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WEAK COMPACTNESS CRITERIA FOR SET VALUED INTEGRALS AND RADON NIKODYM THEOREM FOR VECTOR VALUED MULTIMEASURES

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Abstract. Some criteria for weak compactness of set valued integrals are given. Also we show some applications to the study of multimeasures on Banach spaces with the Radon-Nikodym property.

Keywords: weak compactness, measurable multifunctions, Radon-Nikodym property, multimeasures

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1. Introduction

The theory of measurable multifunctions has shown to be useful in many mathematical fields such as Control Theory [1], Convex Analysis [6], Abstract Evolution Equations [15], etc.

It is the purpose of this paper to provide some results about the weak compactness of measurable selections of a measurable multifunction, and to use them to show a Radon-Nikodym Theorem for multimeasures.

2. Preliminaries

In this section we state some notation and definitions that we are using in the paper.

For a Banach space X, its dual will be denoted by X^*.

We will also denote by $P_f(X)$, $P_{fc}(X)$, $P_k(X)$, $P_{kc}(X)$, $P_{\omega k}(X)$ and $P_{\omega kc}(X)$ the sets of nonempty subsets of X that are closed, closed convex, compact, compact convex, weakly compact, and weakly compact convex, respectively.

For a subset A of X we set

$$|A| = \sup_{a \in A} \|a\|;$$
$$\mathrm{cl}(A) = \text{the norm closure of } A;$$
$$\overline{c_0}(A) = \text{the closed convex hull of } A;$$

It has been standard to define measurable multifunctions as follows:

Given a separable Banach space X and a measurable space (Ω, Σ), a multifunction $F\colon \Omega \to P_f(X)$ is called *measurable* if for each $z \in X$ the function

$$f(\omega) = d(z, F(\omega)) = \inf_{y \in F(\omega)} \|z - y\|$$

is measurable; by Castaing Representation ([6]), a closed valued multifunction $F\colon \Omega \to X$ is measurable if and only if there is a sequence $f_n\colon \Omega \to X$ of measurable functions such that $F(\omega) = \mathrm{cl}\{f_n(\omega)\}$ for each $\omega \in \Omega$.

Interested in dealing with integration in non separable Banach spaces, the authors of [3], inspired by Castaing Representation, defined μ-measurability of multifunctions in arbitrary Banach spaces in the following way: Given a complete finite measure space (Ω, Σ, μ) and a Banach space X, a multifunction $F\colon \Omega \to P_f(X)$ is called μ-*measurable*, if there is a μ-null set $N \in \Sigma$ and a sequence of μ-measurable functions $f_n\colon \Omega \to X$ such that

$$F(\omega) = \mathrm{cl}\,\{f_n(\omega)\} \qquad \text{for all } \omega \in \Omega \setminus N.$$

This definition allows us to deal with considerable generality in all our results.

Given a measurable multifunction $F\colon \Omega \to P_f(X)$, we denote by S_F^p the set

$$S_F^p = \{f\colon \Omega \to X\colon\ f \in L_X^p(\mu);\ f(\omega) \in F(\omega)\ \mu\text{-a.e.}\},$$

and for $E \in \Sigma$ we denote

$$\int_E F\,\mathrm{d}\mu = \left\{\int_E f\,\mathrm{d}\mu\colon\ f \in S_F^1\right\}.$$

We say that a measurable multifunction F is *integrably bounded* if $|F(.)| \in L^1(\mu)$. We recall that a subset K in $L_X^1(\mu)$ is *uniformly integrable* if for each $\varepsilon > 0$, there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E \|f\|\,\mathrm{d}\mu < \varepsilon\ \forall f \in K$. A sequence F_n of integrably bounded multifunctions is *uniformly integrable* if the sequence $\{|F_n(.)|\}$ is uniformly integrable. Following [19, 20], for $\{A_n, A\} \subset P_f(X)$, we say that $A_n's$ *weakly converges* to A ($A_n \xrightarrow{\omega} A$) if, $\sigma(x^*, A_n) \to \sigma(x^*, A)$ for each $x^* \in X^*$ where

$\sigma(x^*, B) = \sup\{\langle x^*, x\rangle \colon x \in B\}$ for any non-empty subset B of X. A sequence of measurable multifunctions $\{F_n\}_{n=1}^{\infty}$ is said to be *weakly convergent* to F in $L^1_X(\mu)$ ($F_n \xrightarrow{\omega} F$), if

$$\int_\Omega \sigma(x^*(\omega), F_n(\omega))\,\mathrm{d}\mu(\omega) \to \int_\Omega \sigma(x^*(\omega), F(\omega))\,\mathrm{d}\mu(\omega)$$

for each $x^* \in (L^1_X(\mu))^*$.

A *multimeasure* is a function $M\colon \Sigma \to P(X)$ satisfying

(i) $M(\emptyset) = \{0\}$;

(ii) if $E_1, E_2 \in \Sigma$ with $E_1 \cap E_2 = \emptyset$, then $M(E_1 \cup E_2) = M(E_1) + M(E_2)$;

(iii) if $\{E_n\}_{n=1}^{\infty}$ is a sequence in Σ with $E_i \cap E_j = \emptyset\ \forall i \neq j$ then

$$\begin{aligned} M\Bigl(\bigcup_{n=1}^{\infty} E_n\Bigr) &= \sum_{n=1}^{\infty} M(E_n) \\ &= \{x \in X\colon \text{ for each } n \in \mathbb{N},\ \text{ there is } x_n \in M(E_n) \\ &\qquad \text{such that } \textstyle\sum x_n \text{unconditionally converges to } x\}. \end{aligned}$$

The multimeasure M is called to have *bounded variation* if

$$\|M\| = \sup \sum_{i=1}^{n} \|M(A_i)\|$$

is finite where the sup is taken over all finite partition of Ω.

For a fixed measurable space $(\Omega, \Sigma), c_a(X)$ will denote the Banach space of all X valued countably additive, bounded variation vector measures endowed with the norm of total variation.

3. Weak compactness criteria for S^p_F in $L^p_X(\mu)$

The following result can be found in [3].

Theorem 3.1. *Let $F\colon \Omega \to P_{fc}(X)$ be an integrably bounded multifunction. Then S^1_F is weakly compact in L^1_X if and only if for almost every $F(\omega)$ is weakly compact $\omega \in \Omega$.*

A small refinement of the above theorem yields the following one.

Theorem 3.2. *If $1 \leqslant p < \infty$ and $F\colon \Omega \to P_f(X)$ is a measurable multifunction, then the following statements are equivalent:*

(a) *S^p_F is relatively weakly compact in $L^p_X(\mu)$.*

(b) *S^p is bounded in $L^p_X(\mu)$ and the multifunction $G\colon \Omega \to P_{fc}(X)$ defined by $G(\omega) = \overline{c_0}F(\omega)$ takes weakly compact values μ-a.e.*

P r o o f. (a $\Rightarrow$ b). Suppose $p = 1$. If S^1_F is relatively weakly compact in $L^1_X(\mu)$ then it is bounded, and by [13] (Theorem 3.2) F is integrably bounded. Furthermore, given a sequence $\{f_n\} \subseteq S^1_F$, there is a sequence $g_n \in \overline{c_0}\{f_k\colon k \geqslant n\}$ ([8], Theorem 2.1) such that $g_n(\omega)$ is norm convergent in X μ-a.e. This implies $\overline{c_0}F(\omega)$ weakly compact μ-a.e.

(b $\Rightarrow$ a). If S^1_F is bounded and $\overline{c_0}F(\omega)$ is weakly compact μ-a.e., being F measurable, there is a null set $N_0 \in \Sigma$ and a sequence $f_n\colon \Omega \to X$ of measurable functions such that $\mu(N_0) = 0$ and $F(\omega) = \operatorname{cl}(f_n(\omega))$ $\forall \omega \in \Omega \setminus N_0$. Applying the Pettis measurability theorem [9], for each $n \in \mathbb{N}$ there is $N_n \in \Sigma$ with $\mu(N_n) = 0$ and $\operatorname{cl}(f_n(\Omega \setminus N_n))$ is separable.

If we put $N = \bigcup\limits_{n=0}^{\infty} N_n$ we have that $\mu(N) = 0$ and $F(\Omega \setminus N)$ is separable. Let Y be the separable Banach space generated by $F(\Omega \setminus N)$. Then if we set, as in [3],

$$H\colon \Omega \to P_f(Y),$$

$$H(\omega) = \begin{cases} F(\omega) & \text{if } \omega \in \Omega \setminus N, \\ \{0\} & \text{if } \omega \in N, \end{cases}$$

H is a measurable multifunction taking values in a separable Banach space. Applying Theorem 1.5 of [13], we have that $\overline{c_0}H$ is a measurable multifunction. Since $G(\omega) = \overline{c_0}F(\omega) = \overline{c_0}H(\omega)$ μ-a.e., we conclude that G is a measurable multifunction taking values in a separable Banach space. It is easy to see that G is integrably bounded and $G(\omega) \in P_{\omega kc}(X)$ μ-a.e. So by Theorem 3.1, $S^1_{\overline{c_0}F}$ is weakly compact in $L^1_X(\mu)$ and consequently S^1_F is relatively weakly compact.

Let $1 < p < \infty$. Since S^p_F is relatively weakly compact in $L^p_X(\mu)$ and the injection $i\colon L^p_X(\mu) \to L^1_X(\mu)$ is continuous, the set S^p_F is relatively weakly compact in $L^1_X(\mu)$.

If we put $M = \overline{S^p_F}$, we have that M is decomposable, i.e. if $f, g \in M$ and $A \in \Sigma$, then $fX_A + gX_{\Omega\setminus A} \in \Sigma$. Then, according to [13] Theorem 3.1, there is a measurable multifunction $G\colon \Omega \to P_f(X)$ such that $M = S^1_G$. Since $\overline{S^1_G}$ is weakly compact in $L^1_X(\mu)$, we see that $\overline{c_0}G(\omega)$ is weakly compact μ-a.e. Since $S^1_G = S^p_G \supset S^p_F$, Corollary 1.2 from [13] implies the conclusion.

For the converse, suppose $\overline{c_0}F(\omega)$ is weakly compact for almost every $\omega \in \Omega$ and S^p_F is bounded in $L^p_X(\mu)$; then S^p_F is bounded in $L^1_X(\mu)$. It is not hard to see that

$$S^p_F \subset S^p_{\overline{c_0}F} = S^1_{\overline{c_0}F}.$$

By Theorem 3.1, $S^1_{\overline{c_0}F}$ is weakly compact in $L^1_X(\mu)$, which implies that S^p_F is relatively weakly compact in $L^1_X(\mu)$. Applying corollary 3.4 of [8], we conclude that S^p_F is relatively weakly compact in $L^p_X(\mu)$. $\square$

Corollary 3.1. *If $F(\omega)$ is convex and weakly compact μ-a.e. with F a measurable integrably bounded multifunction, then for $1 \leqslant p < \infty$, S_F^p is weakly compact in $L_X^p(\mu)$ if and only if it is bounded.*

Proof. The condition is necessary for S_F^p to be relatively weakly compact.

On the other hand, let $\{f_n\}$ be a sequence in S_F^p converging to f in the weak topology of $L_X^p(\mu)$. By Mazur Theorem there is a sequence relabeled as $\{f_n\}$ converging to f in the strong topology of $L_X^1(\mu)$. So there is a subsequence$\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k(\omega)\longrightarrow f(\omega)}$ for almost every $\omega \in \Omega$. This implies $f(\omega) \in F(\omega)$ μ-a.e., and f is measurable. Therefore $f \in S_F^p$. □

Corollary 3.2. *Let X be a Banach space and $1 \leqslant p < \infty$. For every measurable and integrably bounded multifunction $F\colon \Omega \to P_{fc}(X)$, S_F^p is weakly compact if and only if X is reflexive.*

Proof. It is a consequence of the well known fact that a Banach space X is reflexive if and only if bounded sets and relatively weakly compact ones are the same. □

Remark 3.1. According to Theorem 3.2 above, Theorems 5.2 and 5.5 of [16] hold for any Banach space X and any $p \in [1, +\infty)$. On the other hand, Theorem 5.4 of [16] is false since c_0 does not contain any isomorphic copy of ℓ_1, and the multifunction $F\colon [0,1] \to P_{fc}(c_0)$, defined by $F(t) \equiv B_{c_0} = \{x \in c_0\colon \|x\|_{c_0} \leqslant 1\}$, is μ-measurable and integrably bounded with respect to the Lebesgue measure on [0,1]; but S_F^1 is not weakly compact in $L_{c_0}^1(\mu)$.

If we want Theorem 5.4 of [16] to be true we should add the hypothesis X is weakly sequentially complete, since according to Rosenthal ℓ_1 dichotomy a Banach space X with no copy of ℓ_1 is reflexive if and only if it is sequentially weakly complete and in such a case our Corollary 3.2 can be applied.

Remark 3.2. The weak compactness of S_F^1 plays a key role in the existence of a mild solution of evolution inclusions ([17]) with the hypothesis $F\colon \Omega \to P_{\omega kc}(X)$. In [15], in an attempt of giving a different approach in the context of reflexive Banach spaces, the weak compactness was replaced by closedness and boundedness. According to Corollary 3.2, this is a particular case of [17].

4. Weak limits of sequences of measurable multifunctions

In this section we generalize a result due to Castaing [4] and Papageorgiou [19].

Theorem 4.1. *Let X be a Banach space with X^* having the Radon-Nikodym property. Let $\{F_n\}$ be a uniformly integrable sequence of measurable multifunctions $F_n\colon \Omega \to P_{\omega kc}(X)$ satisfying the following conditions:*

(i) *For every $A \in \Sigma$, the set*

$$H_A = \bigcup_{n=1}^{\infty} \int_A F_n \,\mathrm{d}\mu$$

is relatively weakly compact.

(ii) *Any bounded variation vector measure $m\colon \Sigma \to X$ verifying $m(A) \in \overline{c_0}(H_A)$ for all $A \in \Sigma$ admits a density in $L^1_X(\mu)$. Then there exists $F\colon \Omega \to P_{\omega kc}(X)$ integrably bounded and a subsequence $\{F_{nk}\}$ of $\{F_n\}$ such that $F_{nk} \to F$ in $L^1_X(\mu)$.*

Proof. Since $F_n\colon \Omega \to P_{\omega kc}(X)$ for each $n \in \mathbb{N}$ is a measurable multifunction, we have that for each $n \in \mathbb{N}$ there is a set $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and $F_n(\Omega \backslash N_n)$ is separable. If $N = \bigcup\limits_{n=1}^{\infty} N_n$ then $\mu(N) = 0$ and the closed subspace Y generated by $\bigcup\limits_{n=1}^{\infty} F_n(\Omega \setminus N)$ is separable. Now we define

$$G_n\colon \Omega \to P_{\omega kc}(Y)$$

by

$$G_n(\omega) = \begin{cases} F_n(\omega); & \omega \in \Omega \setminus N \\ \{0\}; & \omega \in N. \end{cases}$$

The sequence G_n is a sequence of measurable multifunctions satisfying

$$\bigcup_{n=1}^{\infty} \int_A G_n \,\mathrm{d}\mu = H_A;$$

since X^* has the Radon-Nikodym property, by [23], every separable subspace of X has a separable dual. So Y^* is separable. Applying Theorem 5.1 of [4] we find a measurable multifunction

$$F\colon \Omega \to P_{\omega kc}(Y) \subset P_{\omega kc}(X)$$

and a subsequence G_{nk} of G_n such that $G_{nk} \xrightarrow{\omega} F$ in $L^1_X(\mu)$. Since for each $n \in \mathbb{N}$; $G_n = F_n$ μ-a.e., we conclude that $F_{nk} \to F$ in $L^1_X(\mu)$.

An operator theoretical application may be interesting. □

Theorem 4.2. *Let X and Y be Banach spaces and $T\colon X \to Y$ a weakly compact operator. If $F_n\colon \Omega \to P_{\omega kc}(X)$ is a sequence of μ measurable multifunctions which is uniformly integrable and bounded in $L^1_X(\mu)$, then there is a subsequence $\{F_{nk}\}$ of $\{F_n\}$ and $G\colon \Omega \to P_{\omega kc}(Y)$ such that $TF_{nk} \xrightarrow{\omega} G$ in $L^1_X(\mu)$.*

Proof. Since $T\colon X \to Y$ is a weakly compact operator, the factorization scheme of [7] provides a reflexive Banach space Z and a pair of bounded linear operators $T_1\colon X \to Z$ and $T_2\colon Z \to Y$ such that $T = T_2 \circ T_1$. If we concentrate ourselves on $T_1F_n\colon \Omega \to P_{\omega kc}(Z)$, we find that $\{T_1F_n\}_{n=1}^{\infty}$ is a sequence of bounded and uniformly integrable multifunctions on $L^1_X(\mu)$.

Hence $\bigcup\limits_{n=1}^{\infty}\{\int_A T_1F_n\,\mathrm{d}\mu\}$ is bounded in Z for each $A \in \Sigma$ and, by reflexivity, relatively weakly compact. Since both Z and Z^* have the Radon-Nikodym property, Theorem 4.1 implies the existence of a measurable multifunction $F\colon \Omega \to P_{\omega kc}(Z)$ and a subsequence $\{F_{nk}\}$ of $\{F_n\}$ such that

$$\int_A \sigma(T_1F_{nk}, z^*)\,\mathrm{d}\mu \to \int_A (F, z^*)\,\mathrm{d}\mu$$

for each $z^* \in Z^*$.

Now, given $y^* \in Y^*, y^*T_2 \in Z^*$, we have

$$\sigma(TF_{nk}, y^*) = \sigma(T_1F_{nk}, y^*T_2)$$

and

$$\sigma(T_2F, y^*) = \sigma(F, y^*T_2).$$

So the conclusion follows with $G = T_2F$. □

We recall that, by applying the above factorization scheme, Papageorgiou [16] has got the following result for separable Banach spaces. Since this result easily extends to arbitary Banach spaces, we state it without the separability assumption:

Theorem 4.3. *Let $F_n\colon \Omega \to P_{fc}(X)$ be a sequence of measurable multifunctions and $W \in P_{\omega kc}(X)$ such that $F_n(\omega) \subseteq W$ μ-a.e. for all $n \in N$. Then there is $F\colon \Omega \to P_{\omega kc}(X)$ and a subsequence $\{F_{nk}\}$ of $\{F_n\}$ such that $F_{nk} \xrightarrow{\omega} F$ in $L^1_X(\mu)$.*

Definition 1. Let $M\colon \Sigma \to P_{\omega kc}(X)$ be a multimeasure, and $\mu\colon \Sigma \to [0,\infty)$ a positive measure. M is called *μ-representable* if there is a μ-measurable multifunction $F\colon \Omega \to P_{\omega kc}(X)$, integrably bounded and such that

$$M(A) = \int_A F\,\mathrm{d}\mu \quad \forall A \in \Sigma.$$

In this case we say that M is μ-representable by F.

We say that M is absolutely continuous with respect to $\mu(M \ll \mu)$ if $\mu(E) = 0$ implies $M(E) = \{0\}$.

Proposition 5.1. *Let $M\colon \Sigma \to P_{\omega kc}(X)$ be a multifunction μ-representable by F. Then*

(a) $M(\Sigma) = \bigcup_{A\in\Sigma} M(A)$ *is separable;*

(b) *F is essentially unique in the sense that if G is another multifunction representing M then $F = G$ μ.c.s.*

P r o o f. a) If there is a μ-measurable multifunction $F\colon \Omega \to P_{\omega kc}(X)$ such that F is integrably bounded and $\int_A F\,\mathrm{d}\mu = M(A)$ $\forall A \in \Sigma$, then by definition there is $N \in \Sigma$ such that $\mu(N) = 0$ and $\bigcup_{\omega\in\Omega\setminus N} F(\omega)$ is separable. Let Y be the separable subspace generated by $\bigcup_{\omega\in\Omega\setminus N} F(\omega)$. Then for each selector f of F we have $\int_A f\,\mathrm{d}\mu \in Y$, which implies that $\bigcup_{A\in\Sigma} M(A)$ is separable.

b) By (a), we can suppose X separable. Now we apply Theorem III.35 of [6] to get the conclusion. □

Theorem 5.1. *Let X be a Banach space. The following statements are equivalent:*

(a) *Both X and X^* have the Radon-Nikodym property.*

(b) *For every complete finite measure space (Ω,Σ,μ) and any μ continuous bounded variation multimeasure $M\colon \Sigma \to P_{\omega kc}(X)$ with $M(\Sigma)$ separable, there is a μ-measurable integrably bounded multifunction $F\colon \Omega \to P_{\omega kc}(X)$ such that $M(A) = \int_A F\,\mathrm{d}\mu$ $\forall A \in \Sigma$.*

If μ is non-atomic, (a) *and* (b) *are equivalent to*

(c) *For every probability space (Ω,Σ,μ) and every μ-continuous bounded variation multimeasure $M\colon \Sigma \to P_{\omega k}(X)$ with $M(\Sigma)$ separable, there is an integrably* *bounded multifunction* $F\colon \Omega \to P_{\omega k}(X)$ *such that*

$$M(A) = \int_A F \,\mathrm{d}\mu, \quad \forall A \in \Sigma.$$

Proof. (a $\Rightarrow$ b). Since $M(\Sigma)$ is separable, there is no loss of generality in assuming X separable. Since X^* has the Radon-Nikodym property, it is separable and the proof follows as either in [5] or [12].

(b $\Rightarrow$ a). If X does not have the Radon-Nikodym property, then there is a separable subspace Y of X which lacks this a property. So there is a $m\colon \Sigma \to Y$ vector measure bounded variation and $m \ll \mu$, which is not μ-representable where $\Omega = [0,1]$ Σ is the Borel σ-algebra and μ is the Lebesgue measure. Therefore the Radon-Nikodym property on X is a sufficient condition.

Suppose X^* lacks the Radon-Nikodym property. By the proposition in [11], if $\Omega = \{-1,1\}^{\mathbb{N}}$ is the Cantor group and μ the normalized Haar measure on Ω, there is a subset $H \subseteq L^1_X(\mu)$ such that

(i) H is uniformly bounded;
(ii) $\{\int_A f \,\mathrm{d}\mu\}_{f\in H}$ is relatively weakly compact for each $A \in \Sigma$;
(iii) H is not relatively weakly compact in $L^1_X(\mu)$.

Now we define

$$G = \left\{ f = \sum_{i=1}^{n} g_i X_{Ai}; g_i \in H, A_i \in \Sigma; A_i \cap A_j = \emptyset \ \forall i \neq j \& \bigcup_{i=1}^{n} A_i = \Omega \right\}.$$

Since G is a bounded decomposable subset of $L^1_X(\mu)$, so is $\overline{G}$. So there is a μ-measurable integrably bounded multifunction $F'\colon \Omega \to P_f(X)$ such that $S^1_{F'} = \overline{G}$. Take $F = \overline{c_0}F'$. Then F is integrably bounded and by the summation technique used in the proof of Theorem II.3.8 of [9] we get that $M(A) = \int_A F \,\mathrm{d}\mu \subset c_0(\int_A f \,\mathrm{d}\mu)_{f\in H}$ and by Krein-Smulyan $M(A) = \left\{\int_A F \,\mathrm{d}\mu\right\}$ is relatively weakly compact for each $A \in \Sigma$. Since F is closed convex valued, so is M. In conclusion, $M(.) = \int_{(.)} F \,\mathrm{d}\mu$ is a weakly compact convex multimeasure. Since $H \subseteq S^1_F$, this set is not relatively weakly compact and by Theorem 3.2, $F(\omega)$ is not weakly compact μ-a.e.

(a $\Rightarrow$ c). Take $M\colon \Sigma \to P_{\omega k}(X)$ with $M \ll \mu$ and $M(\Sigma)$ separable. Since X has the Radon-Nikodym property, by [24], $\operatorname{cl} M(A)$ is convex for each $A \in \Sigma$. Thus $M(A)$ is convex and weakly compact for each $A \in \Sigma$. Therefore, we have reduced the problem to the implication a $\Rightarrow$ b.

(c $\Rightarrow$ a). If $M\colon \Sigma \to P_{\omega k}(X)$ is a multimeasure such that $\forall A \in \Sigma$,

$$M(A) = \int_A F \,\mathrm{d}\mu$$

for some $F\colon \Omega \to P_{\omega k}(X)$, integrably bounded, then by corollary I of [18], $\operatorname{cl} M(A)$ is convex for each $A \in \Sigma$. So

$$M(A) = \overline{c_0}\left(\int_A F\,\mathrm{d}\mu\right) = \int_A \overline{c_0} F\,\mathrm{d}\mu$$

and by the implication b $\Rightarrow$ a, the proof is complete. □

Remark 5.1. The equivalence (a) $\Leftrightarrow$ (b) is found in [14] (Theorem 5.3) with a different proof.

If $S_M = \{m\colon \Sigma \to X; m \in c_a(X), m(A) \in M(A) \forall A \in \Sigma\}$ with M a compact valued multimeasure then the following holds.

Theorem 5.2. *For a Banach space X, the following statements are equivalent:*

(a) *X has the Radon-Nikodym property.*

(b) *If $M\colon \Sigma \to P_k(X)$ is a μ-continuous bounded variation multimeasure such that S_M is compact in $c_a(X)$ then there is an integrably bounded multifunction $F\colon \Omega \to P_{kc}(X)$ such that*

$$M(A) = \int_A F\,\mathrm{d}\mu.$$

P r o o f. Suppose X has the Radon-Nikodym property. Then by [24] Theorem 2.7, $M(\Sigma)$ is relatively compact in X. Therefore $M(\Sigma)$ is separable.

For each $m \in S_M$ there is $f_m \in L^1_X(\mu)$ such that

$$m(A) = \int_A f_m\,\mathrm{d}\mu \quad \forall A \in \Sigma$$

and S_M is isomorphic to $\{f_m\}_{m \in S_M} \subseteq L^1_X(\mu)$. Furthermore, by [10] we have that for each $A \in \Sigma$,

$$M(A) = \left\{\int_A f_m\,\mathrm{d}\mu\right\}_{m \in S_M}.$$

Since $\{f_m\}_{m \in S_M}$ is a decomposable compact subset of $L^1_X(\mu)$ we have that $\{f_m\}_{m \in S_M}$ is also separable in $L^1_X(\mu)$; hence we can suppose X separable. So by [13], there is an integrably bounded multifunction $F\colon \Omega \to P_{fc}(X)$ such that $S^1_F = \{f_m\}_{m \in S_M}$. Therefore

$$M(A) = \int_A F\,\mathrm{d}\mu \quad \text{for each } A \in \Sigma$$

with S^1_F compact in $L^1_X(\mu)$. This implies $F(\omega)$ is weakly compact μ-a.e. and by [2] Proposition 7, $F(\omega)$ is compact μ-a.e.

Conversely, if (b) holds, it holds for any single vector measure, which is the definition of the Radon Nikodym property. □

Since the unit ball of $L^\infty([0,1])$ is not compact in $L^1[0,1]$, the multimeasure M can be represented by a compact valued multifunction without S_M being a compact subset of $c_a(X)$, as is shown in the next theorem.

Theorem 5.3. *Let X be a Banach space. The following assertions are equivalent:*

(a) *For every $F\colon [0,1] \to P_{\omega kc}(X)$ μ-measurable respect with to the Lebesgue measure, with $|F| \in L^\infty(\mu)$, $M(A) = \int_A F\,\mathrm{d}\mu$ is compact for each $A \in \Sigma$.*

(b) *X is finite dimensional.*

Proof. (b $\Rightarrow$ a). If X is finite dimensional, then for each $A \in \Sigma$, $\int_A F\,\mathrm{d}\mu \subset B(0,M)$, where $M = \sup\operatorname{ess}|F|$. This implies $M(A)$ is compact.

(a $\Rightarrow$ b). Suppose X is infinite dimensional. Then there is a convex separable subset W in B_X such that W is not compact, which implies the existence of a sequence $\{x_k\} \subset W$ without any convergent subsequence. Put $F\colon [0,1] \to P_{\omega kc}(X)$ such that $F(\omega) \equiv W (\omega \in [0,1])$. Then for each $k \in \mathbb{N}$, $f_k \equiv x_k$ is a measurable selection of F and, if μ is the Lebesgue measure on $[0,1]$, then for any $t > 0$, $\{\int_0^t f_k\,\mathrm{d}\mu\}$ is not compact in X, which implies that $M\colon [0,1] \to P_{\omega kc}(X)$ is not compact valued. □

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