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UNIQUELY COVERED RADICAL CLASSES OF ℓ-GROUPS

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Abstract. It is proved that a radical class σ of lattice-ordered groups has exactly one cover if and only if it is an intersection of some σ-complement radical class and the big atom over σ.

Keywords: radical class, atom, unique covering question, quasi-complement radical class, σ-homogeneous

MSC 2000: 06E08

Let g and S be the classes of all ℓ-groups and of all radical classes of ℓ-groups, respectively. Let $\sigma, \tau \in S$. If $\sigma < \tau$ and there does not exist any $\varrho \in S$ such that $\sigma < \varrho < \tau$, then we say that τ is an atom over σ or that τ covers σ. Denote by $A(\sigma)$ the class of all atoms over σ. For $\sigma \in S$, $G \in g$, the symbol $\sigma(G)$ stands for the largest convex ℓ-subgroup of G which belongs to σ. Denote by $T(G)$ the least radical class containing G. Write $R(G) = \{\sigma(G) \mid \sigma \in S\}$ (see [2]).

Let $G \in g$. Let α be an infinite cardinal and $\omega(\alpha)$ be the least ordinal having cardinality α. For any $i \in \omega(\alpha)$, set $G_i = Z$, the additive group of integers under the usual order. Write $G(\alpha) = (\vec{\otimes} G_i) \vec{\otimes} G$ for the lexico-product of these G_i and G, $i \in \omega(\alpha)$, with order from left to right. Both $G(\alpha)$ and $T(G(\alpha))$ are called the regular atoms over G or $T(G)$ (cf. [1], [3]).

Let $\sigma \in S$. Suppose $A(\sigma) \neq \emptyset$. Put $\varepsilon(\sigma, 1) = \sup A(\sigma)$ and for any ordinal α, define inductively

$$\varepsilon(\sigma,\alpha) = \begin{cases} \sup A(\varepsilon(\sigma,\alpha-1)) & \text{when } \alpha \text{ is nonlimit,}\\ \bigvee_{\beta<\alpha} \varepsilon(\sigma,\beta) & \text{when } \alpha \text{ is limit.} \end{cases}$$

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Form $Z(\sigma) = \bigvee_\alpha \varepsilon(\sigma, \alpha)$, where α runs over all ordinals. Call $Z(\sigma)$ the big atom over σ (cf. [1], [3]).

In 1977, J. Jakubík raised the following question of unique covering of radical classes: whether there exists $\alpha \in S$ such that $A(\sigma)$ is a one-element class.

The first author of the present paper answered "yes" in [3] and gave several sufficient and necessary conditions for a radical class to have a unique covering under the condition "non-superatom", i.e. "not containing $A_0 = \sup A(0)$". In this short note, we prove a theorem for all radical classes having a unique covering by using the notion of σ-homogenity. Recall that an ℓ-group G is called homogeneous if $T(G)$ is an atom.

Definition 1. An ℓ-group G is called quasi-homogeneous if there exists a largest radical class which does not contain G. If it is the case, then denote this radical class by T^G and call it the quasi-complement radical class of G or $T(G)$.

Let $\sigma \in S$. Recall that the complement radical class of σ, denoted by σ', is the largest radical class meeting σ in 0.

In the sequel, the appearence of T^G always suggests that G be q.h. A homogeneous ℓ-group G is clearly q.h. and T^G coincides with $T(G)'$ in this case (cf. [3] for detail). The ℓ-group $\{0\}$ is trivialy non-q.h. The cardinal sum of Z and Q (the rationals with the usual order and the usual addition) provides a non-trivial example of non-q.h. ℓ-group.

Definition 2. Let $\sigma \in S$. An ℓ-group G is called σ-homogeneous of $\sigma(G)$ is maximal in $R(G) \setminus \{G\}$. In this case, call $\sigma^G = \sup\{\tau \in S \mid \tau(G) \leqslant \sigma(G)\}$ the associated σ-complement radical class of G or σ-complement of G for short.

Remark.

1. If G is σ-homogeneous, then $G \in g \setminus \sigma$.
2. If G is homogeneous, then for each $\sigma \in S$ with $G \in g \setminus \sigma$, G is σ-homogeneous and $\sigma^G = T^G$.
3. If G is an σ-homogeneous ℓ-group, then $A(\sigma) \neq \emptyset$ (since $\sigma \vee T(G) \in A(\sigma)$).
4. If G is quasi-homogeneous, then there is some $\sigma \in S$ such that G is σ-homogeneous, this is guaranteed by the following proposition.

Proposition 1. *Let $0 \neq G \in g$. G is quasi-homogeneous if and only if $R(G) \setminus \{G\}$ possesses a largest element.*

P r o o f. To show the condition is necessary, let $H \in R(G) \setminus \{G\}$. Then $T(H) < T(G)$ (otherwise, $H = T(H)(G) = T(G)(G) = G$, a contradiction). Thus $T(H) < T^G$ and $H \leqslant T^G(G)$. This asserts that $T^G(G)$ is the largest element in $R(G) \setminus \{G\}$.

Conversely, let H be the largest element in $R(G) \setminus \{G\}$. Put $\tau = \bigvee_{\sigma \in C} \sigma$, where C is the collection of all radical classes which do not contain G. Then $\tau(G) = \bigvee_{\sigma \in C} \sigma(G) \leqslant \bigvee_{\sigma \in C} \sigma(H) = H \neq G$. Hence $G \in g \setminus \tau$. Therefore $\tau = T^G$. $\square$

Proposition 2. *Let $G \in g$, $\sigma \in S$. Then G is σ-homogeneous if and only if $\sigma \vee T(G) \in A(\sigma)$. Therefore $A(\sigma) = \{\sigma \vee T(G) \mid G \text{ is } \sigma\text{-homogeneous}\}$.*

Proof. We have that $\sigma \vee T(G)$ covers σ if and only if $T(G)$ covers $T(\sigma(G))$ if and only if $\sigma(G)$ is a maximal element in $R(G) \setminus \{G\}$ if and only if G is σ-homogeneous. $\square$

Corollary. *$A(\sigma) = \emptyset$ if and only if there is no σ-homogeneous ℓ-group.*

Proposition 3. *Let G be σ-homogeneous, then the σ-complement of G has exactly one cover.*

Proof. Firstly, $\sigma^G(G) = (\bigvee_{\tau(G) \leqslant \sigma(G)} \tau)(G) = \bigvee \tau(G) \leqslant \sigma(G)$. Since $\sigma^G \geqslant \sigma$, we infer that $\sigma^G(G) = \sigma(G)$. Hence the projectivity of $[\sigma^G, T(G) \vee \sigma^G]$ and $[T(\sigma(G)), T(G)]$ implies that $T(G) \vee \sigma^G$ covers σ^G. On the other hand, if $(T(H) \vee \sigma^G) \in A(\sigma^G)$ and $T(H) \vee \sigma^G \neq T(G) \vee \sigma^G$, then

$$\sigma^G = (T(H) \vee \sigma^G) \wedge (T(G) \vee \sigma^G) = \sigma^G \vee (T(H) \wedge T(G)).$$

Hence

$$\sigma(G) = \sigma^G(G) \geqslant (T(H) \wedge T(G))(G) = T(H)(G).$$

Thus $T(H) \leqslant \sigma^G$, a contradiction. Therefore $A(\sigma^G)$ is a one-element class. $\square$

Corollary 1. *T^G has exactly one cover.*

Corollary 2. *Each regular atom $G(\alpha)$ over a nonzero ℓ-group G is q.h. and has exactly one cover. Moreover, $T^{G(\alpha)} \geqslant A_0$ and is therefore a superatom which is not a polar (cf.* [3]*).*

Theorem 4. *Let $\sigma \in S$. Then $A(\sigma)$ is a one-element class if and only if there is a σ-homogeneous ℓ-group G such that $\sigma = \sigma^G \wedge Z(\sigma)$.*

Proof. For the sufficiency, suppose that $\sigma = \sigma^G \wedge Z(\sigma)$, where G is a σ-homogeneous ℓ-group. We then have $\sigma \vee T(G)$ covers σ. If there exists some $H \in g$ with $\sigma \vee T(H) \in A(\sigma)$ and $\sigma \vee T(H) \neq \sigma \vee T(G)$, then, since $T(H)(G) \neq G$, only two cases may occur: either $T(H)(G) \leqslant \sigma(G)$ or $T(H)(G)$ is not comparable with $\sigma(G)$. For the former, we have $T(H) \leqslant \sigma^G$ and $T(H) \leqslant \sigma^G \wedge Z(\sigma) = \sigma$, which contradicts $T(H) \in A(\sigma)$. For the latter, we have $T(H)(G) \vee \sigma(G) = G$, therefore $T(H) \vee T(\sigma(G)) \geqslant T(G)$. Thus $\sigma \vee T(H) = \sigma \vee T(\sigma(G)) \vee T(H) \geqslant \sigma \vee T(G)$. Hence $\sigma \vee T(H) = \sigma \vee T(G)$, which is not the case. So σ is uniquely covered by $\sigma \vee T(G)$.

To show the condition is also necessary, let $A(\sigma)$ be a one-element class with the unique element ϱ. Then, from Proposition 2, we have a σ-homogeneous ℓ-group G with $\varrho = \sigma \vee T(G)$. Clearly, $\sigma \leqslant \sigma^G \wedge Z(\sigma)$. Now, let $H \in g \setminus \sigma$, then either $H \in g \setminus \sigma^G$ or $H \in \sigma^G$. The first case implies that $H \in g \setminus \sigma^G \wedge Z(\sigma)$. For the second case, we infer that $G \in g \setminus T(H)$ and $T(H)(G) \leqslant \sigma^G(G) = \sigma(G)$, then $\sigma \vee T(H)$ is not comparable with ϱ. This implies that $\sigma \vee T(H)$ is not contained in $Z(\sigma)$, since ϱ is the unique atom over σ, thus $H \in g \setminus Z(\sigma)$ and $H \in g \setminus \sigma^G \wedge Z(\sigma)$, therefore $\sigma = \sigma^G \wedge Z(\sigma)$. This finishes the proof. □

Remark. In general, $\sigma^G \wedge Z(\sigma) \neq \sigma^G$. For instance, let $G = Z \oplus Q$, $\sigma = T(Q)$, then $\sigma^G = T^Z \neq \sigma = \sigma^G \wedge Z(\sigma)$.

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