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MEASURE OF NONCOMPACTNESS OF LINEAR OPERATORS BETWEEN SPACES OF SEQUENCES THAT ARE $(\overline{N}, q)$ SUMMABLE OR BOUNDED

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Abstract. In this paper we investigate linear operators between arbitrary BK spaces X and spaces Y of sequences that are $(\overline{N}, q)$ summable or bounded. We give necessary and sufficient conditions for infinite matrices A to map X into Y. Further, the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for A to be a compact operator.

Keywords: BK spaces, bases, matrix transformations, measure of noncompactness

MSC 2000: 40H05, 46A45, 47B07

1. Introduction and well-known results

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ and φ, l_∞, c and c_0 for the sets of all finite, bounded, convergent sequences and sequences convergent to naught, respectively, and finally, for $1 \leqslant p < \infty$,

$$l_p = \left\{ x \in \omega \colon \sum_{k=0}^{\infty} |x_k|^p < \infty \right\}.$$

By e and $e^{(n)}$ $(n = 0, 1, \ldots)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

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A *BK space* is a Banach sequence space with continuous coordinates.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a *(Schauder) basis* if for each $x \in X$ there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

A BK space $X \supset \varphi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{n=0}^{\infty} x_n e^{(n)}$.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers and $x \in \omega$. Then we write

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (n = 0, 1, \ldots) \quad \text{and} \quad A(x) = (A_n(x))_{n=0}^{\infty}.$$

For any subset X of ω, the set

$$X_A = \{x \in \omega \colon \ A(x) \in X\}$$

is called the *matrix domain of* A *in* X. For instance, if E is the matrix defined by $e_{nk} = 1$ $(0 \leqslant k \leqslant n)$ and $e_{nk} = 0$ $(k > n)$ for all $n = 0, 1, \ldots$, then $cs = c_E$ and $bs = (l_\infty)_E$ are the sets of convergent and bounded series.

2. Sets of sequences that are $(\overline{N}, q)$-summable or bounded and their β-duals

Let $(q_k)_{k=0}^{\infty}$ be a positive sequence and Q the sequence with $Q_n = \sum_{k=0}^{n} q_k$ $(n = 0, 1, \ldots)$.

Further, let the matrix $\overline{N}_q$ be defined by

$$(\overline{N}_q)_{n,k} = \begin{cases} \dfrac{q_k}{Q_n} & (0 \leqslant k \leqslant n) \\ 0 & (k > n) \end{cases} \qquad (n = 0, 1, \ldots).$$

Then we define sets

$$(\overline{N}, q)_0 = (c_0)_{\overline{N}_q}, \quad (\overline{N}, q) = (c)_{\overline{N}_q} \quad \text{and} \quad (\overline{N}, q)_\infty = (l_\infty)_{\overline{N}_q}$$

of sequences that are $(\overline{N}, q)$ *summable to naught, summable* and *bounded*, respectively.

Proposition 2.1. (cf. [2, Corollary 1]) *Each of the sets* $(\overline{N}, q)_0$, $(\overline{N}, q)$ *and* $(\overline{N}, q)_\infty$ *is a BK space with respect to the norm* $\|\cdot\|_{\overline{N}_q}$ *defined by*

$$\|x\|_{\overline{N}_q} = \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \right|.$$

Further, if $Q_n \to \infty$ $(n \to \infty)$, *then* $(\overline{N}, q)_0$ *has AK, and every sequence* $x = (x_k)_{k=0}^{\infty} \in (\overline{N}, q)$ *has a unique representation*

$$x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)} \quad \textit{where } l \in \mathbb{C} \textit{ is such that } x - le \in (\overline{N}, q)_0.$$

We need the following notations:

For any two sequences x and y, let $xy = (x_k y_k)_{k=0}^{\infty}$.

If X and Y are arbitrary subsets of ω and z is any sequence, then we write

$$z^{-1} * X = \{x \in \omega\colon\ xz \in X\} \quad \text{and} \quad M(X,Y) = \bigcap_{x \in X} x^{-1} * Y.$$

In the special case, when $Y = cs$, the set

$$X^{\beta} = M(X, cs) = \left\{a \in \omega\colon\ \sum_{k=0}^{\infty} a_k x_k \text{ converges for all } x \in X\right\}$$

is called the β*-dual of* X. By $\mathcal{U}$ we denote the set of all sequences u such that $u_k \neq 0$ $(k = 0, 1, \ldots)$. For $u \in \mathcal{U}$, let $1/u = (1/u_k)_{k=0}^{\infty}$. Finally, let the operator $\Delta^+\colon\ \omega \to \omega$ be defined by

$$\Delta^+ x = \left((\Delta^+ x)_k\right)_{k=0}^{\infty} = (x_k - x_{k+1})_{k=0}^{\infty}.$$

Proposition 2.2. (cf. [2, Theorem 6]) *We put*

$$\begin{aligned}
\mathcal{N}_0 &= (1/q)^{-1} * \left(\left(Q^{-1} * l_1\right)_{\Delta^+} \cap \left(Q^{-1} * l_\infty\right)\right) \\
&= \left\{a \in \omega\colon\ \sum_{k=0}^{\infty} Q_k \left|\frac{a_k}{q_k} - \frac{a_{k+1}}{q_{k+1}}\right| < \infty \text{ and } Qa/q \in l_\infty\right\}, \\
\mathcal{N} &= (1/q)^{-1} * \left(\left(Q^{-1} * l_1\right)_{\Delta^+} \cap \left(Q^{-1} * c\right)\right)
\end{aligned}$$

and

$$\mathcal{N}_\infty = (1/q)^{-1} * \left(\left(Q^{-1} * l_1\right)_{\Delta^+} \cap \left(Q^{-1} * c_0\right)\right).$$

Then $(\overline{N}, q)_0^{\beta} = \mathcal{N}_0$, $\quad (\overline{N}, q)^{\beta} = \mathcal{N}$ $\quad$ *and* $\quad (\overline{N}, q)_\infty^{\beta} = \mathcal{N}_\infty$.

3. Matrix transformations

Let X and Y be two Banach spaces. By $B(X,Y)$, we denote the set of all continuous linear operators from X into Y, and we write

$$\|L\| = \sup\{\|L(x)\|\colon\ \|x\| = 1\}$$

for the operator norm of L. In the special case when $Y = \mathbb{C}$, the complex numbers, we write $X^* = B(X,\mathbb{C})$ for the set of all continuous linear functionals on X, and

$$\|f\| = \sup\{|f(x)|\colon\ \|x\| = 1\} \quad (f \in X^*)$$

for the norm of the continuous linear functional f.

If X is a BK space and $a \in \omega$, then we put

$$\|a\|^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right|\colon\ \|x\| = 1\right\}$$

provided the term on the right exists and is finite. This is the case whenever $a \in X^\beta$ (cf. [10, Theorem 7.2.9, p. 107]).

Proposition 3.1. *On any of the spaces $(\overline{N},q)_0^\beta$, $(\overline{N},q)^\beta$ and $(\overline{N},q)_\infty^\beta$, we have*

$$\|a\|^* = \sup_n \left(\sum_{k=0}^{n-1} Q_k \left|\frac{a_k}{q_k} - \frac{a_{k+1}}{q_{k+1}}\right| + \left|\frac{a_n Q_n}{q_n}\right|\right).$$

Proof. Given any sequence x we write

$$x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)} \quad \text{and} \quad \tau_k^{[n]} = \tau_k(x^{[n]}) = \frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j^{[n]} \quad (k,n = 0,1,\ldots).$$

Let $a \in \mathcal{N}_0$ and let n be a nonnegative integer. We define the sequence $b^{[n]}$ by

$$b_k^{[n]} = \begin{cases} Q_k \Delta^+(a/q)_k & (0 \leqslant k \leqslant n) \\ \dfrac{a_n Q_n}{q_n} & (k = n) \\ 0 & (k > n) \end{cases}$$

and put

$$\|a\|_{\mathcal{N}} = \sup_n \|b^{[n]}\|_1 = \sup_n \left(\sum_{k=0}^{\infty} |b_k^{[n]}|\right).$$

Then

$$\left|\sum_{k=0}^{\infty} a_k x_k^{[n]}\right| = \left|\sum_{k=0}^{n} \frac{a_k}{q_k} \Delta(Q\tau^{[n]})_k\right| \leqslant \sum_{k=0}^{n-1} \left|Q_k \tau_k^{[n]} \Delta^+(a/q)_k\right| + \left|\frac{a_n Q_n}{q_n}\right| |\tau_n^{[n]}|$$

$$\leqslant \sup_k |\tau_k^{[n]}| \cdot \left(\sum_{k=0}^{n-1} \left|Q_k \Delta^+(a/q)_k\right| + \left|\frac{a_n Q_n}{q_n}\right|\right)$$

$$= \|x^{[n]}\|_{\overline{N}_q} \|b^{[n]}\|_1 = \|a\|_{\mathcal{N}} \|x^{[n]}\|_{\overline{N}_q}.$$

Thus

$$\|a\|^* \leqslant \|a\|_{\mathcal{N}}. \tag{3.1}$$

To prove the converse inequality let n be an arbitrary integer. We define the sequence $x^{(n)}$ by

$$\tau_k(x^{(n)}) = \operatorname{sign}(b_k^{[n]}) \quad (k = 0, 1, \ldots).$$

Then

$$\tau_k(x^{(n)}) = 0 \text{ for } k > n, \text{ i. e. } x^{(n)} \in (\overline{N}, q)_0, \quad \|x^{(n)}\|_{\overline{N}_n} = \|\tau(x^{(n)})\|_\infty \leqslant 1$$

and

$$\left|\sum_{k=0}^{\infty} a_k x_k^{(n)}\right| = \left|\sum_{k=0}^{n} b_k^{[n]} x_k^{(n)}\right| = \sum_{k=0}^{n} |b_k^{[n]}| \leqslant \|a\|^*.$$

Since n was arbitrary, we have

$$\|a\|_{\mathcal{N}} \leqslant \|a\|^*. \tag{3.2}$$

Now inequalities (3.1) and (3.2) yield the conclusion. □

If A is an infinite matrix of complex numbers, then we write A_n for the sequence in the n^{th} row of A. For any two subsets X and Y of ω, (X, Y) denotes the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n, and $A(x) \in Y$ for all $x \in X$.

The following results are well known.

Proposition 3.2. (cf. [7, Theorem 1]) *Let X and Y be BK spaces. Then $(X, Y) \subset B(X, Y)$, i. e. every $A \in (X, Y)$ defines an element $L_A \in B(X, Y)$ where*

$$L_A(x) = A(x) \quad (x \in X).$$

Further, $A \in (X, l_\infty)$ if and only if

$$\|A\|^* = \sup_n \|A_n\|^* = \|L_A\| < \infty.$$

Finally, if $(b^{(k)})_{k=0}^{\infty}$ is a basis of X, Y and Y_1 are FK spaces with Y_1 a closed subspace of Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all $k = 0, 1, \ldots$.

Proposition 3.3. (cf. [8, Proposition 3.4]) *Let T be a triangle.*

(a) *Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

(b) *Further, if X and Y are BK spaces and $A \in (X, Y_T)$, then*

$$\|L_A\| = \|L_B\|. \tag{3.3}$$

As a corollary of Propositions 3.1 and 3.2, we obtain

Corollary 3.4. *Let $q = (q_k)_{k=0}^{\infty}$ be a positive sequence and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ $(n \to \infty)$.*

(a) *Then $A \in ((\overline{N}, q)_\infty, l_\infty)$ if and only if*

$$M((\overline{N}, q)_\infty, l_\infty) = \sup_{m,n}\left(\sum_{k=0}^{m-1} Q_k \left|\frac{a_{nk}}{q_k} - \frac{a_{n,k+1}}{q_{k+1}}\right| + |Q_m a_{nm}/q_m|\right) < \infty \tag{3.4}$$

and

$$A_n Q/q \in c_0 \quad \text{for all } n = 0, 1, \ldots. \tag{3.5}$$

(b) *Then $A \in ((\overline{N}, q), l_\infty)$ if and only if condition (3.4) holds and*

$$A_n Q/q \in c \quad \text{for all } n = 0, 1, \ldots. \tag{3.6}$$

(c) *Then $A \in ((\overline{N}, q)_0, l_\infty)$ if and only if condition (3.4) holds.*

(d) *Then $A \in ((\overline{N}, q)_0, c_0)$ if and only if condition (3.4) holds and*

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for all } k = 0, 1, \ldots. \tag{3.7}$$

(e) *Then $A \in ((\overline{N}, q)_0, c)$ if and only if condition (3.4) holds and*

$$\lim_{n\to\infty} a_{nk} = l_k \quad \text{for all } k = 0, 1, \ldots. \tag{3.8}$$

(f) *Then $A \in ((\overline{N}, q), c_0)$ if and only if conditions (3.4), (3.6) and (3.7) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0. \tag{3.9}$$

(g) *Then $A \in ((\overline{N}, q), c)$ if and only if conditions (3.4), (3.5) and (3.8) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = l. \tag{3.10}$$

As a corollary of Propositions 2.1 and 3.3, we obtain

Corollary 3.5. *Let X be a BK space, $(p_k)_{k=0}^{\infty}$ a positive sequence and $P_n = \sum_{k=0}^{n} p_k$ $(n = 0, 1, \ldots)$. Then $A \in (X, (\overline{N}, p)_\infty)$ if and only if*

$$M(X, (\overline{N}, p)_\infty) = \sup_m \left\| \frac{1}{P_m} \sum_{n=0}^{m} p_n A_n \right\|^* < \infty. \tag{3.11}$$

Further, if $(b^{(k)})_{k=0}^{\infty}$ is a basis of X, then $A \in (X, (\overline{N}, p)_0)$ if and only if condition (3.11) *holds and*

$$\lim_{m\to\infty} \left(\frac{1}{P_m} \sum_{n=0}^{m} p_n A_n(b^{(k)}) \right) = 0 \quad \text{for all } k = 0, 1, \ldots, \tag{3.12}$$

and $A \in (X, (\overline{N}, p))$ if and only if condition (3.12) *holds and*

$$\lim_{m\to\infty} \left(\frac{1}{P_m} \sum_{n=0}^{m} p_n A_n(b^{(k)}) \right) = l_k \quad \text{for all } k = 0, 1, \ldots. \tag{3.13}$$

Remark 1. (a) If $X = l_r$ $(1 \leqslant r < \infty)$ and Y is any one of the spaces $(\overline{N}, p)_\infty$, $(\overline{N}, p)$ and $(\overline{N}, p)_0$, then the conditions for $A \in (X, Y)$ follow from the respective ones in Corollary 3.5 by replacing the norm $\|\cdot\|^*$ in condition (3.11) by the natural norm on l_s where $s = \infty$ for $r = 1$ and $s = r/(r-1)$ for $1 < r < \infty$, i.e.

$$M(l_r, (\overline{N}, p)_\infty) = \begin{cases} \sup_{m,k} \left| \frac{1}{P_m} \sum_{n=0}^{m} p_n a_{nk} \right| & (r = 1) \\ \sup_m \left(\sum_{k=0}^{\infty} \left| \frac{1}{P_m} \sum_{n=0}^{m} p_n a_{nk} \right|^s \right)^{1/s} & (1 < r < \infty), \end{cases}$$

and by replacing the terms $A_n(b^{(k)})$ in conditions (3.12) and (3.13) by the terms a_{nk}.

(b) We consider the conditions

$$\begin{aligned} &M((\overline{N}, q)_\infty, (\overline{N}, p)_\infty) \\ &\quad = \sup_{m,n} \left(\sum_{k=0}^{n-1} Q_k \left| \frac{1}{P_m} \sum_{l=0}^{m} p_l \left(\Delta^+ A_l/q\right)_k \right| + \left| \frac{Q_n}{q_n P_m} \sum_{l=0}^{m} p_l a_{ln} \right| \right) < \infty, \end{aligned} \tag{3.14}$$

$$(3.15)\qquad \left(\frac{a_{nk}Q_k}{q_k}\right)_{k=0}^{\infty} \in c_0 \qquad (n=0,1,\ldots),$$

$$(3.16)\qquad \left(\frac{a_{nk}Q_k}{q_k}\right)_{k=0}^{\infty} \in c \qquad (n=0,1,\ldots),$$

$$(3.17)\qquad \lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right) = 0 \qquad (k=0,1,\ldots),$$

$$(3.18)\qquad \lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right) = l_k \qquad (k=0,1,\ldots),$$

$$(3.19)\qquad \lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n \left(\sum_{k=0}^{\infty} a_{nk}\right)\right) = 0 \qquad (k=0,1,\ldots),$$

$$(3.20)\qquad \lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n \left(\sum_{k=0}^{\infty} a_{nk}\right)\right) = l_k \qquad (k=0,1,\ldots).$$

Then

$A \in ((\overline{N}, q)_\infty, (\overline{N}, p)_\infty)$ if and only if (3.14) and (3.15);
$A \in ((\overline{N}, q), (\overline{N}, p)_\infty)$ if and only if (3.14) and (3.16);
$A \in ((\overline{N}, q)_0, (\overline{N}, q)_\infty)$ if and only if (3.14);
$A \in ((\overline{N}, q)_0, (\overline{N}, p)_0)$ if and only if (3.14) and (3.17);
$A \in ((\overline{N}, q)_0, (\overline{N}, p))$ if and only if (3.14) and (3.18);
$A \in ((\overline{N}, q), (\overline{N}, p)_0)$ if and only if (3.14), (3.16), (3.17) and (3.19);
$A \in ((\overline{N}, q), (\overline{N}, p))$ if and only if (3.14), (3.16), (3.18) and (3.20).

4. Measure of noncompactness and transformations

If X and Y are metric spaces, then $f\colon X \mapsto Y$ is a compact map if $f(Q)$ is relatively compact (i.e., if the closure of $f(Q)$ is a compact subset of Y) subset of Y for each bounded subset Q of X. In this section we investigate, among other things, when in some special cases (see Corollary 4.3), an operator L_A is compact. Our investigations use the measure of noncompactness. Recall that if Q is a bounded subset of a metric space X, then the *Hausdorff measure of noncompactness* of Q is denoted by $\chi(Q)$, and

$$\chi(Q) = \inf\{\varepsilon > 0\colon\ Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function χ is called the *Hausdorff measure of noncompactness*, and for its properties see [1], [3] or [9]. Denote by $\overline{Q}$ the closure of Q. For the convenience of the reader, let us mention the following facts: If Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\begin{aligned}\chi(Q)=0 \Longleftrightarrow Q \quad &\text{is a totally bounded set},\\ \chi(Q)&=\chi(\overline{Q}),\\ Q_1\subset Q_2 &\Longrightarrow \chi(Q_1)\leqslant\chi(Q_2),\\ \chi(Q_1\cup Q_2)&=\max\{\chi(Q_1),\chi(Q_2)\},\\ \chi(Q_1\cap Q_2)&\leqslant\min\{\chi(Q_1),\chi(Q_2)\}.\end{aligned}$$

If our space X is a normed space, then the function $\chi(Q)$ has some additional properties connected with the linear structure. We have e.g.

$$\begin{aligned}\chi(Q_1+Q_2)&\leqslant\chi(Q_1)+\chi(Q_2),\\ \chi(\lambda Q)&=|\lambda|\chi(Q)\quad\text{for each}\quad\lambda\in\mathbb{C}.\end{aligned}$$

If X and Y are normed spaces, then for $A\in B(X,Y)$ the Hausdorff measure of noncompactness of A, denoted by $\|A\|_\chi$, is defined by $\|A\|_\chi=\chi(AK)$, where $K=\{x\in X\colon \|x\|\leqslant 1\}$ is the unit ball in X. Further, A is compact if and only if $\|A\|_\chi=0$, and $\|A\|_\chi\leqslant\|A\|$. Recall the following well known result (see e.g. [3, Theorem 6.1.1] or [1, 1.8.1]).

Proposition 4.1. *Let X be a Banach space with a Schauder basis $\{e_1,e_2,\dots\}$, Q a bounded subset of X, and $P_n\colon X\mapsto X$ the projector onto the linear span of $\{e_1,e_2,\dots,e_n\}$. Then*

$$\begin{aligned}\frac{1}{a}\limsup_{n\to\infty}\Bigl(\sup_{x\in Q}\|(I-P_n)x\|\Bigr)&\leqslant\chi(Q)\\ &\leqslant\inf_n\sup_{x\in Q}\|(I-P_n)x\|\leqslant\limsup_{n\to\infty}\Bigl(\sup_{x\in Q}\|(I-P_n)x\|\Bigr),\end{aligned}\tag{4.1}$$

where $a=\limsup\limits_{n\to\infty}\|I-P_n\|$.

Let us mention that concerning the number a in Proposition 4.1, if $X=c_0$, then $a=1$, but if $X=c$, then $a=2$ (see e.g. [3, p. 22]).

Concerning Corollary 3.4 and the measures of noncompactness we have

Theorem 4.2. *Let A be as in Corollary 3.4, and for any integer n, r, $n>r$, set*

$$\|A\|^{(r)}=\sup_{n>r}\sup_m\Biggl(\sum_{k=0}^{m-1}Q_k\left|\frac{a_{nk}}{q_k}-\frac{a_{n,k+1}}{q_{k+1}}\right|+|Q_m a_{nm}/q_m|\Biggr).\tag{4.2}$$

Let X be either $(\overline{N},q)_0$ or $X=(\overline{N},q)$, and let $A\in(X,c_0)$. Then we have

$$\|L_A\|_\chi=\lim_{r\to\infty}\|A\|^{(r)}. \tag{4.3}$$

Let X be either $(\overline{N},q)_0$ or $X=(\overline{N},q)$, and let $A\in(X,c)$. Then we have

$$\frac{1}{2}\cdot\lim_{r\to\infty}\|A\|^{(r)}\leqslant\|L_A\|_\chi\leqslant\lim_{r\to\infty}\|A\|^{(r)}. \tag{4.4}$$

Let X be either $(\overline{N},q)_0$, $(\overline{N},q)$ or $X=(\overline{N},q)_\infty$, and let $A\in(X,l_\infty)$. Then we have

$$0\leqslant\|L_A\|_\chi\leqslant\lim_{r\to\infty}\|A\|^{(r)}. \tag{4.5}$$

Proof. Let us remark that the limits in (4.3), (4.4) and (4.5) exist. Set $K=\{x\in X\colon\|x\|\leqslant 1\}$. In the case $A\in(X,c_0)$ for $X=(\overline{N},q)_0$ or $X=(\overline{N},q)$, by Proposition 4.1 we have

$$\|L_A\|_\chi=\chi(AK)=\lim_{r\to\infty}\Big[\sup_{x\in K}\|(I-P_r)Ax\|\Big], \tag{4.6}$$

where $P_r\colon c_0\mapsto c_0$, $r=1,2,\ldots$, is the projector on the first $r+1$ coordinates, i.e., $P_r(x)=(x_0,x_1,x_2,\ldots,x_r,0,0,\ldots)$, $x=(x_k)\in c_0$ (let us remark that $\|I-P_r\|=1$, $r=0,1,2,\ldots$). Further, by Proposition 3.2 and Corollary 3.4 we have

$$\|A\|^{(r)}=\sup_{x\in K}\|(I-P_r)Ax\|, \tag{4.7}$$

and by (4.6) we get (4.3). To prove (4.4) let us remark that every sequence $x=(x_k)_{k=0}^\infty\in c$ has a unique representation

$$x=le+\sum_{k=0}^\infty(x_k-l)e^{(k)}\quad\text{where } l\in\mathbb{C}\text{ is such that } x-le\in c.$$

Let us define $P_r\colon c\mapsto c$ by $P_r(x)=le+\sum_{k=0}^{r}(x_k-l)e^{(k)}$, $r=0,1,2,\ldots$. It is easy to prove that $\|I-P_r\|=2$, $r=0,1,2,\ldots$. Now the proof of (4.4) is similar to the case (4.3), and we omit it. Let us prove (4.5). Define $P_r\colon l_\infty\mapsto l_\infty$ by $P_r(x)=(x_0,x_1,x_2,\ldots,x_r,0,0,\ldots)$, $x=(x_k)\in l_\infty$, $r=0,1,2,\ldots$. It is clear that

$$AK\subset P_r(AK)+(I-P_r)(AK).$$

Now, by the elementary properties of the function χ we have

$$\begin{aligned}\chi(AK)&\leqslant\chi(P_r(AK))+\chi((I-P_r)(AK))=\chi(I-P_r)(AK)\\&\leqslant\sup_{x\in K}\|(I-P_r)Ax\|.\end{aligned}$$

Finally, by Proposition 3.2 and Corollary 3.4 we get (4.5). □

As a corollary of the above theorem, we have

Corollary 4.3. *Let A be as in Theorem 4.2. Then if $A \in (X, c_0)$ for $X = (\overline{N}, q)_0$ or $X = (\overline{N}, q)$, or if $A \in (X, c)$ for $X = (\overline{N}, q)_0$ or $X = (\overline{N}, q)$, then in all cases we have*

$$L_A \text{ is compact if and only if } \lim_{r\to\infty} \|A\|^{(r)} = 0. \tag{4.8}$$

Further, if $A \in (X, l_\infty)$ for $X = (\overline{N}, q)_0$, $X = (\overline{N}, q)$ or $X = (\overline{N}, q)_\infty$, then we have

$$L_A \text{ is compact if } \lim_{r\to\infty} \|A\|^{(r)} = 0. \tag{4.9}$$

The following example shows that it is possible for L_A in (4.9) to be compact in the case $\lim\limits_{r\to\infty} \|A\|^{(r)} > 0$, and hence in general in (4.9) we have just "if".

Example 4.4. Let the matrix A be defined by $A_n = e^{(0)}$ $(n = 0, 1, \ldots)$ and $q_n = 2^n$, $n = 0, 1, 2, \ldots$. Then $M((\overline{N}, q)_\infty, l_\infty) = \sup\limits_n [1 + (2 - 2^{-n})] < 3$, and by Corollary 3.4 we know that $A \in ((\overline{N}, q)_\infty, l_\infty)$. Further,

$$\|A\|^{(r)} = \sup_{n>r} \Big[1 + \Big(2 - \frac{1}{2^n}\Big)\Big] = 3 - \frac{1}{2^{r+1}} \quad \text{for all } r,$$

whence

$$\lim_{r\to\infty} \|A\|^{(r)} = 3 > 0.$$

Since $A(x) = x_0 e_0$ for all $x \in (\overline{N}, q)_\infty$, L_A is a compact operator.

Now we continue with the following auxiliary result.

Lemma 4.5. *Let $q_k > 0$ $(k = 0, 1, \ldots)$ and $Q_n = \sum\limits_{k=0}^{n} q_k \to \infty$ $(n \to \infty)$. We put*

$$\tau_n(x) = \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \quad \text{for all } x \in \omega.$$

Let $r \geqslant 0$ and let the operators $B^{(r,0)}\colon (\overline{N}, q)_0 \to (\overline{N}, q)_0$ and $B^{(r)}\colon (\overline{N}, q) \to (\overline{N}, q)$ be defined by

$$B^{(r,0)}(x) = \sum_{k=r+1}^{\infty} x_k e^{(k)} \quad (x \in (\overline{N}, q)_0), \tag{4.10}$$

$$B^{(r)}(x) = \sum_{k=r+1}^{\infty} (x_k - l)e^{(k)} \quad (x \in (\overline{N}, q)) \tag{4.11}$$

where $l = \lim_{n\to\infty} \tau_n(x)$. *Then*

$$\|B^{(r,0)}\| = 1 + \frac{Q_r}{Q_{r+1}} \tag{4.12}$$

and

$$\|B^{(r)}\| = 2. \tag{4.13}$$

P r o o f. First we show identity (4.12). Let $x \in (\overline{N}, q)_0$. Since

$$\tau_n(B^{(r,0)}(x)) = 0 \quad \text{for } 0 \leqslant n \leqslant r$$

and, for $n \geqslant r+1$,

$$\begin{aligned}\left|\tau_n(B^{(r,0)}(x))\right| &= \left|\frac{1}{Q_n}\sum_{k=r+1}^{n} q_k x_k\right| = \left|\tau_n(x) - \frac{Q_r}{Q_n}\tau_r(x)\right| \\ &\leqslant \left(1 + \frac{Q_r}{Q_{r+1}}\right)\|x\|_{(\overline{N},q)_\infty},\end{aligned}$$

it follows that

$$\left\|B^{(r,0)}(x)\right\|_{(\overline{N},q)_\infty} \leqslant \left(1 + \frac{Q_r}{Q_{r+1}}\right)\|x\|_{(\overline{N},q)_\infty},$$

and consequently

$$\|B^{(r,0)}\| \leqslant 1 + \frac{Q_r}{Q_{r+1}}. \tag{4.14}$$

Defining the sequence x by

$$x_k = \begin{cases} -1 & (0 \leqslant k \leqslant r) \\ \dfrac{Q_r + Q_{r+1}}{q_{r+1}} & (k = r+1) \\ -\dfrac{Q_r + Q_{r+1}}{q_{r+2}} & (k = r+2) \\ 0 & (k \geqslant r+3), \end{cases}$$

we conclude

$$\tau_n(x) = -1 \quad (0 \leqslant n \leqslant r),$$
$$\tau_{r+1}(x) = -\frac{Q_r}{Q_{r+1}} + \frac{Q_r}{Q_{r+1}} + 1 = 1$$

and

$$\begin{aligned}\tau_n(x) &= \frac{1}{Q_n}\left(-Q_r + Q_r + Q_{r+1} - (Q_r + Q_{r+1})\right)\\ &= -\frac{Q_r}{Q_n} \quad (n \geqslant r+2).\end{aligned}$$

Since $Q_n \to \infty$ $(n \to \infty)$, we have

$$x \in (\overline{N}, q)_0 \quad \text{and} \quad \|x\|_{(\overline{N},q)_\infty} = 1.$$

Further,

$$\tau_{r+1}(B^{(r,0)}(x)) = \frac{1}{Q_{r+1}}\left(Q_r + Q_{r+1}\right) = 1 + \frac{Q_r}{Q_{r+1}}$$

and

$$\tau_n(B^{(r,0)}(x)) = 0 \quad \text{for } n \neq r+1.$$

Therefore

$$\left\|B^{(r,0)}(x)\right\|_{(\overline{N},q)_\infty} = 1 + \frac{Q_r}{Q_{r+1}} = \left(1 + \frac{Q_r}{Q_{r+1}}\right)\|x\|_{(\overline{N},q)_\infty}$$

and

$$\|B^{(r,0)}\| \geqslant 1 + \frac{Q_r}{Q_{r+1}}. \tag{4.15}$$

Now (4.14) and (4.15) together yield identity (4.12). Now we prove identity (4.13). Let $x \in (\overline{N}, q)$. We have

$$\tau_n(B^{(r)}(x)) = 0 \quad \text{for } 0 \leqslant n \leqslant r$$

and, for $n \geqslant r+1$,

$$\begin{aligned}\left|\tau_n(B^{(r)}(x))\right| &= \left|\frac{1}{Q_n}\sum_{k=r+1}^{n} q_k(x_k - l)\right| = \left|\tau_n(x) - \frac{Q_r}{Q_n}\tau_r(x) - l + \frac{Q_r}{Q_n}l\right|\\ &\leqslant \left|1 + \frac{Q_r}{Q_n}\right| \|x\|_{(\overline{N},q)_\infty} + \left|1 - \frac{Q_r}{Q_n}\right| |l|.\end{aligned}$$

Since $|l| = \lim_{n\to\infty} |\tau_n(x)| \leqslant \|x\|_{(\overline{N},q)_\infty}$, we have

$$\left|\tau_n(B^{(r)}(x))\right| \leqslant 2\|x\|_{(\overline{N},q)_\infty} \quad \text{for } n \geqslant r+1,$$

and consequently

$$\|B^{(r)}\| \leqslant 2. \tag{4.16}$$

Defining the sequence x by

$$x_k = \begin{cases} -1 & (0 \leqslant k \leqslant r) \\ 2\dfrac{Q_{r+1}}{q_{r+1}} - 1 & (k = r+1) \\ -1 & (k \geqslant r+2), \end{cases}$$

we conclude

$$\begin{aligned} \tau_n(x) &= -1 \quad (0 \leqslant n \leqslant r), \\ \tau_{r+1}(x) &= \frac{1}{Q_{r+1}}\,(-Q_r + 2Q_r - q_{r+1}) = 1 \end{aligned}$$

and

$$\begin{aligned} \tau_n(x) &= \frac{1}{Q_n}\left(-Q_r + 2Q_{r+1} - \sum_{k=r+1}^{n} q_k\right) = \frac{1}{Q_n}\,(-Q_n + 2Q_{r+1}) \\ &= -1 + 2\frac{Q_{r+1}}{Q_n} \leqslant 1 \quad (n \geqslant r+2). \end{aligned}$$

Hence

$$\|x\|_{(\overline{N},q)_\infty} = 1 \quad \text{and} \quad \lim_{n\to\infty} \tau_n(x) = -1, \text{ i. e. } x \in (\overline{N},q).$$

Finally,

$$\begin{aligned} \tau_n(B^{(r)}(x)) &= 0 \quad (0 \leqslant n \leqslant r) \\ \tau_{r+1}(B^{(r)}(x)) &= \frac{q_{r+1}}{Q_{r+1}}(x_{r+1} + 1) = 2 \end{aligned}$$

and

$$\tau_n(B^{(r)}(x)) = 2\frac{Q_{r+1}}{Q_n} \leqslant 2 \quad (n \geqslant r+2).$$

This implies

$$\|B^{(r)}\| \geqslant 2. \tag{4.17}$$

Now (4.16) and (4.17) together yield (4.13). □

Concerning Corollary 3.5 and the measures of noncompactness we have

Theorem 4.6. *Let X be a BK space, let A be as in Corollary 3.5, and let $P_m \to \infty$, $(m \to \infty)$. Then for any integer m, r, $m > r$, set*

$$\|A\|^{(r)}_{(\overline{N},p)_\infty} = \sup_{m>r}\left\|\frac{1}{P_m}\sum_{n=0}^{m} p_n A_n\right\|^{*}. \tag{4.18}$$

Further, if X has a Schauder basis and $A \in (X, (\overline{N}, p)_0)$, then we have

$$\frac{1}{b}\cdot\lim_{r\to\infty}\|A\|^{(r)}_{(\overline{N},p)_\infty} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(r)}_{(\overline{N},p)_\infty}, \tag{4.19}$$

where $b = \limsup\limits_{n\to\infty}(2 - p_n/P_n)$. If X has a Schauder basis and $A \in (X, (\overline{N}, p))$ then we have

$$\frac{1}{2}\cdot\lim_{r\to\infty}\|A\|^{(r)}_{(\overline{N},p)_\infty} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(r)}_{(\overline{N},p)_\infty}. \tag{4.20}$$

Finally, if $A \in (X, (\overline{N}, p)_\infty)$, then we have

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(r)}_{(\overline{N},p)_\infty}. \tag{4.21}$$

P r o o f. Let us remark that the limits in (4.19), (4.20) and (4.21) exist. Set $K = \{x \in X\colon \|x\| \leqslant 1\}$. Suppose that $A \in (X, (\overline{N}, p)_0)$. Let $B^{(r,0)}\colon (\overline{N}, p)_0 \mapsto (\overline{N}, p)_0$ be the projector defined in Lemma 4.5. Then by (4.12) we have that $\|B^{(r,0)}\| = 2 - p_r/P_r$. Now, to prove (4.19), by Propositions 2.1 and 4.1 we have

$$\frac{1}{b}\limsup_{r\to\infty}\Big(\sup_{x\in K}\|B^{(r,0)}Ax\|\Big) \leqslant \chi(AK) \leqslant \limsup_{r\to\infty}\Big(\sup_{x\in K}\|B^{(r,0)}Ax\|\Big), \tag{4.22}$$

where $b = \limsup\limits_{r\to\infty}\|B^{(r,0)}\|$. Thus, since

$$\sup_{x\in K}\|B^{(r,0)}Ax\| = \|A\|^{(r)}_{(\overline{N},p)_\infty},$$

we prove (4.19). To prove (4.20) let us remark (see Proposition 2.1) that $(\overline{N}, p)$ has the Schauder basis $e, e^{(k)}$, $k = 0, 1, \ldots$, and every $(x_k)_{k=0}^\infty \in (\overline{N}, q)$ has a unique representation

$$x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)},$$

where $l \in \mathbb{C}$ is such that $x - le \in (\overline{N}, p)_0$. Let $B^{(r)}\colon (\overline{N}, p)_0 \mapsto (\overline{N}, p)_0$ be the projector defined by (see Lemma 4.5)

$$B^{(r)}(x) = \sum_{k=r+1}^{\infty} (x_k - l)e^{(k)}.$$

Then by (4.13) we have that $\|B^{(r)}\| = 2$. Now the proof of (4.20) is similar to the case (4.19), and we omit it. Let us prove (4.21). Define $\mathcal{P}_r\colon (\overline{N}, p)_\infty \mapsto (\overline{N}, p)_\infty$ by $\mathcal{P}_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$, $x = (x_i) \in (\overline{N}, p)_\infty$, $r = 1, 2, \ldots$. It is clear that

$$AK \subset \mathcal{P}_r(AK) + (I - \mathcal{P}_r)(AK).$$

By Remark 1 (b) it follows that $\mathcal{P}_r$ is a bounded operator, and since it has obviously finite-rank, it is a compact one. Now, by the elementary properties of the function χ we have

$$\begin{aligned} \chi(AK) &\leqslant \chi(\mathcal{P}_r(AK)) + \chi((I - \mathcal{P}_r)(AK)) = \chi((I - \mathcal{P}_r)(AK) \\ &\leqslant \sup_{x \in K} \|(I - \mathcal{P}_r)Ax\| = \|A\|^{(r)}_{(\overline{N},p)_\infty}. \end{aligned} \tag{4.23}$$

□

As a corollary of the above theorem we have

Corollary 4.7. *Let X be a BK space and let A and $\|A\|^{(r)}_{(\overline{N},p)}$ be as in Theorem 4.6. If X has a Schauder basis, and either $A \in (X, (\overline{N}, p)_0)$ or $A \in (X, (\overline{N}, p))$, then*

$$L_A \text{ is compact if and only if } \lim_{r\to\infty} \|A\|^{(r)}_{(\overline{N},p)} = 0. \tag{4.24}$$

Further, if $A \in (X, (\overline{N}, p)_\infty)$, then we have

$$L_A \text{ is compact if } \lim_{r\to\infty} \|A\|^{(r)}_{(\overline{N},p)} = 0. \tag{4.25}$$

Now, concerning Remark 1, we get several corollaries.

Corollary 4.8. *If either $A \in (l^u, (\overline{N}, p)_0)$ $(1 < u < \infty)$, or $A \in (l^u, (\overline{N}, p))$ $(1 < u < \infty)$, then*

L_A is compact if and only if

$$\lim_{r\to\infty} \left[\sup_{m>r} \left(\sum_{k=0}^{\infty} \left| \frac{1}{P_m} \sum_{n=0}^{m} p_n a_{nk} \right|^v \right)^{1/v} \right] = 0, \quad v = u/(u-1). \tag{4.26}$$

Further, if either $A \in (l^1, (\overline{N}, p)_0))$ *or* $A \in (l^1, (\overline{N}, p))$, *then*

$$L_A \text{ is compact if and only if}$$

$$\lim_{r\to\infty}\left(\sup_{n>r,k}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|\right) = 0. \tag{4.27}$$

If $A \in (l^u, (\overline{N}, p))$ $(1 < u < \infty)$, *then*

L_A *is compact if*

$$\lim_{r\to\infty}\left[\sup_{m>r}\left(\sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|^v\right)^{1/v}\right] = 0, \quad v = u/(u-1). \tag{4.28}$$

Finally, if $A \in (l^1, (\overline{N}, p))$, *then*

L_A *is compact if*

$$\lim_{r\to\infty}\left(\sup_{n>r,k}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|\right) = 0. \tag{4.29}$$

From Corollary 4.7, Proposition 3.3 and Remark 1 (b), we have

Corollary 4.9. *If* $A \in (X, (\overline{N}, p)_0)$ *for* $X = (\overline{N}, q)_0$ *or* $X = (\overline{N}, q)$, *or if* $A \in (X, (\overline{N}, p))$ *for* $X = (\overline{N}, q)_0$ *or* $X = (\overline{N}, q)$, *then in all cases we have*

L_A *is compact if an only if*

$$\lim_{r\to\infty}\left[\sup_{m>r,n}\left(\sum_{k=0}^{n-1} Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m} p_l(\Delta^+ A_l/q)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m} p_l a_{ln}\right|\right)\right] = 0. \tag{4.30}$$

Further, if $A \in (X, (\overline{N}, p)_\infty)$ *for* $X = (\overline{N}, q)_\infty$, $X = (\overline{N}, q)_0$ *or* $X = (\overline{N}, q)$, *then we have*

L_A *is compact if*

$$\lim_{r\to\infty}\left[\sup_{m>r,n}\left(\sum_{k=0}^{n-1} Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m} p_l(\Delta^+ A_l/q)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m} p_l a_{ln}\right|\right)\right] = 0. \tag{4.31}$$

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