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# A COMPARISON ON THE COMMUTATIVE NEUTRIX CONVOLUTION OF DISTRIBUTIONS AND THE EXCHANGE FORMULA

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*Abstract.* Let $\tilde{f}$, $\tilde{g}$ be ultradistributions in $\mathscr{Z}'$ and let $\tilde{f}_n = \tilde{f} * \delta_n$ and $\tilde{g}_n = \tilde{g} * \sigma_n$ where $\{\delta_n\}$ is a sequence in $\mathscr{Z}$ which converges to the Dirac-delta function $\delta$. Then the neutrix product $\tilde{f} \diamond \tilde{g}$ is defined on the space of ultradistributions $\mathscr{Z}'$ as the neutrix limit of the sequence $\{\frac{1}{2}(\tilde{f}_n\tilde{g} + \tilde{f}\tilde{g}_n)\}$ provided the limit $\tilde{h}$ exist in the sense that

$$\operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n\tilde{g} + \tilde{f}\tilde{g}_n, \psi\rangle = \langle \tilde{h}, \psi\rangle$$

for all $\psi$ in $\mathscr{Z}$. We also prove that the neutrix convolution product $f \circledast g$ exist in $\mathscr{D}'$, if and only if the neutrix product $\tilde{f} \diamond \tilde{g}$ exist in $\mathscr{Z}'$ and the exchange formula

$$F(f \circledast g) = \tilde{f} \diamond \tilde{g}$$

is then satisfied.

*Keywords*: distributions, ultradistributions, delta-function, neutrix limit, neutrix product, neutrix convolution, exchange formula

*MSC 2000*: 46F10

In the following, $\mathscr{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathscr{D}'$ denotes the space of distributions defined on $\mathscr{D}$.

The convolution product of certain pairs of distributions in $\mathscr{D}'$ is usually defined as follows, see for example Gel'fand and Shilov [5].

**Definition 1.** Let $f$ and $g$ be distributions in $\mathscr{D}'$ satisfying either of the following conditions:

(a) either $f$ or $g$ has bounded support,

(b) the supports of $f$ and $g$ are bounded on the same side. Then the *convolution product* $f * g$ is defined by the equation

$$\langle (f*g)(x), \varphi(x)\rangle = \langle g(y), \langle f(x), \varphi(x+y)\rangle\rangle \tag{1}$$

for arbitrary test function $\varphi$ in $\mathscr{D}$.

It follows that if the convolution product $f * g$ exists by Definition 1 then the following equations hold:

$$f * g = g * f, \tag{2}$$

$$(f * g)' = f * g' = f' * g. \tag{3}$$

Definition 1 is rather restrictive and in order to define further convolution products of distributions, Jones in [6] gave the following definition.

**Definition 2.** Let $f$ and $g$ be distributions in $\mathscr{D}'$ and let $\tau$ be an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leqslant \tau(x) \leqslant 1$,
(iii) $\tau(x) = 1, |x| \leqslant \frac{1}{2}$,
(iv) $\tau(x) = 0, |x| \geqslant 1$.

Let

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

for $n = 1, 2, \ldots$. Then the *convolution product* $f * g$ is defined as the limit of the sequence $\{f_n * g_n\}$, providing the limit $h$ exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g_n, \varphi\rangle = \langle h, \varphi\rangle$$

for all $\varphi$ in $\mathscr{D}$.

Note that in this definition the convolution product $f_n * g_n$ exists in the sense of Definition 1 since $f_n$ and $g_n$ both have bounded supports. It is clear that if the convolution product $f * g$ exists by this definition, then equation (2) holds. However, equations (3) need not necessarily hold since Jones proved that

$$1 * \operatorname{sgn} x = x = \operatorname{sgn} x * 1$$

and

$$(1 * \operatorname{sgn} x)' = 1, \quad 1' * \operatorname{sgn} x = 0, \quad 1 * (\operatorname{sgn} x)' = 2.$$

Many convolution products could still not be defined in the sense of Definition 2 and the following modification of Definition 2 was given in [2]:

**Definition 3.** Let $f$ and $g$ be distributions in $\mathscr{D}'$, let

$$\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n + n^{n+1}), & x < -n, \end{cases}$$

where $\tau$ is as in Definition 2 and let $f_n = f\tau_n$. Then the *neutrix convolution product* $f \boxed{*} g$ is defined to be the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit $h$ exists in the sense that

$$\underset{n\to\infty}{\text{N-lim}} \langle f_n * g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for all $\varphi$ in $\mathscr{D}$, where $N$ is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero as $n$ tends to infinity.

The convolution product $f_n * g_n$ in this definition is again in the sense of Definition 1, the supports of $f_n$ and $g_n$ being bounded. The neutrix convolution product $f \boxed{*} g$ clearly satisfies equation (2) if it exists, although it does not necessarily satisfy equations (3). A non-commutative neutrix convolution product, denoted by $f \circledast g$ was defined in [2].

In the following definition we will also give commutative neutrix convolution product which differs from Definition 3.

**Definition 4.** Let $f$ and $g$ be distributions in $\mathscr{D}'$, let

$$\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n + n^{n+1}), & x < -n, \end{cases}$$

where $\tau$ is as in Definition 2 and let $f_n = f\tau_n$, $g_n = g\tau_n$. Then the *neutrix convolution product* $f \diamond g$ is defined to be the neutrix limit of the sequence $\frac{1}{2}\{f_n * g + g_n * f\}$, provided the limit $h$ exists in the sense that

$$\underset{n\to\infty}{\text{N-lim}} \frac{1}{2} \langle f_n * g + g_n * f, \varphi \rangle = \langle h, \varphi \rangle$$

for all $\varphi$ in $\mathscr{D}$. The neutrix convolution product $f \diamond g$ clearly commutative and satisfies equations (2) and (3) if it exists.

It can be shown that if the convolution product $f*g$ exists in the sense of Definition 1, then the neutrix convolution product $f \circledast g$ exists and

$$2f * g = f \circledast g.$$

As in Gel'fand and Shilov [5], we define the Fourier transform of a function $\varphi$ in $\mathscr{D}$ by

$$F(\varphi)(\sigma) = \tilde{\varphi}(\sigma) = \int_{-\infty}^{\infty} \varphi(x)\mathrm{e}^{\mathrm{i}x\sigma}\,\mathrm{d}x.$$

Here $\sigma = \sigma_1 + i\sigma_2$ is a complex variable and it is well known that $\tilde{\varphi}(\sigma)$ is an entire analytic function with the property

$$|\sigma|^q|\tilde{\varphi}(\sigma)| \leqslant C_q\mathrm{e}^{a|\sigma_2|} \tag{4}$$

for some constants $C_q$ and $a$ depending on $\tilde{\varphi}$. The set of all analytic functions $\mathscr{Z}$ with property (4) is in fact the space

$$F(\mathscr{D}) = \{\psi : \exists\varphi \in \mathscr{D}, F(\varphi) = \psi\}.$$

The Fourier transform $\tilde{f}$ of a distribution $f$ in $\mathscr{D}'$ is an ultradistribution in $\mathscr{Z}'$, i.e. a continuous linear functional on $\mathscr{Z}$. It is defined by Parseval's equation

$$\langle \tilde{f}, \tilde{\varphi}\rangle = 2\pi\langle f, \varphi\rangle.$$

The *exchange formula* is the equality

$$F(f * g) = F(f)\cdot F(g). \tag{5}$$

It is well known that the exchange formula holds for all convolution products of distributions $f$ and $g$ satisfying Definition 1, provided $f$ and $g$ both have compact support, see for example Treves [7].

We now consider the problem of defining multiplication in $\mathscr{Z}'$. To do this we need the Fourier transform $F(\tau_n)$ of $\tau_n$ and write

$$\delta_n(\sigma) = \frac{1}{2\pi}F(\tau_n),$$

which is a function in $\mathscr{Z}$. Putting $\psi = \tilde{\varphi}$, we have from Parseval's equation

$$\langle \tau_n, \varphi\rangle = \frac{1}{2\pi}\langle F(\tau_n), F(\varphi)\rangle = \langle \delta_n, \psi\rangle.$$

Since

$$\lim_{n\to\infty}\langle\tau_n,\varphi\rangle = \lim_{n\to\infty}\int_{-\infty}^{\infty}\tau_n(x)\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}x = \langle 1,\varphi\rangle$$

for all $\varphi$ in $\mathscr{D}$ and since $F(1) = 2\pi\delta$, we obtain

$$\lim_{n\to\infty}\langle\delta_n,\psi\rangle = \langle\delta,\psi\rangle$$

for all $\psi$ in $\mathscr{Z}$. Thus $\{\delta_n\}$ is a sequence in $\mathscr{Z}$ converging to the Dirac delta function $\delta$.

If $f$ is an arbitrary distribution in $\mathscr{D}'$, then since $\delta_n$ is a function in $\mathscr{Z}$, the convolution product $\tilde{f}*\delta_n$ is defined by

$$\langle(\tilde{f}*\delta_n)(\sigma),\psi(\sigma)\rangle = \langle\tilde{f}(\nu),\langle\delta_n(\sigma),\psi(\sigma+\nu)\rangle\rangle \tag{6}$$

for arbitrary $\psi$ in $\mathscr{Z}$. If $\psi = \tilde{\varphi}$, we have

$$\psi(\sigma+\nu) = F[\mathrm{e}^{\mathrm{i}x\nu}\varphi(x)]$$

and it follows from Parseval's equation that

$$\begin{aligned}\langle\delta_n(\sigma),\psi(\sigma+\nu)\rangle &= \frac{1}{2\pi}\langle F(\tau_n)(\sigma),F(\mathrm{e}^{\mathrm{i}x\nu}\varphi)(\sigma)\rangle = \langle\tau_n(x),\mathrm{e}^{\mathrm{i}x\nu}\varphi(x)\rangle\\ &= \int_{-\infty}^{\infty}\tau_n(x)\mathrm{e}^{\mathrm{i}x\nu}\varphi(x)\,\mathrm{d}x\\ &\to \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}x\nu}\varphi(x)\,\mathrm{d}x = \psi(\nu).\end{aligned} \tag{7}$$

Thus

$$\lim_{n\to\infty}\langle\tilde{f}*\delta_n,\psi\rangle = \langle\tilde{f},\psi\rangle$$

for arbitrary $\psi$ in $\mathscr{Z}$ and it follows that $\{\tilde{f}*\delta_n\}$ is a sequence of infinitely differentiable functions converging to $\tilde{f}$ in $\mathscr{Z}'$.

This leads us to the following definition:

**Definition 5.** Let $f$ and $g$ be distributions in $\mathscr{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathscr{Z}'$ and let $\tilde{f}_n = \tilde{f}*\delta_n$ and $\tilde{g}_n = \tilde{g}*\delta_n$. Then the *neutrix product* $\tilde{f}\diamond\tilde{g}$ is defined to be the neutrix limit of the sequence $\frac{1}{2}\{\tilde{f}_n\cdot\tilde{g}+\tilde{g}_n\cdot\tilde{f}\}$, provided the limit $\tilde{h}$ exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty}\frac{1}{2}\langle\tilde{f}_n\cdot\tilde{g}+\tilde{g}_n\cdot\tilde{f},\psi\rangle = \langle\tilde{h},\psi\rangle$$

for all $\psi$ in $\mathscr{Z}$.

In this definition we use $\tilde{f} \diamond \tilde{g}$ to denote the neutrix product of $\tilde{f}$ and $\tilde{g}$ to distinguish it from the usual definition of the product $\frac{1}{2}\{\tilde{f}_n \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}\}$. If

$$\lim_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}, \psi\rangle = \langle \tilde{h}, \psi\rangle$$

for all $\psi$ in $\mathscr{Z}$, we simply say that the *product* $\tilde{f} \cdot \tilde{g}$ exists and equals $\tilde{h}$. We then of course have

$$\tilde{f} \diamond \tilde{g} = \tilde{f} \cdot \tilde{g}.$$

It is immediately obvious that if the neutrix product $\tilde{f} \diamond \tilde{g}$ exists then the neutrix product is commutative.

The product of ultradistributions in $\mathscr{Z}'$ also has the following property:

**Theorem 1.** *Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $\mathscr{Z}'$ and suppose that the neutrix products $\tilde{f} \diamond \tilde{g}$ and $\tilde{f} \diamond \tilde{g}'$ (or $\tilde{f}' \diamond \tilde{g}$) exist. Then the neutrix product $\tilde{f}' \diamond \tilde{g}$ (or $\tilde{f} \diamond \tilde{g}'$) exists and*

$$(\tilde{f} \diamond \tilde{g})' = \tilde{f}' \diamond \tilde{g} + \tilde{f} \diamond \tilde{g}'. \tag{8}$$

Proof. Let $\psi$ be an arbitrary function in $\mathscr{Z}$. Then

$$\langle \tilde{f}' \diamond \tilde{g}, \psi\rangle = \operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n' \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}', \psi\rangle, \quad \langle \tilde{f} \diamond \tilde{g}', \psi\rangle = \operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n \cdot \tilde{g}' + \tilde{g}_n' \cdot \tilde{f}, \psi\rangle.$$

Further,

$$\begin{aligned}
\langle (\tilde{f} \diamond \tilde{g})', \psi\rangle &= -\langle \tilde{f} \diamond \tilde{g}, \psi'\rangle \\
&= -\operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}, \psi'\rangle \\
&= -\operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\{\langle \tilde{f}_n, \tilde{g}\psi'\rangle + \langle \tilde{g}_n, f\psi'\rangle\} \\
&= -\operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n, (\tilde{g}\psi)'\rangle - \operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{g}_n, (\tilde{f}\psi)' - \tilde{g}'\psi\rangle \\
&= \operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n \cdot \tilde{g}' + \tilde{f}_n' \cdot \tilde{g}, \psi\rangle + \operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{g}_n' \cdot \tilde{f} + \tilde{g}_n \cdot \tilde{f}', \psi\rangle
\end{aligned}$$

and so

$$\operatorname*{N-lim}_{n\to\infty} \frac{1}{2}\langle \tilde{f}_n' \cdot \tilde{g} + \tilde{g}_n \cdot f', \psi\rangle = \langle (\tilde{f} \diamond \tilde{g})', \psi\rangle - \langle \tilde{f} \diamond \tilde{g}', \psi\rangle.$$

Hence the neutrix product $\tilde{f}' \diamond \tilde{g}$ exists and equation (8) follows.

It follows similarly that if $\tilde{f}' \diamond \tilde{g}$ exists then $\tilde{f} \diamond \tilde{g}'$ exists. □

We can now prove the exchange formula.

**Theorem 2.** *Let $f$ and $g$ be distributions in $\mathscr{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathscr{Z}'$. Then the neutrix convolution product $f \mathbin{\Diamond\!\!\!\!\ast} g$ exists in $\mathscr{D}'$, if and only if the neutrix product $\tilde{f} \diamond \tilde{g}$ exists in $\mathscr{Z}'$ and the exchange formula*

$$F(f \mathbin{\Diamond\!\!\!\!\ast} g) = \tilde{f} \diamond \tilde{g}$$

*is then satisfied.*

Proof. We have from equation (7) that

$$\langle \delta_n(\sigma), \psi(\sigma + \nu) \rangle = F(\tau_n \varphi)$$

and then from equation (6) that

$$\begin{aligned} \langle \tilde{f}_n, \psi \rangle = \langle \tilde{f} * \delta_n, \psi \rangle &= \langle \tilde{f}, F(\tau_n \varphi) \rangle = 2\pi \langle f, \tau_n \varphi \rangle \\ &= 2\pi \langle f_n, \varphi \rangle = \langle F(f_n), \psi \rangle. \end{aligned}$$

Analogusly we have,

$$\begin{aligned} \langle \tilde{g}_n, \psi \rangle = \langle \tilde{g} * \delta_n, \psi \rangle &= \langle \tilde{g}, F(\tau_n \varphi) \rangle = 2\pi \langle g, \tau_n \varphi \rangle \\ &= 2\pi \langle g_n, \varphi \rangle = \langle F(g_n), \psi \rangle \end{aligned}$$

on using Parseval's equation twice. It follows that $F(f_n) = \tilde{f}_n$. Similarly, we have $F(g_n) = \tilde{g}_n$. Now since the convolution product $\frac{1}{2}[f_n * g + g_n * f]$ exists in the sense of Definition 1 and $f_n$ and $g_n$ both have compact support

$$F(\frac{1}{2}(f_n * g + g_n * f)) = \frac{1}{2}[F(f_n) \cdot F(g) + F(g_n) \cdot F(f)] = \frac{1}{2}\{\tilde{f}_n \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}\}$$

and so on using Parseval's equation again

$$\pi \langle f_n * g + g_n * f, \varphi \rangle = \frac{1}{2} \langle F(f_n * g + g_n * f), \psi \rangle = \frac{1}{2} \langle \tilde{f}_n \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}, \psi \rangle$$

Suppose the neutrix convolution product $f \mathbin{\Diamond\!\!\!\!\ast} g$ exists. Then

$$\begin{aligned} 2\pi \langle f \mathbin{\Diamond\!\!\!\!\ast} g, \varphi \rangle &= \operatorname*{N-lim}_{n \to \infty} \pi \langle f_n * g + g_n * f, \varphi \rangle \\ &= \operatorname*{N-lim}_{n \to \infty} \frac{1}{2} \langle F(f_n * g + g_n * f), \psi \rangle \\ &= \operatorname*{N-lim}_{n \to \infty} \langle \tilde{f}_n \cdot \tilde{g} + \tilde{g}_n \cdot \tilde{f}, \psi \rangle = \langle \tilde{f} \diamond \tilde{g}, \psi \rangle \end{aligned}$$

for arbitrary $\varphi$ in $\mathscr{D}$ and $F\varphi$ in $\mathscr{Z}$, proving the existence of the neutrix product $\tilde{f} \diamond \tilde{g}$ and the exchange formula.

Conversely, if the neutrix product $\tilde{f} \diamond \tilde{g}$ exists then the argument can be reversed to prove the existence of the neutrix convolution product $f \mathbin{\Diamond\!\!\!\!\ast} g$ and the exchange formula. This completes the proof of the theorem. □

The following Definition and Theorems were given in [4].

**Definition 6.** Let $f$ and $g$ be distributions in $\mathscr{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathscr{Z}'$ and let $\tilde{f}_n = \tilde{f} * \delta_n$ and $\tilde{g}_n = \tilde{g} * \delta_n$. Then the *neutrix product* $\tilde{f} \,\Box\, \tilde{g}$ is defined to be the neutrix limit of the sequence $\{\tilde{f}_n . \tilde{g}_n\}$, provided the limit $\tilde{h}$ exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty} \langle \tilde{f}_n . \tilde{g}_n, \psi \rangle = \langle \tilde{h}, \psi \rangle$$

for all $\psi$ in $\mathscr{Z}$.

**Theorem 3.** *Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $\mathscr{Z}'$ and suppose that the neutrix products $\tilde{f} \,\Box\, \tilde{g}$ and $\tilde{f} \,\Box\, \tilde{g}'$ (or $\tilde{f}' \,\Box\, \tilde{g}$) exist. Then the neutrix product $\tilde{f}' \,\Box\, \tilde{g}$ (or $\tilde{f} \,\Box\, \tilde{g}'$) exists and*

$$(\tilde{f} \,\Box\, \tilde{g})' = \tilde{f}' \,\Box\, \tilde{g} + \tilde{f} \,\Box\, \tilde{g}'. \tag{9}$$

**Theorem 4.** *Let $f$ and $g$ be distributions in $\mathscr{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathscr{Z}'$. Then the neutrix convolution product $f \boxed{*} g$ exists in $\mathscr{D}'$, if and only if the neutrix product $\tilde{f} \,\Box\, \tilde{g}$ exists in $\mathscr{Z}'$ and the exchange formula*

$$F(f \boxed{*} g) = \tilde{f} \,\Box\, \tilde{g}$$

*is then satisfied.*

We finally give an example where the two commutative neutrix products differ

It was proved in [3] that

$$x^s \boxed{*} x_+^r = (-1)^{r+s+1} B(r+1, s+1) x_-^{r+s+1} \tag{10}$$

for $r, s = 0, 1, \ldots$, where $B$ denotes the Beta function and it can be proved easily that

$$x^s \diamondsuit x_+^r = \frac{r!s!}{(r+s+1)!} x^{r+s+1} \tag{11}$$

for $r, s = 0, 1, \ldots$. It follows from Definition 5 and Definition 6 respectively that

$$\mathrm{i}e^{\mathrm{i}r\pi/2}(\sigma + \mathrm{i}0)^{-r-1} \diamond \delta^{(s)}(\sigma) = \frac{(-i)^{r+1}s!}{(r+s+1)!}\delta^{(r+s+1)}(\sigma) \tag{12}$$

$$\mathrm{i}e^{\mathrm{i}(2r+s)\pi/2} 2\pi(-i)^s(\sigma + \mathrm{i}0)^{-r-1} \,\Box\, \delta^{(s)}(\sigma) = \frac{(-1)^{r+s+1}s!}{(r+s+1)!}(\sigma + \mathrm{i}0)^{-r-s-2} \tag{13}$$

for $r, s = 0, 1, 2, \ldots$, since $F(x^s) = 2(-i)^s\pi\delta^{(s)}(\sigma)$,

$$F(x_+^r) = \mathrm{i}e^{\mathrm{i}r\pi/2}\Gamma(r+1)(\sigma+\mathrm{i}0)^{-r-1}$$

and

$$F(x_-^{r+s+1}) = (-\mathrm{i})e^{-\mathrm{i}(r+s)\pi/2}\Gamma(r+1)(\sigma-\mathrm{i}0)^{-r-s-2}$$

for $r, s = 0, 1, 2, \ldots$, see Gel'fand and Shilov [5].

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