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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 3, 449–461

Czechoslovak Mathematical Journal, 51 (126) (2001), 449–461

ALGEBRAS AND SPACES OF DENSE CONSTANCIES

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(Received March 22, 1995)

Abstract. A DC-space (or space of dense constancies) is a Tychonoff space X such that for each $f \in C(X)$ there is a family of open sets $\{U_i\colon\ i \in I\}$, the union of which is dense in X, such that f, restricted to each U_i, is constant. A number of characterizations of DC-spaces are given, which lead to an algebraic generalization of the concept, which, in turn, permits analysis of DC-spaces in the language of archimedean f-algebras. One is led naturally to the notion of an almost DC-space (in which the densely constant functions are dense), and it is shown that all metrizable spaces have this property.

Keywords: space and algebra of dense constancy, c-spectrum

MSC 2000: 54C05, 54G99, 06F25, 16S90

1. Introduction

In this article all topological spaces are, unless the contrary is specified, Tychonoff spaces; a Hausdorff space X is *Tychonoff* if the *cozerosets* $\operatorname{coz}(f) = \{x \in X\colon\ f(x) \neq 0\}$, $f \in C(X)$, form a base for the topology. $C(X)$ denotes the ring of all continuous, real-valued functions defined on X. Recall (see [7]) that if X is Tychonoff, then each point $p \in X$ has a base of zeroset neighbourhoods. As is customary, βX denotes the Stone-Čech compactification of X.

$C(X)$ is an *f-algebra*; that is, a real algebra which supports a lattice-ordering (given by the pointwise supremum and infimum) such that $a \wedge b = 0$ implies that $a \wedge bc = 0$, for each $c \geqslant 0$.

We shall consider spaces over which the continuous real-valued functions are *densely constant*; that is, for each $f \in C(X)$ there exist open sets $\{U_i\colon\ i \in I\}$ which are pairwise disjoint, the union of which is dense in X, and such that f is constant when restricted to each of the U_i (although not necessarily the same constant over the various U_i.) Obviously, any space X which contains a dense subset of isolated points has this property. Any space with this property will be called a *space of dense constancies* or, with regard for brevity, a *DC-space*.

In the special case when, for each $f \in C(X)$ there is a family of *clopen* sets $\{U_i\colon i \in I\}$ which are pairwise disjoint, the union of which is dense in X, so that f, restricted to each U_i is constant, we speak of a *Specker space*. Specker spaces are considered in detail in [4] and [11]. We simply observe here that a Specker space is a DC-space having a π-base of clopen sets. (A family of non void open sets $\mathcal{B}$ is called a *π-base* if each non-void open set in X contains a member of $\mathcal{B}$.)

We close this introduction with some preliminary facts about DC-spaces.

Let X be a space. Call a point $p \in X$ a *DC-point* if, for each $0 < f \in C(X)$ and each neighbourhood V of p, there is a non-empty open set U contained in V on which f is constant. We immediately recall two other definitions: $p \in X$ is a *P-point* if each zeroset which contains p is a neighbourhood of p. (Note: a *P-space* is one in which every point is a P-point; see [7], pp. 62–63, and also Theorem 14.29.) We say that $p \in X$ is an *almost P-point* if every zeroset containing p has interior.

It should be clear that a P-point is an almost P-point, and it is not too hard to see that every almost P-point is a DC-point. Here now is the fundamental connection between DC-spaces and DC-points.

Proposition 1.1. *Let X be a Tychonoff space. Then the following are equivalent.*

(a) *X is a DC-space.*

(b) *For each $f \in C(X)$, $f \neq 0$, and each non-void open set V on which f does not vanish identically, there is a non-empty open set $U \subseteq V$, so that f is constant and non-zero on U.*

(c) *Every point of X is a DC-point.*

Proof. Suppose first that X is a DC-space and $f \in C(X)$, with $f \neq 0$. Suppose also that V is a non-empty open set on which f does not vanish. By definition, there is a family $\{U_i\colon i \in I\}$ of pairwise disjoint open sets, the union of which is dense in X, and such that f restricted to each U_i is constant. Let $x \in V$ be a point for which $f(x) > 0$; then for some $j \in I$, $V \cap \operatorname{coz}(f) \cap U_j \neq \emptyset$. It should be clear that on $V \cap \operatorname{coz}(f) \cap U_j$ f is constant, and this proves that (a) implies (b).

Now let us assume that (b) holds, and pick $p \in X$. Suppose that $f > 0$ in $C(X)$, and V is a neighbourhood of p. Without loss of generality, $V = \operatorname{coz}(g)$, for some $0 < g \in C(X)$. We consider $f \wedge g$; if $f \wedge g = 0$ then f vanishes on $\operatorname{coz}(g)$, and we are done. So suppose that $f \wedge g > 0$; then, by assumption, there is an open set $W \subseteq \operatorname{coz}(g)$ on which f is identically a non-zero constant. This proves that p is a DC-point, whence (b) implies (c).

On the other hand, suppose that each $p \in X$ is a DC-point. Pick $f \in C(X)$; to show that X is a DC-space, it clearly suffices to consider the case $f > 0$. Consider a maximal family of pairwise disjoint open sets $\{U_i \colon i \in I\}$ such that f restricted to each of the U_i is constant. We are to prove that the union U of these sets is dense in X.

Now, if $p \in X$ is not in the closure of U, let V be an open set, containing p, which misses U. Obviously, f cannot vanish identically on V, for if so the maximality of the U_i would be violated. Applying (c), we obtain a non-void open set $W \subseteq V$, on which f is both constant and non-zero, which once again contradicts maximality. Therefore, U is dense in X, and this shows that (c) implies (a), and the proposition is proved. □

The next proposition gives some idea about the clustering of DC-points. It is essentially a consequence of the proof (that (c) implies (a)) above.

Proposition 1.2. *The set of DC-points in a Tychonoff space is closed.*

Proof. Suppose that $p \in \mathrm{cl}(D)$, where D is the set of all DC-points of the space X. Suppose that $f > 0$ in $C(X)$ and V is a neighbourhood of p. There is a DC-point $q \in \mathrm{int}(V)$, and so we can locate a non-void open set $W \subseteq V$ on which f is constant. This shows that p is a DC-point. □

2. Algebras of dense constancy

In this section the goal is to develop the ideas about dense constancies from an algebraic point of view. For that we introduce f-rings and f-algebras. We shall also need the concept of a maximal ring of quotients. All rings in this article are assumed to be commutative and possess an identity. By "subring" we mean *subring inheriting the identity.* We shall write $A \leqslant B$ when A is a subring or subalgebra of B.

Definition 2.1. If A is a commutative ring and a subring of B, we say that B is a *quotient ring* of A if, for each pair b_1 and b_2 in B, with $b_1 \neq 0$, there is an $a \in A$ such that $ab_1, ab_2 \in A$, and $ab_1 \neq 0$. This concept is due to Utumi [13]. Lambek [9] gives an account of the *maximal ring of quotients* QA of a commutative ring A; it is a quotient ring in the sense just introduced, and every quotient ring of A can be embedded in QA.

Definition 2.2. We are interested in certain lattice-ordered algebras. Let us, for completeness, remind the reader that a *lattice-ordered* group G is one with an underlying lattice such that $x \leqslant y$ implies that $a + x \leqslant a + y$ and $x + a \leqslant y + a$.

A *lattice-ordered ring* $(R, +, \cdot, \vee, \wedge)$ is a ring structure such that $(R, +, \vee, \wedge)$ is a lattice-ordered group in which $a, b \geqslant 0$ implies that $ab \geqslant 0$. A lattice-ordered ring R is an *f-ring* if $a \wedge b = 0$ implies that $ac \wedge b = 0$, for each $c \geqslant 0$. The term "f-algebra" should be clear to the reader. (For background on lattice-ordered groups and rings, we refer the reader to [2] and [5].)

If G is any lattice-ordered group and X is any subset of G, then $X^\perp$ denotes the *polar* of X; that is,

$$X^\perp = \{g \in G\colon\ |g| \wedge |x| = 0,\ \ \forall x \in X\}.$$

$X^\perp$ is a convex lattice-subgroup of G, and if A is an f-ring then $X^\perp$ is also an ideal. If $X = \{a\}$, for some $a \in A$, we write $X^\perp = a^\perp$; the meaning of the symbols $X^{\perp\perp}$ and $a^{\perp\perp}$ should be self-evident.

Definition 2.3. Recall that a commutative ring A is said to be *semiprime* if it has no non-zero nilpotent elements; equivalently, if the intersection of all the prime ideals of A is zero. In a semiprime f-ring the notion of "polar" and "annihilator ideal" coincide.

For semiprime rings, Banaschewski (in [3]) gave a construction of the maximal ring of quotients, as a direct limit of rings of local quotients; we refer the reader to the construction, but we shall not bring it up here. In [10] it is used to realize QA as an f-ring when A is an f-ring. F. W. Anderson, in [1], showed that the maximal ring of quotients (even of certain not-necessarily commutative rings) can be given an (essentially unique) ordering which traces on the one given in A, and it is also shown in Corollary 3.2 of [1] that if A is a commutative ring with identity, then this canonical ordering makes QA an f-ring whenever A is an f-ring.

In QA we have the *classical ring of quotients* of A, denoted qA; it consists of all fractions a/d, where d is a non-divisor of zero, where $a/d = a'/d'$ if and only if $ad' = a'd$; addition and multiplication of fractions is carried out as in the quotient field of an integral domain. Clearly, qA is a quotient ring of A in the sense of Utumi, and the canonical ordering of QA traces on qA as $a/d \geqslant 0$ (with $d > 0$, without loss of generality, as $a/d = ad/d^2$, and $d^2 > 0$ if A is an f-ring), if and only if $a \geqslant 0$. Then A is an f-subring of qA, which is an f-subring of QA.

Definition 2.4. An f-ring A is archimedean if for each $a, b \geqslant 0$ in A there is a positive integer n such that $na \nleqslant b$.

Definition 2.5. Recall that a lattice-ordered group is *laterally complete* if every set of pairwise disjoint elements has a supremum. H is the *lateral completion* of G if H is laterally complete, G is *dense* in H (meaning that for each $0 < h \in H$ there is an element $g \in G$ so that $0 < g \leqslant h$) and no proper lattice-subgroup of H which contains G is laterally complete. We denote the lateral completion of G by G^L.

The following appears as Corollary 1.7.1 in [10].

Theorem 2.6. *Suppose that A is an archimedean f-ring. Then*

$$QA = (qA)^L = q(A^L).$$

From this point onward we shall restrict ourselves to real algebras. Furthermore, all algebras shall be semiprime unless otherwise stated. Let us observe that if A is a real algebra, then so is QA, and A lies in it as a subalgebra.

We now come to the (new and) central definition of the section.

Definition 2.7. We say that A is an *algebra of dense constancy* (or a *DC-algebra*) if for each $0 \neq a \in A$ and each $0 < x \in A$ for which $ax \neq 0$, there exists a $b \in A$, with $0 < b \leqslant x$, and a non-zero real number r such that $ab = rb$. Observe straightaway that one gets an equivalent definition by formulating the preceding for $0 < a \in A$. In addition, if $0 < a \in A$ and $ab = rb > 0$, as stated, then, as $(a-r)b = 0$ and a is not disjoint to b, we may choose $b \leqslant a$, if we desire.

It should be evident, from Proposition 1.1, that $C(X)$ is a DC-algebra if and only if X is a DC-space.

Our next three propositions give a fairly good idea of the force of dense constancy.

Proposition 2.8. *Any dense f-subalgebra of a DC-algebra is also a DC-algebra.*

Proof. This is immediate from the definition of density and the comment in Definition 2.7. □

Recall that the *Jacobson radical* of a ring is the intersection of its maximal ideals.

Proposition 2.9. *If A is a DC-algebra then it is archimedean and the Jacobson radical is zero.*

Proof. Let's prove the second assertion first. If $0 < a \in A$ locate a positive $b \in A$ and a positive real number r such that $ab = rb$. Since A is semiprime, there is a minimal prime ideal P such that $b \notin P$. Then $a \equiv r1 \mod P$, and the same is true modulo any maximal ideal which contains P. Clearly, a cannot lie in any such maximal ideal.

Suppose, by way of contradiction, that $0 < na_1 < a_2$, for each natural number n. There exists a positive element $b \in A$ and a non-zero real number r such that

$a_1b = rb$. Now, as $a_2b \neq 0$, we may choose a $c \in A$ such that $0 < c < b$, and $a_2c = sc$. However, since $a_1c = rc$, we get that $nrc < sc$, for each $n \in \mathbb{N}$, which is contradiction. Hence, A is archimedean, and we are done. □

We turn now to enlargements of DC-algebras. In preparation for this, we need one more excursion.

Definition 2.10. Recall that if A is a semiprime ring, then QA is self-injective, and indeed the injective hull of A as an A-module (see [9]). Thus, QA is the A-essential closure of A; let us explain. If N is an A-module and M is an A-submodule, then it is said that N is an *A-essential extension* of M if every non-zero A-submodule of N intersects M non-trivially. The point is that QA is an A-essential extension of A, and that every A-essential extension of A can be embedded as an A-submodule of QA.

Theorem 2.11. *If A is a DC-algebra then so is QA. Moreover, QA is the largest extension of A which is a DC-algebra and in which A is dense.*

Proof. If A is DC-algebra, then to show that QA is one, it suffices to prove that qA and A^L are DC-algebras. We leave it to the reader to verify that qA is a DC-algebra.

Let us now verify that if A is a DC-algebra then A^L is too. We use the characterization, by Roger Bleier [6], of orthocompletions, and we recall that for archimedean f-algebras, the orthocompletion and the lateral completion coincide, owing to a theorem of Bernau (see [2], Theorem 8.2.4).

If $f \in A^L$, $f \neq 0$, then for each $0 < a \in A$ for which $fa \neq 0$, there exist b and c in A, with $a > b > 0$, such that $fb = cb \neq 0$. As A is a DC-algebra, there is an element $d > 0$ in A, such that $d < b$ and $cd = rd$, for a suitable non-zero real number r. Notice, incidentally, that $fd = cd$, proving that $fd = rd$. Since A is dense in A, this is enough to prove that A^L is a DC-algebra. Invoking Theorem 2.6, we get that QA is a DC-algebra whenever A is one.

As to maximality, suppose that B is an extension of A, in which A is dense, so that B is a DC-algebra. Then for each $b \in B$, $b \neq 0$, there exist $a \in A$, $a \neq 0$, and $r \in \mathbb{R}$, $r \neq 0$, such that $ba = ra$. This says that B is an A-essential extension of A. Since QA is the A-essential hull of A, it follows that $B \leqslant QA$. □

One then gets the following nice-sounding corollary.

Corollary 2.11.1. *For an f-algebra A the following are equivalent:*

(1) *A is DC-algebra and maximal, among DC-algebras, with respect to dense extension.*

(2) *A is a DC-algebra and rationally complete; (meaning: $QA = A$).*
(3) *A is the lateral completion of an f-algebra which is generated (as a vector space) by its idempotents.*

Proof. The equivalence of (1) and (2) follows immediately from Theorem 2.11.

Let us suppose that (2) holds. Let $S(A)$ be the subalgebra of A generated by its idempotents. $S(A)$ is a DC-algebra, by definition, and therefore, since $S(A)$ is obviously dense in A, and $S(A) = qS(A)$, we may conclude that $S(A) \leqslant A \leqslant S(A)^L$, by Theorem 2.11, whence it follows that $A = S(A)^L$, because $A = QA$.

Conversely, if A is the lateral completion of S, which is generated by its idempotents, then, as in the previous paragraph, $Q(S) = A$. S is obviously a DC-algebra, and so by Theorem 2.11, (1) follows. □

3. The c-spectrum of an algebra

We remind the reader that all algebras discussed in this section are semiprime. We now proceed to define the constancy-spectrum of an element, and the DC-subalgebra of an f-algebra.

Suppose that A is a f-algebra and $a \in A$. We consider a family of pairs

$$\{(c_i, r_i)\colon\ c_i \in A,\ \ c_i c_j = 0,\ \ \forall i \neq j,\ \ r_i \in \mathbb{R}\}$$

such that $ac_i = r_i c_i$, and the set is maximal with respect to the conditions just announced. Obviously, there is no loss of generality in picking the c_i to be positive. Suppose that $\{(d_k, s_k)\colon\ k \in K\}$ is another such maximal family. Then, for each index i there exists an index k such that $c_i d_k > 0$; but then $ac_i d_k = r_i c_i d_k = s_k c_i d_k$. Hence $r_i = s_k$. Reversing the roles of i and k, we see that the sets $\{r_i\colon\ i \in I\}$ and $\{s_k\colon\ k \in K\}$ coincide. For this reason it makes sense to speak of a *constancy spectrum* or *c-spectrum* for a. If the family $\{c_i\colon\ i \in I\}$ is in fact maximal with respect to pairwise disjointness in A, we say that a has a *full c-spectrum.*

If $\{(c_i, r_i)\colon\ i \in I\}$ is a c-spectrum for a, we call the c_i a *c-spectrum support* for a; if $\{(c_i, r_i)\colon\ i \in I\}$ is a full c-spectrum, we refer to the c_i as a *full c-spectrum support* for a. Note that $\{(1, 0)\}$ is a full c-spectrum for 0, and $\{(1, 1)\}$ is a full c-spectrum for $1 \in A$. We emphasize that in a c-spectrum $\{(c_i, r_i)\colon\ i \in I\}$ of an element a, the real values r_i are uniquely determined by a, but the members of the c-spectrum support c_i, in general, are not.

In terms of constancy spectra, here is a characterization of DC-algebras:

Proposition 3.1. *An f-algebra A is a DC-algebra if and only if every element of A has a full c-spectrum.*

P r o o f. Suppose A is a DC-algebra and $0 < a \in A$, with c-spectrum given by the pairs (c_i, r_i). The objective is to show that the c_i are maximal disjoint. So suppose that $0 < x \in A$ is disjoint from all the c_i; then $ax = 0 = 0x$ implies that the pair $(x, 0)$ can be addeed to the (c_i, r_i), violating the maximality. Thus, $ax > 0$, and since A is a DC-algebra, there exist $0 < y \leqslant x$ in A and a positive real number s, such that $ay = sy$. But y too is disjoint from all the c_i, once again contradicting the maximality with the pair (y, s). Therefore, the given c-spectrum is full.

Conversely, suppose that $0 < a \in A$ has a full c-spectrum, given by the pairs (d_i, s_i). If $x > 0$ in A and $ax > 0$, then for some index j, $axd_j > 0$, because the c-spectrum is full. Then $axd_j = s_j xd_j$; after observing that s_j must be positive, the proof is complete. □

The c-spectrum serves to introduce the concept of a "DC-subalgebra". Let $\mathbf{dc}A$ be the subset of the f-algebra A consisting of all the elements of A which have a full c-spectrum. It is a routine matter to verify that $\mathbf{dc}A$ is indeed an f-subalgebra of A.

In the next result we collect the basic properties of $\mathbf{dc}A$; we refer to this subalgebra as the *DC-subalgebra* of A, or, in more elaborate terms, as the *subalgebra of A of dense constancy.*

Proposition 3.2. *Suppose that A is a semiprime f-algebra.*

(a) *$0 < a \in A$ belongs to $\mathbf{dc}A$ if and only if, for each $x > 0$ in A such that $ax > 0$, there exists a $y > 0$, with $x \geqslant y$, and a positive real number r, such that $ay = ry$.*

(b) *If B is a DC-algebra which is an f-subalgebra of A, then $B \leqslant \mathbf{dc}A$.*

P r o o f. The proof of (a) is none other than the proof of Proposition 3.1 just completed.

As for (b), we have from Proposition 3.1, that each $b \in B$ has a full c-spectrum (in B), say, $\{(x_i, r_i) \colon i \in I\}$. Let J be the subset of all indices i such that $r_i = 0$. Extend $\{x_i \colon i \in J\}$ to a maximal pairwise disjoint subset in $b^{\perp}$. Let's denote the new elements by $\{x_k \colon k \in K\}$, and set $r_k = 0$, for each $k \in K$. We leave it to the reader to verify that $\{(x_i, r_i) \colon i \in I \cup K\}$ constitutes a full c-spectrum support in A for b. □

Propositon 3.2 yields an immediate corollary.

Corollary 3.2.1. *Suppose that B is a dense f-subalgebra of A. Then $\mathbf{dc}B = B \cap \mathbf{dc}A$. Moreover, if $\mathbf{dc}B$ is dense in B then $\mathbf{dc}A$ is dense in A.*

Definition 3.3. Suppose now that X is an arbitrary Tychonoff space. We denote the DC-subalgebra $\mathbf{dc}C(X)$ by $\mathbf{dc}(X)$, and will say that X is an *almost DC-space* if $\mathbf{dc}(X)$ is dense in $C(X)$. A function belonging to $\mathbf{dc}(X)$ is said to be *densely constant.*

Example 3.4. Let A be the algebra of piecewise polynomials on $\mathbb{R}$. The only densely constant functions in A are, clearly, the constant ones; therefore $\mathbf{dc}A$ is not dense in A. On the other hand, A is dense in $C(\mathbb{R})$, and, as we will demonstrate in the next section, $\mathbb{R}$ is an almost DC-space. This means that the converse to Corollary 3.2.1 is false.

Having pointed that out, we also remind the reader that A in this example is not uniformly complete. It is reasonable to ask if the converse of Corollary 3.2.1 is true for uniformly complete algebras.

Following Definition 3.7, we give a formulation of this question for compact spaces.

Recall that, for any topological space X, $C(\beta X) \cong C^*(X)$, the subalgebra of bounded continuous functions. Half of the following proposition is a consequence of Corollary 3.2.1.

Proposition 3.5. *X is a DC-space if and only if βX is.*

Proof. If X is a DC-space then $C(X)$ is a DC-algebra, and then Corollary 3.2.1 guarantees that $C^*(X)$ is a DC-algebra as well, from which we conclude, in turn, that βX is a DC-space.

Conversely, suppose that βX is a DC-space. Let $0 < f \in C(X)$; write $f = (f \wedge 1)(f \vee 1)$. Note that $g = (f \vee 1)^{-1} \in C^*(X)$, and both $f \wedge 1$ and g are in $\mathbf{dc}(X)$. But it should be clear that if g is densely constant then so is its inverse, $f \vee 1$. This shows that f lies in $\mathbf{dc}(X)$, proving that $C(X)$ is a DC-algebra. □

This line of argument also works for almost DC-spaces. We leave the verification to the reader, and summarize, in the following proposition.

Proposition 3.6. *X is an almost DC-space if and only if βX is.*

Definition 3.7. Suppose that X and Y are compact spaces. Recall now that a continuous surjection $f\colon Y \longrightarrow X$ is *irreducible* if X is not the image of any proper closed subset of Y. It is well known that f is irreducible if and only if the map $A \mapsto f(A)$ is a boolean isomorphism from the algebra $\mathcal{R}(Y)$, of all regular closed sets of Y, onto $\mathcal{R}(X)$. (For the necessity, see 6.5 (d) in [12]; the sufficiency is clear, once it's remembered that in any regular space the regular closed subsets form a base for the closed sets.) This implies that $f\colon Y \longrightarrow X$ is irreducible if and only if the functorially induced embedding $C(f)\colon C(X) \longrightarrow C(Y)$ is dense.

From Proposition 3.1 and Corollary 3.2.1 we get the following result on the behavior of dense constancies with regard to irreducible maps.

Proposition 3.8. *Suppose that $f\colon Y \longrightarrow X$ is an irreducible map between compact spaces. Then*
(1) *If Y is a DC-space it follows that X is one, too.*
(2) *If X is an almost DC-space then so is Y.*

Remark 3.9. The formulation of the question raised in Example 3.4 is, for compact spaces, precisely the converse to (2) in Proposition 3.8. If this converse were valid then, since every extremally disconnected space is almost DC, it would follow that every Tychonoff space is an almost DC-space.

In the next section of this article we show that every metrizable space is an almost DC-space, but for now let us adduce some preliminary evidence that the conjecture that every compact space is almost DC is not as wild as it might seem at first glance.

The following are clearly almost DC-spaces:
(a) all DC-spaces;
(b) all spaces with a π-base of clopen sets.

Additional remark 3.10. The concept of "irreducible map" extends to that of "perfect and irreducible map" (see [8] or [12]). Without going through the details, we point out that, by joining Proposition 3.6 and 3.8 (1), one can generalize 3.8 (1) to the not-necessarily-compact case, for perfect and irreducible maps.

4. Metrizable spaces are almost DC-spaces

The main theorem of this section is the following.

Theorem 4.1. *Every normal space with a σ-discrete π-base is an almost DC-space.*

Before proceeding with the proof, let us give the definition of a σ-discrete π-base.

Definition 4.2. Let X be a topological space. A collection $\mathcal{D}$ of subsets of X is said to be *discrete* if each point of X has a neighbourhood which intersects at most one of the members of $\mathcal{D}$. This is evidently equivalent to requiring that the members of $\mathcal{D}$ are pairwise disjoint, and that the subset $\bigcup \mathcal{D}$, endowed with the subspace topology, is the topological coproduct of the members of $\mathcal{D}$.

A π-base $\mathcal{B}$ of X is *σ-discrete* if $\mathcal{B} = \bigcup\limits_n \mathcal{B}_n$, with each $\mathcal{B}_n$ a discrete family of sets. It should be clear, for instance, that a metric space has a σ-discrete π-base. It is well known that a metric space is normal.

P r o o f of Theorem 4.1. Let $\mathcal{B} = \bigcup\limits_n \mathcal{B}_n$ be a π-base of X, and assume that each $\mathcal{B}_n$ is a discrete family. We pick $0 < f \in C(X)$; the task at hand is to find a densely constant function g so that $0 < g \leqslant f$. Let us denote $\mathcal{B}_n = \{B_{n,i}\colon\ i \in I_n\}$. Passing to smaller sets, if necessary, we may assume without loss of generality that the variation of f on $B_{1,i}$ is less than $\frac{1}{2}$, for each $i \in I_1$. For each $i \in I_1$, select an open set $U_{1,i}$ so that $\operatorname{cl} U_{1,i} \subseteq B_{1,i}$. Let $B_1 = \bigcup\limits_{i \in I_1} B_{1,i}$. Since X is normal, we may apply the Tietze Extension Theorem, and the function f' defined on $(\bigcup\{U_{1,i}\colon\ i \in I\}) \cup (X \setminus B_1)$, by

$$f'(x) = \begin{cases} \inf f(B_{1,i}) & \text{if } x \in \operatorname{cl} U_{1,i}, \\ f(x) & \text{if } x \in X \setminus B_1, \end{cases}$$

can be extended continuously to a function $f_1 \in C(X)$. By replacing f_1, restricted to $B_{1,i}$, with $\inf(f(B_{1,i}))e_{1,i} \vee (f_1 \wedge f)$, for each $i \in I_1$, we may assume that $0 \leqslant f_1 \leqslant f$, and, in fact, $f - \frac{1}{2} \leqslant f_1 \leqslant f$.

Let I_2' denote the subset of I_2 consisting of all j for which

$$B_{2,j} \setminus \Bigl(\bigcup\{\operatorname{cl} U_{1,i}\colon\ i \in I_1\}\Bigr) \neq \emptyset.$$

Arguing as before, we can construct a function $f_2 \in C(X)$, which coincides with f_1 on the set

$$X \setminus \Bigl(\bigcup\Bigl\{B_{2,j} \setminus \Bigl(\bigcup\{\operatorname{cl} U_{1,i}\colon\ i \in I_1\}\Bigr)\colon\ j \in I_2'\Bigr\}\Bigr),$$

is constant on some open set $U_{2,j}$ such that

$$\operatorname{cl} U_{2,j} \subseteq B_{2,j} \setminus \Bigl(\bigcup\{\operatorname{cl} U_{1,i}\colon\ i \in I_1\}\Bigr),$$

for each $j \in I_2'$, and $0 \leqslant f_2 \leqslant f_1$ with $f_1 - \frac{1}{4} \leqslant f_2 \leqslant f_1$.

Next, we define $I_3' \subseteq I_3$ to be the set of all $k \in I_3$ for which

$$B_{3,k} \setminus \Bigl(\bigcup\{\operatorname{cl} U_{1,i}\colon\ i \in I_1\} \cup \Bigl(\bigcup\{\operatorname{cl} U_{2,j}\colon\ j \in I_2'\}\Bigr)\Bigr) \neq \emptyset.$$

Then the construction proceeds as before, to produce

(a) a Cauchy sequence of functions $f_1 \geqslant f_2 \geqslant \ldots \geqslant f_n \ldots \geqslant 0$ such that $f_{n-1} - \frac{1}{2^n} \leqslant f_n \leqslant f_{n-1}$, for each $n \in \mathbb{N}$;

(b) a family of subsets I_n' of I_n, consisting of those $j \in I_n$, along with open sets $U_{n,j}$ $(j \in I_n')$ such that $\operatorname{cl} U_{n,j}$ misses all the points of $B_{n,j}$ which lie in one of the $\operatorname{cl} U_{m,i}$, for some $m < n$, and $i \in I_m'$; note that $I_1' = I_1$;

(c) f_n is constant on $U_{n,j}$, for each $j \in I_n$, and agrees with f_m (for all $m < n$), on all $U_{m,i}$ ($i \in I_m$).

Let g be the limit of the f_n; then it satisfies $0 < g \leqslant f$, and it is constant on each $U_{n,i}$, for each n and each $i \in I_{n,i}$. To finish the proof one merely has to check that the union of all the $U_{n,i}$ is dense in X. To that end, let V be a non-empty open subset of X. For some $n \in \mathbb{N}$, and some $i \in I_n$, $B_{n,i} \subseteq V$. If $i \in I'_n$, then $U_{n,i} \subseteq B_{n,i}$; otherwise (by construction) $B_{n,i}$ is contained in the union of all the $\operatorname{cl} U_{m,j}$ (all $m < n$, $j \in I'_m$), and therefore meets at least one of these $U_{m,j}$. □

We have the following immediate corollary.

Corollary 4.1.2. *Every metrizable space and every normal space with a countable π-base is an almost DC-space.*

Let us refine Theorem 4.1. For the record, here is a working definition of separation of points.

Definition 4.3. Suppose that $f \in C(X)$; we say that f *separates the points* x and y in X if $f(x) \neq f(y)$. A subalgebra A of $C(X)$ is said to *separate points* if for each pair of (distinct) points $x, y \in X$, there is an $f \in A$ that separates them.

Recall the Stone-Weierstrass Theorem: For X compact, if A is a subalgebra of $C(X)$, which separates points and contains all the constants, then A is uniformly dense in $C(X)$.

By reworking the presentation in the proof of Theorem 4.1 slightly, one gets the following result.

Theorem 4.4. *Suppose that X is a compact space having a σ-discrete π-base. Then $\mathbf{dc}(X)$ separates the points of X, and, consequently, it is uniformly dense in $C(X)$.*

P r o o f. Pick two distinct points x and y in X. Select a closed set K which misses x and is a neighbourhood of y. Now let $g > 0$ be a function in $C(X)$ which is identically 1 on K and such that $g(x) = 0$; without loss of generality, we may also suppose that $g \leqslant 1$.

Before embarking on the process described in the proof of Theorem 4.1, modify the σ-discrete base $\mathcal{B}$ given there by removing all the members of $\mathcal{B}$ which are not contained in $\operatorname{int}(K)$ or in $X \setminus K$. The reader will easily see that the sets left still form a π-base. One then begins the proof of Theorem 4.1, arguing only on the members of the various $\mathcal{B}_n$ which are disjoint from K. The upshot: one gets a densely constant function h, such that $0 < h \leqslant g$, so that h and g agree on K, and $h(x) = 0$. Clearly then, h separates x and y.

Applying the Stone-Weierstrass Theorem, the proof is complete. □

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