defined on $[m,n]$. Sequences r_k, R_k are real-valued and defined on $[m, n+1]$ with $r_k \neq 0$, $R_k \neq 0$ on this interval.

Now we can formulate a statement playing an important role in the proof of Theorem 1. The idea is to rewrite the functional $\mathcal{F}$ associated with the disconjugate equation (1) into a form which in the linear case corresponds to the "completion to the square" which then shows equivalence of the disconjugacy of (1) with the positive definiteness of $\mathcal{F}$. Note that our version of the Picone identity is quite general and we will use only its special case.

Lemma 1 (Picone Type Identity). *Let y_k, z_k be defined on $[m, n+2]$ and let $z_k \neq 0$ for $k \in [m, n+1]$. Then the equality*

$$\begin{aligned}
(4) \qquad \Delta\left\{\frac{y_k}{\Phi(z_k)}[\Phi(z_k) r_k \Phi(\Delta y_k) - \Phi(y_k) R_k \Phi(\Delta z_k)]\right\}
&= (p_k - P_k)|y_{k+1}|^\alpha + (r_k - R_k)|\Delta y_k|^\alpha \\
&\quad + \frac{y_{k+1}}{\Phi(z_{k+1})}\{l[y_k]\Phi(z_{k+1}) - L[z_k]\Phi(y_{k+1})\} \\
&\quad + \left\{R_k|\Delta y_k|^\alpha - \frac{R_k \Phi(\Delta z_k)}{\Phi(z_{k+1})}|y_{k+1}|^\alpha + \frac{R_k \Phi(\Delta z_k)}{\Phi(z_k)}|y_k|^\alpha\right\}
\end{aligned}$$

holds for $k \in [m,n]$.

Proof. For $k \in [m,n]$ we have

$$\begin{aligned}
&\Delta[y_k r_k \Phi(\Delta y_k)] - y_{k+1} l[y_k] \\
&\quad = y_{k+1}\Delta(r_k \Phi(\Delta y_k)) + \Delta y_k r_k \Phi(\Delta y_k) - y_{k+1}\Delta(r_k \Phi(\Delta y_k)) + y_{k+1} p_k \Phi(y_{k+1}) \\
&\quad = p_k |y_{k+1}|^\alpha + r_k |\Delta y_k|^\alpha.
\end{aligned}$$

Further, again for $k \in [m, n]$, we have

$$
\begin{aligned}
&L[z_k]\Phi(y_{k+1})\frac{y_{k+1}}{\Phi(z_{k+1})} - \Delta\left[\frac{y_k}{\Phi(z_k)}\Phi(y_k)R_k\Phi(\Delta z_k)\right] \\
&\quad - R_k|\Delta y_k|^\alpha + \frac{R_k\Phi(\Delta z_k)}{\Phi(z_{k+1})}y_{k+1}\Phi(y_{k+1}) - \frac{R_k\Phi(\Delta z_k)}{\Phi(z_k)}y_k\Phi(y_k) \\
&= \frac{\Delta(R_k\Phi(\Delta z_k))}{\Phi(z_{k+1})}y_{k+1}\Phi(y_{k+1}) - P_k y_{k+1}\Phi(y_{k+1}) - \frac{\Delta y_k\Phi(z_k) - y_k\Delta\Phi(z_k)}{\Phi(z_k)\Phi(z_{k+1})} \\
&\quad \times \Phi(y_k)R_k\Phi(\Delta z_k) - \frac{y_{k+1}}{\Phi(z_{k+1})}\Phi(y_{k+1})\Delta(R_k\Phi(\Delta z_k)) \\
&\quad - \frac{y_{k+1}}{\Phi(z_{k+1})}\Delta\Phi(y_k)R_k\Phi(\Delta z_k) \\
&\quad - R_k|\Delta y_k|^\alpha + \frac{R_k\Phi(\Delta z_k)}{\Phi(z_{k+1})}y_{k+1}\Phi(y_{k+1}) - \frac{R_k\Phi(\Delta z_k)}{\Phi(z_k)}y_k\Phi(y_k) \\
&= -P_k|y_{k+1}|^\alpha - R_k|\Delta y_k|^\alpha \\
&\quad + \frac{1}{\Phi(z_k)\Phi(z_{k+1})}[-R_k\Delta y_k\Phi(y_k)\Phi(z_k)\Phi(\Delta z_k) \\
&\quad + R_k y_k\Phi(y_k)\Phi(\Delta z_k)\Delta\Phi(z_k) - R_k y_{k+1}\Delta\Phi(y_k)\Phi(z_k)\Phi(\Delta z_k) \\
&\quad + R_k y_{k+1}\Phi(y_{k+1})\Phi(z_k)\Phi(\Delta z_k) - R_k y_k\Phi(y_k)\Phi(z_{k+1})\Phi(\Delta z_k)] \\
&= -P_k|y_{k+1}|^\alpha - R_k|\Delta y_k|^\alpha \\
&\quad + \frac{1}{\Phi(z_k)\Phi(z_{k+1})}[R_k y_k\Phi(y_k)\Phi(z_k)\Phi(\Delta z_k) - R_k y_k\Phi(y_k)\Phi(z_k)\Phi(\Delta z_k)] \\
&= -P_k|y_{k+1}|^\alpha - R_k|\Delta y_k|^\alpha.
\end{aligned}
$$

Combining these two equalities we get the desired result. □

The last summand of (4) can be rewritten as $\frac{R_k z_k}{z_{k+1}}G(y, z)$, where

$$G(y, z) := \frac{z_{k+1}}{z_k}|\Delta y_k|^\alpha - \frac{z_{k+1}\Phi(\Delta z_k)}{z_k\Phi(z_{k+1})}|y_{k+1}|^\alpha + \frac{z_{k+1}\Phi(\Delta z_k)}{z_k\Phi(z_k)}|y_k|^\alpha.$$

Using this fact we have the following lemma.

Lemma 2. *Let y_k, z_k be defined on $[m, n+1]$ and let $z_k \neq 0$ on this interval. Then*

$$G(y, z) \geqslant 0$$

for $k \in [m, n]$, where the equality holds if and only if $\Delta y_k = y_k(\Delta z_k/z_k)$.

Proof. It is sufficient to verify the inequality

$$\text{(5)} \qquad \frac{z_{k+1}}{z_k}|\Delta y_k|^\alpha + \frac{z_{k+1}|\Delta z_k|^{\alpha-2}\Delta z_k}{|z_k|^{\alpha-2}z_k^2}|y_k|^\alpha \geqslant \frac{z_{k+1}|\Delta z_k|^{\alpha-2}\Delta z_k}{z_k|z_{k+1}|^{\alpha-2}z_{k+1}}|y_{k+1}|^\alpha$$

for $k \in [m, n]$.

Denote $u_k := z_k/z_{k+1}$. Then inequality (5) assumes the form

$$\frac{|\Delta y_k|^\alpha}{u_k} + \frac{|1-u_k|^{\alpha-2}(1-u_k)}{|u_k|^\alpha}|y_k|^\alpha \geqslant \frac{1}{u_k}|1-u_k|^{\alpha-2}(1-u_k)|y_{k+1}|^\alpha.$$

First we prove the statement of Lemma (2) for $k \in [m,n]$ such that $u_k = 1$. In this case clearly $G(y,z) = |\Delta y_k|^\alpha \geqslant 0$, where equality holds if and only if $\Delta y_k = 0$, which holds if and only if $\Delta y_k = y_k(\Delta z_k/z_k)$.

In the remainder of this proof, when we write u we mean u_k (the same holds for the sequences a, b, v and v_0).

Now, denote

$$\begin{aligned} J_0 &:= \{k \in [m,n];\ z_k \neq z_{k+1}\}, \\ J_1 &:= \{k \in J_0;\ y_k = 0\}, \\ J_2 &:= \{k \in J_0;\ \Delta y_k = 0\}. \end{aligned}$$

Hence we have four cases:

Case I: $k \in J_0 \setminus (J_1 \cup J_2)$.

Putting $a = \Delta y_k$, $b = y_k$ we get $y_{k+1} = a + b$. Inequality (6) now leads to

$$\frac{|a|^\alpha}{u} + \frac{|1-u|^{\alpha-2}(1-u)}{|u|^\alpha}\,|b|^\alpha \geqslant \frac{1}{u}\,|1-u|^{\alpha-2}(1-u)|a+b|^\alpha$$

and by dividing it by $|b|^\alpha$ the inequality (6) assumes the form

$$\left|\frac{a}{b}\right|^\alpha \frac{1}{u} + \frac{1}{|u|^\alpha}\,|1-u|^{\alpha-2}(1-u) \geqslant \frac{1}{u}\,|1-u|^{\alpha-2}(1-u)\left|1+\frac{a}{b}\right|^\alpha.$$

Now, denote

$$H(v;u) := \frac{|v|^\alpha}{u} - \frac{1}{u}\,|1-u|^{\alpha-2}(1-u)|1+v|^\alpha + \frac{1}{|u|^\alpha}\,|1-u|^{\alpha-2}(1-u),$$

where $v := a/b$. For $v = v_0 := (1-u)/u$ the following equality holds:

$$\begin{aligned} H(v_0;u) &= \frac{|1-u|^\alpha}{u|u|^\alpha} - \frac{1}{u|u|^\alpha}|1-u|^{\alpha-2}(1-u) + \frac{1}{|u|^\alpha}|1-u|^{\alpha-2}(1-u) \\ &= \frac{|1-u|^{\alpha-2}}{u|u|^\alpha}\big((1-u)^2 - (1-u) + u - u^2\big) = 0. \end{aligned}$$

Differentiating H with respect to v we obtain

$$H_v(v;u) = \alpha\frac{|v|^{\alpha-1}\operatorname{sgn} v}{u} - \alpha\frac{1}{u}|1-u|^{\alpha-2}(1-u)|1+v|^{\alpha-1}\operatorname{sgn}(1+v)$$

and hence

$$H_v(v_0;u) = \alpha\frac{|1-u|^{\alpha-1}\operatorname{sgn}(1-u)}{|u|^{\alpha-1}u\operatorname{sgn}u} - \alpha\frac{|1-u|^{\alpha-2}(1-u)}{|u|^{\alpha-1}u\operatorname{sgn}u} = 0.$$

Further, we get

$$H_{vv}(v;u) = \alpha(\alpha-1)\frac{1}{u}\big(|v|^{\alpha-2} - |1-u|^{\alpha-2}(1-u)|1+v|^{\alpha-2}\big).$$

Consequently,

$$\begin{aligned}H_{vv}(v_0;u) &= \alpha(\alpha-1)\frac{1}{u}\left(\frac{|1-u|^{\alpha-2}}{|u|^{\alpha-2}} - \frac{|1-u|^{\alpha-2}(1-u)}{|u|^{\alpha-2}}\right)\\ &= \alpha(\alpha-1)\frac{|1-u|^{\alpha-2}}{u|u|^{\alpha-2}}(1-1+u)\\ &= \alpha(\alpha-1)\frac{|1-u|^{\alpha-2}}{|u|^{\alpha-2}} > 0.\end{aligned}$$

Since

$$\begin{aligned}H_v(v;u) = 0 &\Longleftrightarrow |v|^{\alpha-1}\operatorname{sgn}v = |1-u|^{\alpha-2}(1-u)|1+v|^{\alpha-1}\operatorname{sgn}(1+v)\\ &\Longleftrightarrow |v|^{\alpha-1}\operatorname{sgn}v = |(1-u)(1+v)|^{\alpha-1}\operatorname{sgn}[(1-u)(1+v)]\\ &\Longleftrightarrow v = 1+v-u-uv\\ &\Longleftrightarrow v = \frac{1-u}{u}\\ &\Longleftrightarrow v = v_0\end{aligned}$$

holds, hence H_v has just a single zero v_0. Note that this case occurs if and only if $\Delta y_k = y_k(\Delta z_k/z_k)$. In the opposite case, $H(v;u) > 0$.

Case II: $k \in J_1 \setminus J_2$.

Suppose, by contradiction (see the inequality (6)), that

$$\frac{|\Delta y_k|^\alpha}{u} \leqslant \frac{1}{u}|1-u|^{\alpha-2}(1-u)|\Delta y_k|^\alpha.$$

Therefore

$$\frac{1}{u} + |1-u|^{\alpha-2} \leqslant \frac{|1-u|^{\alpha-2}}{u}.$$

Further, we distinguish the following particular cases:

- if $u > 1$, then the following inequality holds:

$$1 + u|1-u|^{\alpha-2} \leqslant |1-u|^{\alpha-2}$$
$$\frac{1}{|1-u|^{\alpha-1}} \leqslant \operatorname{sgn}(1-u),$$

  a contradiction, since the left hand side is positive,

- for $0 < u < 1$ the same computation as above holds and hence we get a contradiction, since $1/|1-u|^{\alpha-1} \not\leqslant 1$, where $0 < |1-u|^{\alpha-1} < 1$,
- for $u < 0$ we have
$$\frac{1}{|1-u|^{\alpha-1}} \geqslant \operatorname{sgn}(1-u),$$
again a contradiction, since $1/|1-u|^{\alpha-1} \not\geqslant 1$, where $|1-u|^{\alpha-1} > 1$.

Case III: $k \in J_2 \setminus J_1$.

Assume, by way of contradiction, that
$$\frac{|1-u|^{\alpha-2}(1-u)}{|u|^{\alpha}}|y_k|^{\alpha} \leqslant \frac{1}{u}|1-u|^{\alpha-2}(1-u)|y_k|^{\alpha}.$$
Consequently,
$$\frac{1-u}{|u|^{\alpha}} \leqslant \frac{1-u}{u}.$$
Similarly as in Case II we have
- $u > 1$: $1/|u|^{\alpha} \geqslant 1/u \Longrightarrow u \geqslant |u|^{\alpha}$,
- $0 < u < 1$: $1/|u|^{\alpha} \leqslant 1/u \Longrightarrow u \leqslant |u|^{\alpha}$,
- $u < 0$: $1/|u|^{\alpha} \leqslant 1/u \Longrightarrow u \geqslant |u|^{\alpha}$.

Obviously in every particular case we again come to a contradiction.

Case IV: $k \in J_1 \cap J_2$.

Here we clearly see that $G(y,z) = 0$, since $y_k = 0$ and $\Delta y_k = 0$. Note that this case occurs if and only if $\Delta y_k = y_k(\Delta z_k/z_k)$. The proof is complete. □

Remark (Linear case). If we put $\alpha = 2$, we get
$$G(y,z) = \left(\Delta y_k - \frac{\Delta z_k}{z_k} y_k\right)^2.$$

3. Roundabout Theorem

In this section we consider equation (1) on the interval $[m,n]$ with $r_k \neq 0$ on $[m,n+1]$. First of all we define and recall some important concepts. An interval $(m,m+1]$ is said to contain *the generalized zero* of a solution y of (1), if $y_m \neq 0$ and $r_m y_m y_{m+1} \leqslant 0$. Equation (1) is said to be *disconjugate* on $[m,n]$ provided any solution of this equation has at most one generalized zero on $(m,n+1]$ and the solution $\widetilde{y}$ satisfying $\widetilde{y}_m = 0$ has no generalized zeros on $(m,n+1]$. Define a class U of the so called *admissible sequences* by
$$U(m,n) = \{\xi \mid \xi\colon\ [m,n+2] \longrightarrow \mathbb{R} \ \text{ such that } \ \xi_m = \xi_{n+1} = 0\}.$$

Then define an "α-degree" functional $\mathcal{F}$ on $U(m,n)$ by

$$\mathcal{F}(\xi; m, n) = \sum_{k=m}^{n} [r_k |\Delta \xi_k|^{\alpha} + p_k |\xi_{k+1}|^{\alpha}].$$

We say $\mathcal{F}$ is *positive definite on* $U(m,n)$ provided $\mathcal{F}(\xi) \geqslant 0$ for all $\xi \in U(m,n)$ and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

Now we are in a position to formulate one of the main results of this paper, the discrete half-linear version of the Reid type Roundabout Theorem.

Theorem 1 (Roundabout Theorem). *The following statements are equivalent:*

(i) *Equation* (1) *is disconjugate on* $[m,n]$.

(ii) *Equation* (1) *has a solution* y *without generalized zeros on* $[m, n+1]$.

(iii) *The generalized Riccati equation*

$$\Delta w_k = p_k - w_k \left(1 - \frac{\Phi(r_k)}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))} \right) \tag{7}$$

or, equivalently,

$$w_{k+1} = p_k + \frac{w_k \Phi(r_k)}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))},$$

where $w_k = r_k \Phi(\Delta y_k)/\Phi(y_k)$ *(the Riccati type substitution) and* Φ^{-1} *is the inverse of* Φ*, has a solution* w_k *on* $[m,n]$ *with* $r_k + w_k > 0$ *on* $[m,n]$.

(iv) $\mathcal{F}$ *is positive definite on* $U(m,n)$.

Proof. (i) $\Longrightarrow$ (ii): The proof of this implication is essentially the same as in the linear case. Indeed, let z_k be a solution of (1) given by the initial conditions

$$z_m = 0, \quad z_{m+1} = 1.$$

It follows that $r_k z_k z_{k+1} > 0$ for $k \in [m+1, n]$. Consider a solution $z_k^{[\varepsilon]}$ satisfying the initial conditions

$$z_m^{[\varepsilon]} = \varepsilon r_m, \quad z_{m+1}^{[\varepsilon]} = 1.$$

Then

$$z_k^{[\varepsilon]} \to z_k \text{ as } \varepsilon \to 0.$$

If we choose $\varepsilon > 0$ sufficiently small, then $y_k \equiv z_k^{[\varepsilon]}$ satisfies

$$r_k y_k y_{k+1} > 0 \text{ for } k \in [m,n],$$

i.e., y has no generalized zero on $[m, n+1]$.

(ii) $\Longrightarrow$ (iii): Assume that z_k is a solution of (1) with $r_k z_k z_{k+1} > 0$ on $[m, n]$. Use the Riccati type substitution $w_k = r_k \Phi(\Delta z_k)/\Phi(z_k)$. Than we have

$$\begin{aligned}\frac{r_k z_{k+1}}{z_k} &= r_k \left(\frac{z_k + \Delta z_k}{z_k}\right) = r_k \left(1 + \frac{\Phi^{-1}(w_k)}{\Phi^{-1}(r_k)}\right) \\ &= \frac{r_k}{\Phi^{-1}(r_k)} + (\Phi^{-1}(r_k) + \Phi^{-1}(w_k)).\end{aligned} \tag{8}$$

Since

$$\Phi\left(\frac{z_k}{z_{k+1}}\right) = \Phi(r_k)\Phi\left(\frac{z_k}{r_k z_{k+1}}\right) = \frac{r_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))},$$

we obtain

$$\Delta w_k = p_k - w_k \left(1 - \frac{\Phi(z_k)}{\Phi(z_{k+1})}\right) = p_k - w_k \left(1 - \frac{r_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))}\right).$$

Now, (8) clearly implies $r_k + w_k > 0$ and hence (iii) holds.

(iii) $\Longrightarrow$ (iv): Assume that w_k is a solution of (7) with $r_k + w_k > 0$. Note that then z_k given by $w_k = r_k \Phi(\Delta z_k)/\Phi(z_k)$, i.e. $\Delta z_k = \Phi^{-1}(w_k/r_k) z_k$, Φ^{-1} being the inverse function of Φ, is a solution of (1). From the Picone identity (4) applied to the case $p_k \equiv P_k$, $r_k \equiv R_k$, $y_k = \xi_k$ and $w_k = r_k \Phi(\Delta z_k)/\Phi(z_k)$ (see equality (9)) we obtain

$$\Delta[\xi_k r_k \Phi(\Delta \xi_k)] - \Delta[|\xi_k|^\alpha w_k] = \xi_{k+1} \Delta(r_k \Phi(\Delta \xi_k)) - p_k |\xi_{k+1}|^\alpha + \widetilde{G},$$

where

$$\widetilde{G} = \widetilde{G}(\xi, w) = r_k |\Delta \xi_k|^\alpha - \frac{w_k \Phi(r_k)}{(r_k + w_k)^{\alpha-1}} |\xi_{k+1}|^\alpha + w_k |\xi_k|^\alpha.$$

Hence

$$r_k |\Delta \xi_k|^\alpha + p_k |\xi_{k+1}|^\alpha = \Delta[w_k |\xi_k|^\alpha] + \widetilde{G}.$$

The summation of the above given equality from m to n yields

$$\mathcal{F}(\xi) = [w_k |\xi_k|^\alpha]_{k=m}^{n+1} + \sum_{k=m}^{n} \widetilde{G}(\xi, w).$$

Then $\mathcal{F}(\xi) \geqslant 0$, since $r_k z_{k+1}/z_k = r_k + w_k > 0$ and Lemma 2 holds. In addition, if $\mathcal{F}(\xi) = 0$, then again by Lemma 2, $\Delta \xi_k = (\Delta z_k / z_k) \xi_k$. Further, we have $\xi_m = 0$ and therefore $\xi \equiv 0$. Consequently, $\mathcal{F}(\xi) > 0$ for all nontrivial admissible sequences.

(iv) $\Longrightarrow$ (i): Suppose, by contradiction, that (1) is not disconjugate on $[m, n]$. Then there exists a nontrivial solution y of (1) such that

$$\begin{aligned} r_M y_M y_{M+1} \leqslant 0, &\quad y_{M+1} \neq 0, \\ r_N y_N y_{N+1} \leqslant 0, &\quad y_N \neq 0, \end{aligned}$$

where $m+1 \leqslant M+1 < N \leqslant n$. Define

$$\xi_k = \begin{cases} 0 & \text{for } k = m, \dots, M, \\ y_k & \text{for } k = M+1, \dots, N, \\ 0 & \text{for } k = N+1, \dots, n+1. \end{cases}$$

Then ξ_k is a nontrivial admissible sequence and hence $\mathcal{F}(\xi) > 0$. Using summation by parts we obtain

$$\begin{aligned}
\mathcal{F}(\xi) = &\sum_{k=m}^{n} [r_k|\Delta\xi_k|^\alpha + p_k|\xi_{k+1}|^\alpha] \\
&= [\xi_k r_k \Phi(\Delta\xi_k)]_{k=m}^{n+1} - \sum_{k=m}^{n} \xi_{k+1} l[\xi_k] = -\sum_{k=M}^{N-1} \xi_{k+1} l[\xi_k] \\
&= y_{M+1}[p_M \Phi(y_{M+1}) - \Delta(r_M \Phi(\Delta\xi_M))] \\
&\quad + y_N[p_{N-1}\Phi(y_N) - \Delta(r_{N-1}\Phi(\Delta\xi_{N-1}))] \\
&= y_{M+1}[\Delta(r_M\Phi(\Delta y_M)) - r_{M+1}\Phi(\Delta\xi_{M+1}) + r_M\Phi(\Delta\xi_M)] \\
&\quad + y_N[\Delta(r_{N-1}\Phi(\Delta y_{N-1})) - r_N\Phi(\Delta\xi_N) + r_{N-1}\Phi(\Delta\xi_{N-1})] \\
&= y_{M+1}[r_{M+1}\Phi(\Delta y_{M+1}) - r_M\Phi(\Delta y_M) - r_{M+1}\Phi(\Delta y_{M+1}) + r_M\Phi(y_{M+1})] \\
&\quad + y_N[r_N\Phi(\Delta y_N) - r_{N-1}\Phi(\Delta y_{N-1}) + r_N\Phi(y_N) + r_{N-1}\Phi(\Delta y_{N-1})] \\
&= G_1 + G_2,
\end{aligned}$$

where

$$G_1 = G_1(y_M, y_{M+1}; r_M) = y_{M+1} r_M \Phi(y_{M+1}) - y_{M+1} r_M \Phi(\Delta y_M)$$

and

$$G_2 = G_2(y_N, y_{N+1}; r_N) = y_N r_N \Phi(\Delta y_N) + y_N r_N \Phi(y_N).$$

To show that $\mathcal{F}(\xi) \leqslant 0$ it remains to verify that $G_1 \leqslant 0$ and $G_2 \leqslant 0$. Let us examine for example the function G_2. It means that we shall to check the inequality

$$y_N r_N \Phi(\Delta y_N) \leqslant -y_N r_N \Phi(y_N).$$

It holds if and only if

$$r_N \Phi\Big(\frac{\Delta y_N}{y_N}\Big) \leqslant -r_N.$$

Now, if $\Delta y_N = 0$, then we get $G_2 = r_N|y_N|^\alpha$. Hence r_N must be negative, since we assume $r_N y_N^2 \leqslant 0$. Consequently, $G_2 < 0$. Further, let $\Delta y_N \neq 0$. Putting $x = y_{N+1}/y_N$ we obtain

$$\widetilde{G}_2(x; r_N) := r_N|x-1|^{\alpha-1}\operatorname{sgn}(x-1) + r_N.$$

Note that $G_2 < 0$ ($G_2 = 0$) if and only if $\widetilde{G}_2 < 0$ ($\widetilde{G}_2 = 0$). If $y_{N+1} = 0$ then $x = 0$ and hence $\widetilde{G}_2(0; r_N) = 0$. Differentiating $\widetilde{G}_2$ with respect to x we obtain

$$\frac{\partial \widetilde{G}_2}{\partial x} = (\alpha - 1) r_N |x - 1|^{\alpha - 2}.$$

Now, we distinguish the following particular cases:

- $x > 0 \Longleftrightarrow y_N y_{N+1} > 0 \Longleftrightarrow r_N < 0 \Longleftrightarrow \partial \widetilde{G}_2 / \partial x < 0$,
- $x < 0 \Longleftrightarrow y_N y_{N+1} < 0 \Longleftrightarrow r_N > 0 \Longleftrightarrow \partial \widetilde{G}_2 / \partial x > 0$.

Therefore we get $G_2 \leqslant 0$.

Similarly one can verify that $G_1 \leqslant 0$ holds. Summarizing the above computations we have $\mathcal{F}(\xi) = G_1 + G_2 \leqslant 0$, a contradiction. Hence (i) holds. □

The end of this section is devoted to Sturmian theory. Consider two equations $l[y_k] = 0$ and $L[z_k] = 0$ (the operators l, L are defined at the beginning of Section 2). Denote

$$\mathcal{F}_{R,P}(\xi) := \sum_{k=m}^{n} [R_k |\Delta \xi_k|^\alpha + P_k |\xi_{k+1}|^\alpha].$$

Then we have the following versions of Sturmian theorems for half-linear difference equations.

Theorem 2 (Sturm's Comparison Theorem). *Suppose that we have $R_k \geqslant r_k$ and $P_k \geqslant p_k$ for $k \in [m, n]$. Then, if $l[y_k] = 0$ is disconjugate on $[m, n]$, then $L[z_k] = 0$ is also disconjugate on $[m, n]$.*

Proof. Suppose that $l[y_k] = 0$ is disconjugate on $[m, n]$. Then Theorem 1 yields $\mathcal{F}(\xi) > 0$ for all admissible sequences ξ. For such a ξ we also have

$$\mathcal{F}_{R,P}(\xi) \geqslant \mathcal{F}(\xi) > 0.$$

Hence $\mathcal{F}_{R,P}(\xi) > 0$ and thus $L[z_k] = 0$ is disconjugate on $[m, n]$ by Theorem 1. □

As far as the separation result is concerned, note that the implication (ii) ⇒ (i) from Theorem 1 is a Sturmian type separation theorem. Hence we have the following statement.

Theorem 3 (Sturm's Separation Theorem). *Two nontrivial solutions $y^{[1]}$ and $y^{[2]}$ of $l[y_k] = 0$, which are not proportional, cannot have a common zero. If $y^{[1]}$ satisfying $y^{[1]}_m = 0$ has a generalized zero in $(n, n+1]$, then $y^{[2]}$ has a generalized zero in $(m, n+1]$. If $y^{[1]}$ has generalized zeros in $(m, m+1]$ and $(n, n+1]$, then $y^{[2]}$ has a generalized zero in $(m, n+1]$.*

P r o o f. It is sufficient to prove the part concerning the common zero of nonproportional solutions since the remaining part follows from Theorem 1. Suppose, by contradiction, that $y_l^{[1]} = 0 = y_l^{[2]}$ for some $l \in [m, n]$. Let $\widetilde{y}$ be a solution of $l[y_k] = 0$ such that $\widetilde{y}_l = 0$, $\widetilde{y}_{l+1} = 1$. Then $y^{[1]} := A\widetilde{y}$ and $y^{[2]} := B\widetilde{y}$, where A, B are suitable nonzero constants, are also nontrivial solutions of $l[y_k] = 0$ satisfying

$$y_l^{[1]} = 0,\ \ y_{l+1}^{[1]} = A \ \text{ and } \ y_l^{[2]} = 0,\ \ y_{l+1}^{[2]} = B,$$

respectively. Hence $y^{[1]} = Cy^{[2]}$, where $C = A/B$, a contradiction. □

4. Oscillation criteria

In this section we give oscillation criteria for equation (1), $k \in [m, \infty)$, with $r_k > 0$ on this interval.

First of all, let us recall some important concepts. Equation (1) is said to be *nonoscillatory* if there exists $K \geqslant m$ such that (1) is disconjugate on $[K, N]$ for every $N > K$. In the opposite case (1) is said to be *oscillatory.* Oscillation of (1) may be equivalently defined as follows. A nontrivial solution of (1) is called *oscillatory* if it has infinitely many generalized zeros. In view of the fact that Sturm's Separation Theorem holds, we have the following equivalence: Any solution of (1) is oscillatory if and only if every solution of (1) is oscillatory. Hence we can speak about *oscillation of equation* (1).

In order to prove our oscillation criteria we need, among other, the following auxiliary statement which is proved in [3].

Lemma 3 (The second mean value theorem of "summation calculus")**.** *Let a sequence a_k be monotonic for $k \in [K, L+1]$. Then for any sequence b_k there exist $N_1, N_2 \in [K, L]$ such that*

$$\sum_{j=K}^{L} a_{j+1} b_j \leqslant a_K \sum_{i=K}^{N_1-1} b_i + a_{L+1} \sum_{i=N_1}^{L} b_i,$$
$$\sum_{j=K}^{L} a_{j+1} b_j \geqslant a_K \sum_{i=K}^{N_2-1} b_i + a_{L+1} \sum_{i=N_2}^{L} b_i.$$

Theorem 4 (Leighton-Wintner type oscillation criterion)**.** *Suppose that*

$$\sum_{j=m}^{\infty} r_j^{1-\beta} = \infty \tag{9}$$

(β *is the conjugate number of* α, *i.e.*, $1/\alpha + 1/\beta = 1$) *and*

$$\sum_{j=m}^{\infty} p_j = -\infty. \tag{10}$$

Then (1) *is oscillatory.*

Proof. According to Theorem 1, it is sufficient to find for any $K \geqslant m$ a sequence y satisfying $y_k = 0$ for $k \leqslant K$ and $k \geqslant N+1$, where $N > K$ (then y is admissible), such that

$$\mathcal{F}(y; K, N) = \sum_{k=K}^{N} [r_k |\Delta y_k|^\alpha + p_k |y_{k+1}|^\alpha] \leqslant 0.$$

Let $K < L < M < N$. Define the sequence y_k by

$$y_k = \begin{cases} 0 & \text{for } k = m, \dots, K, \\ \left(\sum_{j=K}^{k-1} r_j^{1-\beta}\right)\left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{-1} & \text{for } k = K+1, \dots, L+1, \\ 1 & \text{for } k = L+1, \dots, M, \\ \left(\sum_{j=k}^{N} r_j^{1-\beta}\right)\left(\sum_{j=M}^{N} r_j^{1-\beta}\right)^{-1} & \text{for } k = M, \dots, N, \\ 0 & k \geqslant N+1. \end{cases}$$

Using summation by parts we have

$$\begin{aligned} \mathcal{F}(y; K, N) = {} & \sum_{k=K}^{N} [r_k |\Delta y_k|^\alpha + p_k |y_{k+1}|^\alpha] \\ = {} & \sum_{k=K}^{L-1} r_k |\Delta y_k|^\alpha + r_L |\Delta y_L|^\alpha + \sum_{k=L+1}^{M-1} r_k |\Delta y_k|^\alpha + \sum_{k=M}^{N} r_k |\Delta y_k|^\alpha \\ & + \sum_{k=K}^{N} p_k |y_{k+1}|^\alpha \\ = {} & l[y_k r_k \Phi(\Delta y_k)]_{k=K}^{L} - \sum_{j=K}^{L-1} y_{k+1} \Delta(r_k \Phi(\Delta y_k)) \\ & + r_L \bigl(r_L^{1-\beta}\bigr)^\alpha \left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{-\alpha} + [y_k r_k \Phi(\Delta y_k)]_{k=M}^{N+1} \end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=M}^{N} y_{k+1}\Delta(r_k\Phi(\Delta y_k)) + \sum_{k=K}^{N} p_k|y_{k+1}|^{\alpha} \\
&= y_L r_L \Phi(\Delta y_L) + r_L^{1-\beta}\left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{-\alpha} + y_M r_M \Phi(\Delta y_M) \\
&\quad + \sum_{k=K}^{N} p_k|y_{k+1}|^{\alpha} \\
&= \left(\sum_{k=K}^{L} r_k^{1-\beta}\right)^{1-\alpha} + \left(\sum_{k=M}^{N} r_k^{1-\beta}\right)^{1-\alpha} + \sum_{k=K}^{L} p_k|y_{k+1}|^{\alpha} \\
&\quad + \sum_{k=L+1}^{M-1} p_k + \sum_{k=M}^{N} p_k|y_{k+1}|^{\alpha}.
\end{aligned}
$$

Further, the sequence y is strictly monotonic on $[K, L+1]$ and $[M, N+1]$ since

$$
\Delta y_k = r_k^{1-\beta}\left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{-1} > 0 \ \text{ for } \ k \in [K, L]
$$

and

$$
\Delta y_k = \ -r_k^{1-\beta}\left(\sum_{j=M}^{N} r_j^{1-\beta}\right)^{-1} < 0 \ \text{ for } \ k \in [M, N],
$$

and therefore $|y|^{\alpha}$ is also strictly monotonic. Hence, by Lemma 3, there exists $N_1 \in [K, L]$ such that

$$
\sum_{k=K}^{L} p_k|y_{k+1}|^{\alpha} \leqslant |y_K|^{\alpha} \sum_{k=K}^{N_1-1} p_k + |y_{L+1}|^{\alpha} \sum_{k=N_1}^{L} p_k = \sum_{k=N_1}^{L} p_k,
$$

and similarly there exists $N_2 \in [M, N]$ for which

$$
\sum_{k=M}^{N} p_k|y_{k+1}|^{\alpha} \leqslant |y_M|^{\alpha} \sum_{k=M}^{N_2-1} p_k + |y_{N+1}|^{\alpha} \sum_{k=N_2}^{N} p_k = \sum_{k=M}^{N_2-1} p_k.
$$

Using these estimates we have

$$
\mathcal{F}(y; K, N) \leqslant \left(\sum_{k=K}^{L} r_k^{1-\beta}\right)^{1-\alpha} + \left(\sum_{k=M}^{N} r_k^{1-\beta}\right)^{1-\alpha} + \sum_{k=N_1}^{N_2-1} p_k.
$$

Now, denote $A = \Big(\sum\limits_{j=K}^{L} r_j^{1-\beta}\Big)^{1-\alpha}$ and let $\varepsilon > 0$ be arbitrary. According to (10), the integer M can be chosen in such a way that

$$\sum_{j=N_1}^{k} p_j \leqslant -(A+\varepsilon)$$

whenever $k > M$. Since (9) holds,

$$\left(\sum_{k=M}^{N} r_k^{1-\beta}\right)^{1-\alpha} \leqslant \varepsilon$$

if N is sufficiently large.

Summarizing the above estimates, if M, N are sufficiently large, then we have

$$\mathcal{F}(y; K, N) \leqslant A + \varepsilon - (A+\varepsilon) = 0,$$

which completes the proof. □

In the case when $\sum\limits_{j=m}^{\infty} p_j$ is convergent, we can use the following criterion.

Theorem 5 (Hinton-Lewis type oscillation criterion)**.** *Suppose that* (9) *holds and*

$$\lim_{k\to\infty} \left(\sum_{j=m}^{k} r_j^{1-\beta}\right)^{\alpha-1} \left(\sum_{j=k}^{\infty} p_j\right) < -1. \tag{11}$$

Then (1) *is oscillatory.*

P r o o f. Let the sequence y be the same as in the proof of the previous theorem. Hence we have

$$\begin{aligned}
\mathcal{F}(y) &\leqslant \left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{1-\alpha} + \left(\sum_{j=M}^{N} r_j^{1-\beta}\right)^{1-\alpha} + \sum_{j=N_1}^{N_2-1} p_k \\
&= \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{1-\alpha} \left[\left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1} \left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{1-\alpha}\right. \\
&\quad \left. + \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1} \left(\sum_{j=N_1}^{N_2-1} p_j\right) + \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1} \left(\sum_{j=M}^{N} r_j^{1-\beta}\right)^{1-\alpha}\right],
\end{aligned}$$

where $m \leqslant K < L < M < N$, $N_1 \in [K, L]$ and $N_2 \in [M, N]$.

Now, let $\varepsilon > 0$ be such that lim in (11) is less than or equal to $-1-4\varepsilon$. According to (11), K may be chosen in such a way that

$$\left(\sum_{j=m}^{k} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=k}^{\infty} p_j\right) \leqslant -1-3\varepsilon \tag{12}$$

for $k > K$. Obviously there exists $L > K$ such that

$$\left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{1-\alpha} \leqslant 1+\varepsilon.$$

In view of the fact that (12) holds, there exists $M > L$ such that

$$\left(\sum_{j=m}^{k} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=k}^{l} p_j\right) \leqslant -1-2\varepsilon$$

for $l \geqslant M$. Finally, since $\sum\limits_{j=m}^{\infty} r_j^{1-\beta} = \infty$ holds, we have

$$\left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=M}^{N} r_j^{1-\beta}\right)^{1-\alpha} \leqslant \varepsilon$$

if $N > M$ is sufficiently large.

Using these estimates and the fact that $\sum\limits_{j=m}^{k} r_j^{1-\beta}$ is positive and increasing with respect to k, $k \geqslant m$ and $\sum\limits_{j=N_1}^{N_2} p_j$ is negative if N_1, N_2 are sufficiently large, we get

$$\begin{aligned}
\mathcal{F}(y) \leqslant \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{1-\alpha} \Bigg[& \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=K}^{L} r_j^{1-\beta}\right)^{1-\alpha} \\
& + \left(\sum_{j=m}^{N_1} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=N_1}^{N_2-1} p_j\right) + \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=M}^{N} r_j^{1-\beta}\right)^{1-\alpha}\Bigg] \\
\leqslant \left(\sum_{j=m}^{L} r_j^{1-\beta}\right)^{1-\alpha} & [1+\varepsilon-1-2\varepsilon+\varepsilon] = 0,
\end{aligned}$$

which yields the desired result. □

1) A closer examination of the last proof shows that lim in (11) can be replaced by lim sup.

2) In [13] it is proved a "complementary case" of the last criterion—in the sense of the convergence of $\sum\limits_{j=m}^{\infty} r_j^{1-\beta}$. In its proof we use the so called reciprocity principle. This statement is read as follows:

Suppose $p_k < 0$ on $[m, \infty)$. Further, let

$$\sum_{j=m}^{\infty} r_j^{1-\beta} < \infty$$

and

$$\lim_{k\to\infty} \left(\sum_{j=k+1}^{\infty} r_j^{1-\beta} \right)^{\alpha-1} \left(\sum_{j=m}^{k} p_j \right) < -1.$$

Then (1) is oscillatory.

3) Using the Riccati technique in [6] we have proved the following "nonoscillatory suplement" of Theorem 5. Suppose that (9) holds, $\sum\limits_{j=m}^{\infty} p_j$ is convergent and

$$\lim_{k\to\infty} \frac{r_k^{1-\beta}}{\sum\limits_{j=m}^{k-1} r_j^{1-\beta}} = 0.$$

If

$$\liminf_{k\to\infty} \left(\sum_{j=m}^{k-1} r_j^{1-\beta} \right)^{\alpha-1} \left(\sum_{j=k}^{\infty} p_j \right) > \frac{1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1}$$

and

$$\limsup_{k\to\infty} \left(\sum_{j=m}^{k-1} r_j^{1-\beta} \right)^{\alpha-1} \left(\sum_{j=k}^{\infty} p_j \right) < \frac{2\alpha-1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1},$$

then (1) is nonoscillatory.

4) In [15], [16] and in the papers cited therein can be found further oscillation criteria (and other "oscillatory results") for equation (1). For example, we have shown (as a consequence) of more general statement that if $r_k \equiv 1$, then the condition (11)

can be replaced by the weaker one, namely

$$\limsup_{k\to\infty} k^{\alpha-1}\sum_{j=k}^{\infty} p_j < -\frac{1}{\alpha}\Big(\frac{\alpha-1}{\alpha}\Big)^{\alpha-1}.$$

5) Very important role in the oscillation theory of linear differential equations is played by the so called *principal solution*. An extension of this concept to the half-linear differential equation (2) has been already partly done. In [5], [8], [12] the construction of this solution was made and it is based either on the minimality of the solution of generalized Riccati differential equation (since in the linear case the principal solution of linear differential equation generates a minimal solution (near ∞) of the corresponding Riccati equation—the so called *distinguished solution*), or it is based on the generalized Prüfer transformation. Recall that the discrete counterpart of principal solution is called *recessive solution* (for linear equation). Taking into account the above facts we would like to construct recessive solution for euqation (1) and possibly apply it.

6) The fact that we have a theory for differential and also difference equations suggests an idea to develop a unified theory for these equations on arbitrary time scales. This problem was very recently solved in the paper [14].

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References

[1] *R. P. Agarwal*: Difference equations and inequalities, theory, methods, and applications, the second edition. Pure and Appl. Math. M. Dekker, New York-Basel-Hong Kong, 2000.
[2] *C. D. Ahlbrandt and A. C. Peterson*: Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations. Kluwer Academic Publishers, Boston, 1996.
[3] *O. Došlý*: Oscillation criteria for higher order Sturm-Liouville difference equations. J. Differ. Equations Appl. *4* (1998), 425–450.
[4] *O. Došlý*: Oscillation criteria for half-linear second order differential equations. Hiroshima Math. J. *28* (1998), 507–521.
[5] *O. Došlý*: A remark on conjugacy of half-linear second order differential equations. Math. Slovaca *50* (2000), 67–79.
[6] *O. Došlý and P. Řehák*: Nonoscillation criteria for second order half-linear difference equations. Comput. Math. Appl. In press.
[7] *Á. Elbert*: A half-linear second order differential equations. Colloq. Math. Soc. János Bolayi *30* (1979), 158–180.
[8] *Á. Elbert and T. Kusano*: Principal solutions of nonoscillatory half-linear differential equations. Adv. Math. Sci. Appl. (Tokyo) *8* (1998), 745–759.
[9] *J. Jaroš and T. Kusano*: A Picone type identity for second order half-linear differential equations. Acta Math. Univ. Comenian. (N. S.) 68 (1999), 137–151.

[10] *W. G. Kelley and A. Peterson*: Difference Equations: An Introduction with Applications. Acad. Press, San Diego, 1991.
[11] *J. D. Mirzov*: On some analogs of Sturm's and Kneser's theorems for nonlinear systems. J. Math. Anal. Appl. *3* (1976), 418–425.
[12] *J. D. Mirzov*: Principial and nonprincipial solutions of a nonoscillatory system. Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy *31* (1988), 100–117.
[13] *P. Řehák*: Half-linear discrete oscillation theory. In: Proceedings of 6th Colloquium on the qualitative theory of DE, Szeged 1999, http://www.math.u-szeged.hu/ejqtde/index.html. EJQTDE, Szeged, 2000, pp. 1–14.
[14] *P. Řehák*: Half-linear dynamic equations on time scales: IVP and oscillatory properties. Submitted.
[15] *P. Řehák*: Hartman-Wintner type lemma, oscillation and conjugacy criteria for half-linear difference equations. J. Math. Anal. Appl. *252* (2000), 813–827.
[16] *P. Řehák*: Oscillation criteria for second order half-linear difference equations. J. Differ. Equations Appl. In press.

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