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STRICT TOPOLOGIES AS TOPOLOGICAL ALGEBRAS

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Abstract. Let X be a completely regular Hausdorff space, $C_b(X)$ the space of all scalar-valued bounded continuous functions on X with strict topologies. We prove that these are locally convex topological algebras with jointly continuous multiplication. Also we find the necessary and sufficient conditions for these algebras to be locally m-convex.

Keywords: strict topologies, locally convex algebras, locally m-convex algebras

MSC 2000: 46E10, 28C15, 46E25, 46G10, 28B05

1. Introduction and Notations

In this paper X is a completely regular Hausdorff space, K the field of real or complex numbers, $C_b(X)$ the space of all K-valued bounded continuous functions on X. The strict topologies ([11], [13], [6], [7], [5]) β_0, β, β_1, β_p, β_∞, β_g are defined $C_b(X)$ (we will also denote β_0 by β_t, β by β_τ, β_1 by β_σ; the definition of β_g will be taken as given in [5]).

In this paper, considering $C_b(X)$ as an algebra, we first prove that, under the above topologies, it is a topological algebra with jointly continuous multiplication. Also we find necessary and sufficient conditions for these algebras to be locally m-convex.

For locally convex spaces, the notations and results of ([10]) will be used. For the topological spaces we refer to [3]. For topological measure theory notations and results of [13], [14], [11], [6], [7], [5] will be used. All locally convex spaces are assumed to be Hausdorff and over K, the field of real or complex numbers. $X^{\sim}(\nu X)$ will denote the Stone-Čech compactification (real-compactifiaction) of X. We have $X \subset \nu X \subset X^{\sim}$. A topological space is called sham compact if any countable union of its compact subsets is relatively compact ([2]). $P \subset C_b(X)$ is called solid if $f \in P$, $g \in C_b(X)$, $|g| \leqslant f$ implies $g \in P$ ([4]). The topologies β_z $(z = t, \tau, \sigma, g, p)$ are locally solid in the sense that there is a 0-neighbourhood base consisting of absolutely convex solid sets. Let $M(X) = (C_b(X), \|.\|)'$, $M_z(X) = (C_b(X), \beta_z)'$ $(z = t, \tau, \sigma, g, p)$. Solid subsets of $M_z(X)$ are defined in a similar way.

For a $\mu \in M(X)$, we get a $\mu^{\sim} \in M(X^{\sim})$, $\mu^{\sim}(g) = \mu(g\big|_X)$, $g \in C(X^{\sim})$; for a $\mu^{\sim} \in M(X^{\sim})$, $\operatorname{supp}(\mu^{\sim})$ is the smallest compact set C in $X^{\sim}$ such that $|\mu^{\sim}|(C) = |\mu^{\sim}|(X^{\sim})$. For a collection $\{A_\alpha\colon \alpha \in I\}$ of subsets of a vector space E, ΓA_α will denote the absolutely convex hull of $\bigcup\limits_{\alpha\in I} A_\alpha$ ([10]).

An algebra with a locally convex topology is called locally m-convex if it has a 0-neighbourhood base $\{V\colon V \in \mathscr{V}\}$ such that each V is absolutely convex and $VV \subset V$ ([8], [1]).

For each of the β_z, $z = \sigma, \tau, \infty, g, p$, there is a collection $\mathscr{K}_z$ of subsets of $X^{\sim} \setminus X$ such that for each $K \in \mathscr{K}_z$, there is a locally convex topology β_K, generated by the semi-norms p_φ, $\varphi \in C(X^{\sim} \setminus K)$, φ vanishing at infinity, $p_\varphi(f) = \sup\{|f(x)\varphi(x)|\colon x \in X\}$ (in the case of β_p, φ consist of bounded functions on $(X^{\sim} \setminus K)$, vanishing at infinity). The locally convex topology β_z is the infimum of the locally convex topologies β_K ([13]). We denote by κ the topology of uniform convergence on the compact subsets of X.

X is called absolutely Borel measurable in $X^{\sim}$ if for any regular Borel measure ν on $X^{\sim}$, there are Borel sets A, B in $X^{\sim}$, with $A \subset X \subset B$, $\nu(B \setminus A) = 0$ ([13], Def. 8.4).

When X is locally compact, considering $(C_b(X), \beta_t)$ as topological algebra, it is proved in [1] that the finest locally m-convex topology weaker than β_t is the topology of uniform convergence on the compact subsets of X.

2. Main results

Theorem 1. *$(C_b(X), \beta_z)$ is a topological algebra with jointly continuous multiplication, for $z = t, \sigma, \tau, \infty, g, p$.*

P r o o f. We first consider the case $z = t$. Take a bounded and vanishing at infinity $\varphi\colon X \to \mathbb{R}^+$. Then $\sqrt{\varphi}$ is also bounded and vanishes at infinity. Taking $V = \{f \in C_b(X)\colon \|f\varphi\| \leqslant 1\}$ and $U = \{f \in C_b(X)\colon \|f\sqrt{\varphi}\| \leqslant 1\}$, we get $UU \subset V$. This proves the result.

Now we come to the cases $z = \sigma, \tau, \infty, g, p$.

For each $\alpha \in \mathscr{K}_z$, take a $\varphi_\alpha \geqslant 0$ and put $V_\alpha = \{f \in C_b(X)\colon |f\varphi_\alpha| \leqslant 1 \text{ on } X\}$ and $V = \Gamma V_\alpha$. Also take $U_\alpha = \{f \in C_b(X)\colon |f\sqrt{\varphi_\alpha}| \leqslant 1 \text{ on } X\}$ and $U = \Gamma U_\alpha$. It is enough to prove that for every f in U $|f|^2 \in V$. Let $f = \sum\limits_{i=1}^{n} \lambda_i f_i$, $f_i \in U_{\alpha(i)}$, $p = \sum |\lambda_i| \leqslant 1$. Now $\bigl(\sum(\frac{|\lambda_i|}{p})|f_i|\bigr)^2 \leqslant \sum(\frac{|\lambda_i|}{p})(|f_i|)^2 \in V$ implies that $\bigl(\sum|\lambda_i|\,|f_i|\bigr)^2 \in p^2V \subset V$. This proves the result. $\square$

Now we discuss the necessary and sufficient conditions for these topologies to be locally m-convex.

Theorem 2. *The topological algebra $(C_b(X), \beta_t)$ is locally m-convex if and only if X is sham compact. In this case $\beta_t = \kappa$.*

P r o o f. If X is sham compact κ is finer than β_t and so $\kappa = \beta_t$. But κ is locally m-convex and so the result follows.

Conversely suppose β_t is locally m-convex. Take V to be an absolutely convex, solid 0-neighbourhood. Because of locally m-convex property, there exists a bounded and vanishing at infinity $\varphi\colon X \to \mathbb{R}^+$ such that if $U = \{f \in C_b(X)\colon |f\varphi| \leqslant 8\}$ then $\Gamma U^n \subset V$. Let $M = \sup\{\varphi(x)\colon x \in X\}$. Put $K = \{x \in X\colon |\varphi(x) \geqslant 1\}^-$. K is a compact subset of X. Put $W = \{f \in C_b(X)\colon |f| \leqslant \frac{1}{(4(M+1))}$ on $K\}$. We prove that $W \subset V$. Take an $f \in W$. If f is in U, we are done. If not let $K_1 = \{x \in X\colon |f(x)| \leqslant 2\}$ and $K_2 = \{x \in X\colon |f(x)| \geqslant 3\}$. Let $f_1 = \inf(3, |f|)$, $2f_1 \in U$ (note $|\varphi| < 1$ outside K). Define $g_0 \in C_b(X)$, $0 \leqslant g_0 \leqslant 2$, $g_0 = 2$ on K_2, $g_0 = 0$ on K_1. Then $g_0 \in U$. Choose n such that $\frac{1}{2}2f_1 + \frac{1}{2}g_0^n \geqslant |f|$. Since V is solid and $\frac{1}{2}2f_1 + \frac{1}{2}g_0^n \in V$, we get that $f \in V$, which proves that $\beta_t \leqslant \kappa \leqslant \beta_t$. By [4], X is sham compact. □

Theorem 3. *The topological algebra $(C_b(X), \beta_z)$ $(z = \sigma, \infty, g, p)$ is locall m-convex if and only if X is pseudocompact. In this case these topologies coincide with norm topology.*

P r o o f. Suppose X is pseudocompact. In this case $X^\sim = \nu X$. For $z = \sigma$, p, $\mathscr{K}_z$ is void ([12]) and so these topologies become norm topologies which are locally m-convex. Also by [8], $\bigl(C_b(X), \|\ \|\bigr)' = M_g(X) = M_\infty(X)$. Since β_g, β_∞ are Mackey ([5], [4]), these topologies coincide with norm topology and so are locally m-convex.

Conversely suppose β_σ is locally m-convex. Take V to be an absolutely convex, solid 0-neighbourhood in β_σ. Then there exists an absolutely convex 0-neighbourhood U in β_σ such that $\Gamma U^n \subset V$. Fix a zero-set $Z \subset X^\sim \setminus X$. Take a $\varphi \in C(X^\sim \setminus Z)$, φ vanishing at infinity and $U \supset \{f \in C_b(X)\colon |f\varphi| \leqslant 8\}$. $K = \{x \in (X^\sim \setminus Z)\colon |\varphi(x)| \geqslant 1\}$ is a compact subset of $(X^\sim \setminus Z)$. Proceeding as in Theorem 2, we get that $V \supset \{f \in C_b(X)\colon |f| \leqslant \frac{1}{(4(M+1))}$ on $K\}$, where $M = \sup\{\varphi(x)\colon x \in X\}$. Thus β_σ is weaker than the topology of unifrom convergence on the compact subsets of $(X^\sim \setminus Z)$. Since $\bigcup\{Z\colon Z$ a zero set, $Z \subset (X^\sim \setminus X)\} = (X^\sim \setminus \nu X)$, support of $|\mu|^\sim \subset \nu X$, for every $\mu \in M_\sigma(X)$. Take an $f \in C(X)$, $f \geqslant 0$. If f is unbounded, there exists a sequence $\{x_n\} \subset X$ such that $f(x_n) \to \infty$. Since the compact support of the measure $\mu^\sim$, $\mu = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} x_n$, must be subset of νX and f is finite-valued on νX, we get a contradiction. This proves X is pseudocompact.

In other cases, $\bigcup\{C\colon\ C \in \mathscr{K}_z\} \supset (X^{\sim} \setminus \nu X)$, and so proceeding exactly as above we prove that X is pseudocompact. □

Before considering the case of β_τ, we prove the lemma:

Lemma 4. *If* $\operatorname{supp}(\mu^{\sim}) \subset X$, $\forall \mu \in M_\tau(X)$, *then* $\beta_\tau = \beta_t$.

Proof. Take any $P \subset M_\tau^+(X)$, which is $\sigma(M_\tau(X), C_b(X))$-compact. This means P is β_τ-equicontinuous ([12]). We will prove that it is β_t-equicontinuous. For this it is enough to prove that given $\eta > 0$ there is a compact $K \subset X$ such that every measure $\mu \in P$, $|\mu|(X \setminus K) < \eta$. Suppose this is not true. Denoting by $\{\alpha\colon\ \alpha \in I\}$, all compact subsets of X, and ordering them by inclusion, I becomes a directed set. There exists an $\eta > 0$ such that for every $\alpha \in I$ there is a $\mu_\alpha \in P$ with $\mu_\alpha(X \setminus \alpha) \geqslant 2\eta$. By taking subnet if necessary, we assume $\mu_\alpha \to \mu \in P$. Suppose $C = \operatorname{supp}(\mu)$, a compact subset of X and $V = X \setminus C$. Take an increasing net $\{f_\gamma\}$, $\gamma \in J$, $0 \leqslant f_\gamma \leqslant 1$ of continuous functions on X such that $f_\gamma = 0$, on C, $f_\gamma \uparrow \chi_V$. Since β_τ is locally solid and P is β_τ-equicontinuous, solid hull of P is also β_τ-equicontinuous. This means the net $\{f_\gamma \mu_\alpha\}$ (the ordering being point-wise) is relatively $\sigma(M_\tau(X), C_b(X))$-compact. By talking subnet if necessary, we assume this net is convergent to some $\nu \in M_\tau^+(X)$. We claim $\nu = 0$. Fix a $q > 0$. There is (γ_0, α_0) such that $|\mu_\alpha(f_\gamma) - \nu(1)| \leqslant q$ for every $(\gamma, \alpha) \geqslant (\gamma_0, \alpha_0)$. Keeping γ fixed and taking limit over α and using the fact that $\mu(f_\gamma) = 0$, we get that $\nu(1) \leqslant q$. This proves $\nu = 0$. Take (γ_1, α_1) such that $\mu_\alpha(f_\gamma) < \eta$ for every $(\gamma, \alpha) \geqslant (\gamma_1, \alpha_1)$. Now take a compact $\alpha_2 \in I$, $\alpha_2 \geqslant (C \cup \alpha_1)$. This means $\mu_{\alpha_2}(f_\gamma) < \eta$, $\forall \gamma$. Taking limit over γ, we get $\mu_{\alpha_2}(V) \leqslant \eta$. Since $V \supset (X \setminus \alpha_2)$, this is a contradiction. Thus we have proved that $\beta_t = \beta_\tau$. □

Theorem 5. *The topological algebra $(C_b(X), \beta_\tau)$ is locally m-convex if and only if X is absolutely Borel measurable in $X^{\sim}$ and sham compact. In this case $\beta_\tau = \kappa$.*

Proof. Suppose β_τ is locally m-convex. Fix a compact $C \subset (X^{\sim} \setminus X)$. Proceeding exactly as in Theroem 3, we prove that the support of $|\mu|^{\sim} \subset X$, for every $\mu \in M_\tau(X)$. By Lemma 3, $\beta_t = \beta_\tau$. By Theorem 2, X is sham compact and by [12], X is absolutely Borel measurable in $X^{\sim}$.

Conversely if X is sham compact and absolutely Borel measurable in $X^{\sim}$, $\kappa = \beta_t$ (Theorem 2) and $M_\tau(X) = M_t(X)$. Thus elements of $M_\tau(X)$ are supported by the compact subsets of X. By Lemma 3, $\beta_\tau = \beta_t$. By Theorem 2, β_τ is locally m-convex. □

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