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A CANONICAL DIRECTLY INFINITE RING

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Abstract. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}$ the ring of integers. Let $\mathcal{B}$ be the ring of $N \times N$ matrices over $\mathbb{Z}$ generated by the following two matrices: one obtained from the identity matrix by shifting the ones one position to the right and the other one position down. This ring plays an important role in the study of directly finite rings. Calculation of invertible and idempotent elements of $\mathcal{B}$ yields that the subrings generated by them coincide. This subring is the sum of the ideal $\mathcal{F}$ consisting of all matrices in $\mathcal{B}$ with only a finite number of nonzero entries and the subring of $\mathcal{B}$ generated by the identity matrix. Regular elements are also described. We characterize all ideals of $\mathcal{B}$, show that all ideals are finitely generated and that not all ideals of $\mathcal{B}$ are principal. Some general ring theoretic properties of $\mathcal{B}$ are also established.

Keywords: directly finite rings, matrix rings

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1. Introduction and summary

A ring R with identity 1 is said to be directly finite if for any $a, b \in R$, $ab = 1$ implies $ba = 1$; otherwise R is directly infinite. Let $\mathbb{N}$ be the set of nonnegative integers. Also, let $\mathcal{B}$ be the ring of $N \times N$ matrices over the ring $\mathbb{Z}$ of integers generated by two elements: one obtained by shifting the ones of the identity matrix one position to the right, say x, and the other one position down, say y. This ring appears in ([2], p. 263). If in the ring R there is a pair of elements a, b such that $ab = 1 \neq ba$, then we have seen in [4] that there exists a homomorphism of $\mathcal{B}$ onto the ring $R(a, b)$ generated by a and b whose kernel does not contain the matrix

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with 1 in the $(0,0)$ position and 0 elsewhere. Conversely, any homomorphism of this kind produces a pair of elements $a, b \in R$ such that $ab = 1 \neq ba$. Therefore such homomorphisms characterize directly infinite rings.

It is the purpose of this paper to study the ring $\mathcal{B}$ especially concerning its ideals as these play a central role in the above argument. We have already collected some of its properties in [4] in order to establish the result mentioned above. The set $\mathcal{F}$ of all matrices in $\mathcal{B}$ with only a finite number of nonzero entries is an ideal of $\mathcal{B}$ which plays an important role in our considerations. The stage is thus set for a more extensive investigation of the ring $\mathcal{B}$ with the aim of characterizing all its ideals with the above property, equivalently those that do not contain $\mathcal{F}$. The ring $\mathcal{B}$ seems to deserve our scrutiny in its own right.

Section 2 contains the minimum of notation and terminology needed in the paper. Invertible and regular elements of the ring $\mathcal{B}$ are characterized in Section 3, the main result of the section being a multiple characterization of the subring of $\mathcal{B}$ generated by its group of units. Section 4 contains a construction of all ideals of $\mathcal{B}$ as well as the proof of the assertion that all ideals are finitely generated. We also show that not all ideals of $\mathcal{B}$ are principal. The paper is concluded in Section 5 with the discussion of some other properties of the ring $\mathcal{B}$.

2. Notation and terminology

For symbolism and concepts in rings, we follow [1]. In addition, we shall need the following.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $n \in \mathbb{N}$ we write $\overline{n} = \{0, \ldots, n\}$. The letter $\mathbb{Z}$ stands for the ring of integers; $\mathbb{Z}_k$ for the ring of integers modulo k; $M_n(\mathbb{Z})$ for the ring of $\overline{n} \times \overline{n}$ matrices over $\mathbb{Z}$; I_n for the identity of $M_n(\mathbb{Z})$; $\mathcal{A}$ for the ring of all $N \times N$ matrices over $\mathbb{Z}$ with only a finite number of nonzero entries in each row and each column; $\langle m, n\rangle$ for the matrix in $\mathcal{A}$ with 1 in the $(m+t, n+t)$-position for $t = 0, 1, 2, \ldots$ and 0 elsewhere; $[m, n]$ for the matrix in $\mathcal{A}$ with 1 in the (m, n)-position and 0 elsewhere. Note that

$$[m, n] = \langle m, n\rangle - \langle m+1, n+1\rangle, \tag{1}$$

which we shall use frequently. Given a matrix $X \in \mathcal{A}$ and $k \in \mathbb{Z}$, we denote by kX the matrix obtained by multiplying each entry of X by k. A matrix of the form $k\langle m, n\rangle$, with $k \in \mathbb{Z} \setminus \{0\}$, is said to be a *ray matrix*. Clearly, $\langle 0, 0\rangle$ is the identity element of $\mathcal{A}$ and will be often denoted by 1. Two ray matrices without a nonzero entry in the same position are said to be *disjoint*; otherwise they are *overlapping*. A usual $\overline{m} \times \overline{n}$ matrix A over $\mathbb{Z}$ is a *finite matrix*; we denote by A^0 the matrix in $\mathcal{A}$ in which A takes up the upper left corner and the rest is filled with zeros. Let

$$\mathcal{F} = \{A^0\,;\ A \text{ is a finite matrix over} \mathbb{Z}\,\}$$

and let $\mathcal{B}$ be the subring of $\mathcal{A}$ generated by the matrices $\langle 0,1\rangle$ and $\langle 1,0\rangle$.

We start with two lemmas from ([4], Section 3).

Lemma 2.1. *For any $m,n,p,q \in \mathbb{N}$, we have*

$$\langle m,n\rangle\langle p,q\rangle = \langle m+p-r, n+q-r\rangle,$$

where $r = \min\{n,p\}$. In particular,

$$B = \{\langle m,n\rangle\,;\ m,n \in \mathbb{N}\}$$

is a bicyclic semigroup.

Lemma 2.2. *The ring $\mathcal{B}$ consists precisely of the elements of the form*

$$A^0 + \sum_{i=1}^{p} k_i\langle m_i,n_i\rangle$$

where A is a finite matrix over $\mathbb{Z}$, $k_i \in \mathbb{Z}$, $m_i,n_i \in \mathbb{N}$, $i=1,\dots,p$ and $p \geqslant 0$. Moreover, the rays $\langle m_i,n_i\rangle$ may be assumed pairwise disjoint.

We will use these lemmas generally without explicit reference.

3. Invertible and regular elements of $\mathcal{B}$

We first find the form of invertible elements of $\mathcal{B}$ including their canonical form. Toward the determination of the subring of $\mathcal{B}$ generated by its group of units, we first prove that the ring $M_n(\mathbb{Z}\,)$ is generated by its group of units. After a multiple characterization of the subring of $\mathcal{B}$ generated by its group of units, we provide an isomorphic copy of this group.

Let X be the element represented in Lemma 2.2 and set

$$S = \{n_i - m_i\,;\ i=1,\dots,p\}.$$

Clearly $S \neq \emptyset$ if and only if $X \notin \mathcal{F}$. We define functions

(2) $$\alpha,\omega\colon\ \mathcal{B}\setminus\mathcal{F} \to \mathbb{Z}\,,\quad \alpha\colon\ X \mapsto \min S,\quad \omega\colon\ X \mapsto \max S.$$

By ([4], Lemma 6.1), $\mathcal{B} \setminus \mathcal{F}$ is a multiplicative subsemigroup of $\mathcal{B}$ and α, ω are homomorphisms of $(\mathcal{B} \setminus \mathcal{F}, \cdot)$ onto $(\mathbb{Z}, +)$, that is,

$$(XY)\alpha = X\alpha + Y\alpha, \quad (XY)\omega = X\omega + Y\omega \quad (X, Y \in \mathcal{B} \setminus \mathcal{F}). \tag{3}$$

Proposition 3.1. *The units of $\mathcal{B}$ are precisely the elements of the form*

$$A^0 + \varepsilon\langle n+1, n+1\rangle$$

where A is a unit of $M_n(\mathbb{Z})$, $n \in \mathbb{N}$ and $\varepsilon = \pm 1$.

Proof. Obviously, all elements of this form are units of $\mathcal{B}$. Conversely, let $X, Y \in \mathcal{B}$ be such that $XY = YX = 1$. Since $\mathcal{F}$ is an ideal of $\mathcal{B}$ and $1 \notin \mathcal{F}$, we have $X, Y \in \mathcal{B} \setminus \mathcal{F}$. It follows from (3) that

$$X\alpha + Y\alpha = 1\alpha = 0, \quad X\omega + Y\omega = 1\omega = 0.$$

Since $X\omega \geqslant X\alpha$ and $Y\omega \geqslant Y\alpha$, we get $X\omega = X\alpha = 0$ and $Y\omega = Y\alpha = 0$ and we may write

$$X = A^0 + \varepsilon\langle n+1, n+1\rangle, \quad Y = B^0 + \eta\langle n+1, n+1\rangle$$

for some $A, B \in M_n(\mathbb{Z})$ and $\varepsilon, \eta \in \mathbb{Z} \setminus \{0\}$. Now

$$A^0B^0 + \varepsilon\eta\langle n+1, n+1\rangle = XY = 1 = YX = B^0A^0 + \eta\varepsilon\langle n+1, n+1\rangle$$

yields that $AB = I_n = BA$ and $\varepsilon, \eta = \pm 1$, as required. □

If A is an $\overline{m} \times \overline{n}$ matrix, we call

$$A^a = \begin{cases} \begin{bmatrix} A \\ 0 \end{bmatrix} & \overline{n} \times \overline{n} \quad \text{if } m < n, \\ [A\ 0] & \overline{m} \times \overline{m} \quad \text{if } m > n, \\ A & \quad\quad\quad \text{if } m = n \end{cases}$$

the *augmentation matrix* of A.

We now construct all regular elements of the ring $\mathcal{B}$:

Proposition 3.2. *The regular elements of $\mathcal{B}$ are precisely the elements of the form*

$$A^0 \quad \text{or} \quad A^0 + \varepsilon\langle m+1, n+1\rangle$$

where A is an $\overline{m} \times \overline{n}$ matrix, A^a is regular in $M_p(\mathbb{Z})$ with $p = \max\{m, n\}$ and $\varepsilon = \pm 1$.

P r o o f. Suppose that $A^0+\varepsilon\langle m+1,n+1\rangle$ satisfies the above conditions. Since A^a is regular in $M_p(\mathbb{Z})$, there exists $B\in M_p(\mathbb{Z})$ such that $A^aBA^a=A^a$. For $k=0,\dots,p$, define $I'_k=(u_{ij})\in M_p(\mathbb{Z})$ by

$$u_{ij}=\begin{cases}1 & \text{if } i=j\leqslant k,\\ 0 & \text{otherwise.}\end{cases}$$

Straightforward checking shows that $A^aI'_n=A^a=I'_mA^a$, hence $A^a(I'_nBI'_m)A^a=A^a$. Since the last $p-n$ rows and the last $p-m$ columns of $I'_nBI'_m$ are filled with zeros, we have $ACA=A$ for some $\overline{n}\times\overline{m}$ matrix C. It follows easily that

$$\begin{aligned}(A^0+\varepsilon\langle m+1,n+1\rangle)(C^0+\varepsilon\langle n+1,m+1\rangle)(A^0+\varepsilon\langle m+1,n+1\rangle)\\ =A^0+\varepsilon\langle m+1,n+1\rangle\end{aligned}$$

and $A^0+\varepsilon\langle m+1,n+1\rangle$ is regular.

Conversely, let $X,Y\in\mathcal{B}$ be such that $XYX=X$. It follows from (3) that

$$X\alpha+Y\alpha+X\alpha=X\alpha,\quad X\omega+Y\omega+X\omega=X\omega.$$

Hence $X\alpha=-(Y\alpha)$ and $X\omega=-(Y\omega)$. Since $X\omega\geqslant X\alpha$ and $Y\omega\geqslant Y\alpha$, we get

$$X\omega=X\alpha=-(Y\alpha)=-(Y\omega)$$

and we may write

$$X=A^0+\varepsilon\langle m+1,n+1\rangle,\quad Y=B^0+\eta\langle n+1,m+1\rangle$$

for some $\overline{m}\times\overline{n}$ matrix A, $\overline{n}\times\overline{m}$ matrix B and $\varepsilon,\eta\in\mathbb{Z}$. Now

$$A^0B^0A^0+\varepsilon\eta\varepsilon\langle m+1,n+1\rangle=A^0+\varepsilon\langle m+1,n+1\rangle$$

yields that $ABA=A$ and $\varepsilon\in\{-1,0,1\}$, as required. $\square$

Let G_n denote the group of units of $M_n(\mathbb{Z})$.

Lemma 3.3. *The ring $M_n(\mathbb{Z})$ is generated by G_n.*

P r o o f. Let S_n denote the subring of $M_n(\mathbb{Z})$ generated by G_n. It suffices to prove that, for all $i,j=0,\dots,n$, we have $[i,j]\in S_n$. This is trivial for $n=0$. Assume $n>0$. Let $i,j=0,\dots,n$ with $i\neq j$. Then

$$(I_n-[i,j])(I_n+[i,j])=I_n=(I_n+[i,j])(I_n-[i,j])$$

and $[i,j]\in S_n$. Therefore we also have $[i,i]=[i,j][j,i]\in S_n$ and $S_n=M_n(\mathbb{Z})$. $\square$

We are now ready for the main result of this section.

Theorem 3.4. *The subring of $\mathcal{B}$ generated by its invertible (or idempotent) elements equals*

$$\{X + n1;\ X \in \mathcal{F},\ n \in \mathbb{Z}\}.$$

P r o o f. Let $R = \{X + n1;\ X \in \mathcal{F},\ n \in \mathbb{Z}\}$ and let R_i (or R_e) be the subring of $\mathcal{B}$ generated by its invertible (respectively, idempotent) elements. Clearly, R is a subring of $\mathcal{B}$ and it contains all units of $\mathcal{B}$ by Proposition 3.1. Hence $R_i \subseteq R$. To prove the reverse inclusion, we only need to show that $[i,j] \in R_i$ for all $i, j \in \mathbb{N}$, and this is performed similarly to the proof of Lemma 3.3. Thus $R = R_i$.

We have seen in ([4], Lemma 6.2) that the idempotents of $\mathcal{B}$ are precisely the elements of $\mathcal{B}$ of the form A^0 or $A^0 + \langle n+1, n+1\rangle$, where A is an idempotent $\overline{n} \times \overline{n}$ matrix. Thus all idempotents of $\mathcal{B}$ lie in R and $R_e \subseteq R$. To prove the reverse inclusion, we only need to show that $[i,j] \in R_e$ for all $i, j \in \mathbb{N}$. This is trivial for $i = j$. If $i \neq j$, then $[i,i] + [i,j]$ is an idempotent of $\mathcal{B}$ and so $[i,j] \in R_e$ as well. Thus $R_e = R$ as required. □

Our description of invertible elements of $\mathcal{B}$ essentially says that they are invertible elements of various $M_n(\mathbb{Z})$ provided with the "missing tail" of ones or minus ones to make them elements of $\mathcal{B}$. Hence the knowledge of invertible elements of $\mathcal{B}$ depends in a transparent way on the knowledge of invertible elements of various $M_n(\mathbb{Z})$. Analogous statements can be made for idempotent and regular elements of $\mathcal{B}$.

However, for the group of units of $\mathcal{B}$ we have the following simple statement.

Proposition 3.5. *Let G be the group of units of $\mathcal{B}$ and*

$$H = \{A^0 + 1;\ A \in M_n(\mathbb{Z}),\ A + I_n \text{ is invertible in } M_n(\mathbb{Z})\}.$$

Then H is a group and the mapping

$$X = A^0 + \varepsilon 1 \to \begin{cases} (X, 0) & \text{if } \varepsilon = 1, \\ (-X, 1) & \text{if } \varepsilon = -1 \end{cases}$$

is an isomorphism of G onto the direct product $H \times \mathbb{Z}_2$.

P r o o f. The straightforward argument may be safely omitted. □

Given a subset S of a ring R, the *centralizer* of S in R is the subring

$$\{r \in R;\ rs = sr \text{ for every } s \in S\}.$$

The centralizer of R in R is the *center* of R.

Proposition 3.6. *The centralizer C of the set of all idempotents in $\mathcal{B}$ is the subring $\{n1;\ n \in \mathbb{Z}\}$.*

Proof. Clearly, all elements of the form $n1$ lie in C. Conversely, let

$$X = (x_{ij}) \in C.$$

Let $m, n \in \mathbb{N}$ with $m \neq n$. Then

$$x_{mn}[m,n] = [m,m]X[n,n] = [m,m][n,n]X = 0,$$

hence $x_{mn} = 0$. Since $[m,m] + [m,n]$ is an idempotent, we have

$$\begin{aligned} x_{m,m}[m,m] + x_{nn}[m,n] &= ([m,m]+[m,n])X = X([m,m]+[m,n]) \\ &= x_{m,m}[m,m] + x_{mm}[m,n] \end{aligned}$$

and so $x_{mm} = x_{nn}$. Hence $X = k1$ for some $k \in \mathbb{Z}$. □

Corollary 3.7. *The center of $\mathcal{B}$ is the subring $\{n1;\ n \in \mathbb{Z}\}$.*

4. Ideals of $\mathcal{B}$

After introducing an appropriate description of principal ideals, we show that all ideals of $\mathcal{B}$ are finitely generated and construct all ideals of $\mathcal{B}$. We also prove that not all ideals of $\mathcal{B}$ are principal.

In [4], we introduced the following ideals of $\mathcal{B}$: for $k = 1, 2, \ldots$, writing $k \mid l$ if k divides l, let

$$\mathcal{I}_k = \{(a_{ij}) \in \mathcal{B};\ k \mid a_{ij} \text{ for } i, j = 0, 1, \ldots\}, \quad \mathcal{F}_k = \mathcal{I}_k \cap \mathcal{F}.$$

Note that $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{I}_1 = \mathcal{B}$.

Lemma 4.1. *For every $X \in \mathcal{B}$, there exist $l \geqslant 1$, $p \geqslant 0$ and $k_0, \ldots, k_p \in l\mathbb{Z}$ such that $l[0,0]$ and $\sum_{i=0}^{p} k_i\langle i, 0\rangle$ generate $\mathcal{B}X\mathcal{B}$.*

Proof. Let $X \in \mathcal{B}$ and $\mathcal{X} = \mathcal{B}X\mathcal{B}$. By ([4], Lemma 4.2), there exists $l \geqslant 1$ such that $\mathcal{X} \cap \mathcal{F} = \mathcal{F}_l$ and $\mathcal{X} \subseteq \mathcal{I}_l$. In view of (1) and Lemma 2.2, we may write

$$X = A^0 + \sum_{i=1}^{r} u_i\langle i, 0\rangle + \sum_{j=1}^{s} v_j\langle 0, j\rangle + t\langle 0, 0\rangle$$

for some $A \in M_n(\mathbb{Z})$, $r, s \geqslant 0$, $u_i, v_j, t \in \mathbb{Z}$, $i = 1, \ldots, r$, $j = 1, \ldots, s$. Let $m = \max\{n+1, s\}$ and $Y = X\langle m, 0\rangle$. Then

$$Y = \sum_{i=1}^{r} u_i \langle i+m, 0\rangle + \sum_{j=1}^{s} v_j \langle m-j, 0\rangle + t\langle m, 0\rangle.$$

Since $Y = X\langle m, 0\rangle \in \mathcal{X} \subseteq \mathcal{I}_l$, we have $u_i, v_j, t \in l\mathbb{Z}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Let $\mathcal{Y}$ be the ideal of $\mathcal{B}$ generated by $l[0,0]$ and Y. Since $l[0,0] \in \mathcal{F}_l \subseteq \mathcal{X}$ and $Y \in \mathcal{X}$, we have $\mathcal{Y} \subseteq \mathcal{X}$. On the other hand, we get

$$X = X(I_{m-1}^0 + \langle m, m\rangle) = XI_{m-1}^0 + X\langle m, 0\rangle\langle 0, m\rangle = XI_{m-1}^0 + Y\langle 0, m\rangle.$$

Since $XI_{m-1}^0 \in \mathcal{X} \cap \mathcal{F} = \mathcal{F}_l$ and $\mathcal{F}_l$ is clearly the principal ideal generated by $l[0,0]$, it follows that $XI_{m-1}^0 \in \mathcal{Y}$ and so $X \in \mathcal{Y}$. Thus $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} = \mathcal{Y}$ as required. □

We are now able to prove the first main result of this section.

Theorem 4.2. *Every ideal of $\mathcal{B}$ is finitely generated.*

P r o o f. Let J be an ideal of $\mathcal{B}$. By Lemma 4.1, J is generated by a set of the form

$$\{l_\lambda[0,0];\ \lambda \in \Lambda\} \cup \left\{\sum_{i=0}^{p_\lambda} k_{\lambda i}\langle i, 0\rangle;\ \lambda \in \Lambda\right\},$$

where $l_\lambda \geqslant 1$, $p_\lambda \geqslant 0$ and $k_{\lambda 0}, \ldots, k_{\lambda p_\lambda} \in l_\lambda\mathbb{Z}$ for every $\lambda \in \Lambda$. Let l denote the greatest common divisor of all l_λ. Clearly, J is also generated by

$$\{l[0,0]\} \cup \left\{\sum_{i=0}^{p_\lambda} k_{\lambda i}\langle i, 0\rangle;\ \lambda \in \Lambda\right\}.$$

Write $x = \langle 0, 1\rangle$ and $y = \langle 1, 0\rangle$, and let $y^0 = 1$. Since $[0,0] = 1 - \langle 1, 0\rangle\langle 0, 1\rangle$ and $\langle i, 0\rangle = \langle 1, 0\rangle^i$, we can state in this notation that J is generated by

$$\{l(1-yx)\} \cup \left\{\sum_{i=0}^{p_\lambda} k_{\lambda i} y^i;\ \lambda \in \Lambda\right\}.$$

Since $\mathbb{Z}$ is a Noetherian ring, it follows from the Hilbert basis theorem that the polynomial ring $\mathbb{Z}[y]$ is itself Noetherian (cf. [6], p. 395). As $\mathbb{Z}[y]$ is a free unitary ring on $\{y\}$ ([6], Chapter 1.3), it follows that every unitary ring generated by a single element is Noetherian. In particular, the unitary subring U of $\mathcal{B}$ generated by $y = \langle 1, 0\rangle$ is Noetherian and so every ideal of U is finitely generated. Hence there exist $X_1, \ldots, X_n \in U$ such that $UX_1U + \ldots UX_nU$ is the ideal of U generated by $\left\{\sum_{i=0}^{p_\lambda} k_{\lambda i} y^i;\ \lambda \in \Lambda\right\}$. It follows that J is generated by $\{l(1-yx)\} \cup \{X_1, \ldots, X_n\}$ and so J is finitely generated. □

Corolary 4.3. *Every homomorphic image of $\mathcal{B}$ is finitely presented.*

Proof. By ([4], Theorem 4.4), $\mathcal{B}$ can be presented as a unitary ring by $\langle x, y;\ xy = 1\rangle$. Let $X = \{x, y\}$ and let X^* denote the free monoid on X. The semigroup ring $\mathbb{Z}(X^*)$ is a free unitary ring on X ([6], Chapter 1.3). Let J be an ideal of $\mathcal{B}$. Interpreting x as $\langle 0, 1\rangle$ and y as $\langle 1, 0\rangle$, Theorem 4.2 yields that there exist $u_1, \ldots, u_n \in \mathbb{Z}(X^*)$ such that the set $\{u_1, \ldots, u_n\}$ generates J when interpreted as a subset of J. Clearly, $\mathcal{B}/J$ can be presented as a unitary ring by

$$\langle x, y;\ xy = 1,\ u_1 = 0, \ldots, u_n = 0\rangle.$$

Thus every homomorphic image of $\mathcal{B}$ is finitely presented. □

The situation is of course completely different when we consider the multiplicative semigroup of $\mathcal{B}$. For any ring R, we denote by $\mathbf{M}(R)$ its multiplicative semigroup.

Proposition 4.4. *The semigroup $\mathbf{M}(\mathcal{B})$ is not finitely generated.*

Proof. Since $\mathcal{B}$ can be presented as a unitary ring by $\langle x, y;\ xy = 1\rangle$, the mapping $x \mapsto 1$, $y \mapsto 1$ induces a ring homomorphism of $\mathcal{B}$ onto $\mathbb{Z}$ and so $\mathbf{M}(\mathbb{Z})$ is a homomorphic image of $\mathbf{M}(\mathcal{B})$. Since $\mathbf{M}(\mathbb{Z})$ is not finitely generated, $\mathbf{M}(\mathcal{B})$ is not finitely generated, either. □

Given $l \geqslant 1$ and a $\overline{p} \times \overline{q}$ matrix $K = (k_{ij})$ over $l\mathbb{Z}$, we define

$$I(l, K) = \mathcal{F}_l + \Big\{\sum_{t=1}^{r}\sum_{j=0}^{q}\sum_{i=0}^{p} n_{tj}k_{ij}\langle i + u_{tj}, v_{tj}\rangle;\ r \geqslant 0,\ n_{tj} \in \mathbb{Z},\ u_{tj}, v_{tj} \in \mathbb{N},$$
$$i = 0, \ldots, p,\ j = 0, \ldots, q,\ t = 1, \ldots, r\Big\}.$$

We are now able to describe all ideals of $\mathcal{B}$.

Theorem 4.5. *For every $l \geqslant 1$ and every $\overline{p} \times \overline{q}$ matrix $K = (k_{ij})$ over $l\mathbb{Z}$, $I(l, K)$ is a nonzero ideal of $\mathcal{B}$. Conversely, every nonzero ideal of $\mathcal{B}$ is of this form. In particular, every nonzero principal ideal of $\mathcal{B}$ is of the form $I(l, K)$ for some $l \geqslant 1$ and a $\overline{p} \times \overline{0}$ matrix $K = (k_{ij})$ over $l\mathbb{Z}$.*

Proof. First we show that $I(l, K)$ is an ideal of $\mathcal{B}$. Clearly, $I(l, K)$ is a nonzero additive subgroup of $\mathcal{B}$. Let

$$X = A + \Big\{\sum_{t=1}^{r}\sum_{j=0}^{q}\sum_{i=0}^{p} n_{tj}k_{ij}\langle i + u_{tj}, v_{tj}\rangle\Big\}$$

with $A \in \mathcal{F}_l$ and parameters as specified above. Since $\mathcal{B}$ is generated by $\langle 0,1\rangle$ and $\langle 1,0\rangle$, we only have to show that

$$\langle 0,1\rangle X,\ \langle 1,0\rangle X,\ X\langle 0,1\rangle,\ X\langle 1,0\rangle \in I(l,K).$$

Since $\mathcal{F}_l$ is itself an ideal of $\mathcal{B}$, we can assume that $A = 0$. The distributive law accounts for further simplification, allowing us to assume that $r = 1$. Dropping unnecessary subscripts, we are reduced to the case

$$X = \sum_{j=0}^{q}\sum_{i=0}^{p} n_j k_{ij}\langle i+u_j, v_j\rangle.$$

Again, distributivity provides a final reduction to the case where all n_j but one are zero, and we may even assume that the (unique) nonzero n_j equals 1. So we have worked our way out to the case

$$X = \sum_{i=0}^{p} k_{ij}\langle i+u, v\rangle$$

where $j \in \{0,\dots,q\}$. Writing $\sum$ for $\sum\limits_{i=0}^{p}$, by direct computation we obtain

$$\begin{aligned}
\langle 0,1\rangle X &= \sum k_{ij}\langle 0,1\rangle\langle i+u,v\rangle\\
&= \Big(\sum k_{ij}\langle 0,1\rangle[i+u,v]\Big) + \Big(\sum k_{ij}\langle 0,1\rangle\langle i+u+1,v+1\rangle\Big) \in \mathcal{F}_l\\
&\quad + \sum k_{ij}\langle i+u,v+1\rangle \subseteq I(l,K),\\
\langle 1,0\rangle X &= \sum k_{ij}\langle 1,0\rangle\langle i+u,v\rangle = \sum k_{ij}\langle i+u+1,v\rangle \in I(l,K),\\
X\langle 0,1\rangle &= \sum k_{ij}\langle i+u,v\rangle\langle 0,1\rangle = \sum k_{ij}\langle i+u,v+1\rangle \in I(l,K),\\
X\langle 1,0\rangle &= \sum k_{ij}\langle i+u,v\rangle\langle 1,0\rangle = \Big(\sum k_{ij}[i+u,v]\langle 1,0\rangle\Big)\\
&\quad + \Big(\sum k_{ij}\langle i+u+1,v+1\rangle\langle 1,0\rangle\Big) \in \mathcal{F}_l + \sum k_{ij}\langle i+u+1,v\rangle \subseteq I(l,K).
\end{aligned}$$

Therefore $I(l,K)$ is a nonzero ideal of $\mathcal{B}$.

Conversely, let J be a nonzero ideal of $\mathcal{B}$. By ([4], Lemmma 4.2), there exists $l \geqslant 1$ such that $J \cap \mathcal{F} = \mathcal{F}_l$ and $J \subseteq \mathcal{I}_l$. By Theorem 4.2, the ideal J is finitely generated. By Lemma 4.1, we may assume that J is generated by $l[0,0]$ and $X_0,\dots,X_q$ with $X_j = \sum\limits_{i=0}^{p_j} k_{ij}\langle i,0\rangle$ for $j = 0,\dots,q$, where $p_j \geqslant 0$ and $k_{ij} \in \mathbb{Z}$. Since $X_j \in J \subseteq \mathcal{I}_l$, we have $l \mid k_{ij}$ for all i and j. Adding $k_{ij} = 0$ whenever needed, we may assume that $p_j = p$ for every j. Let K denote the $\overline{p} \times \overline{q}$ matrix whose (i,j)-th entry is k_{ij}. Clearly, K is a matrix over $l\mathbb{Z}$.

It is immediate that $l[0,0]$, $X_0, \ldots, X_q \in I(l,K)$, hence $J \subseteq I(l,K)$. Since $\mathcal{F}_l \subseteq J$, to prove the opposite inclusion it suffices to note that

$$\sum_{i=0}^{p} k_{ij} \langle i+u, v\rangle = \sum_{i=0}^{p} k_{ij} \langle i, 0\rangle \langle u, v\rangle = X_j \langle u, v\rangle \in J$$

for $j = 0, \ldots, q$ and $u, v \geqslant 0$. Thus $J = I(l,K)$.

Now let J denote a nonzero principal ideal of $\mathcal{B}$. By Lemma 4.1, J is generated by $l[0,0]$ and $X_0 = \sum\limits_{i=0}^{p} k_{i0}\langle i, 0\rangle$, where $l \geqslant 1$, $p \geqslant 0$ and $k_{i0} \in l\mathbb{Z}$. Viewing $K = (k_{i0})$ as a $\overline{p} \times \overline{0}$ matrix, the above argument yields that $J = I(l,K)$, as required. $\square$

Note that $\mathcal{F}_l \subseteq I(l,K) \subseteq \mathcal{I}_l$ implies that $[0,0] \in I(l,K)$ if and only if $l = 1$. Precise identification of ideals with this property is important in view of the already mentioned characterization of directly infinite rings in [4].

Next we show that not all ideals of $\mathcal{B}$ are principal.

Example 4.6. Let J be the ideal of $\mathcal{B}$ generated by $\langle 0,0\rangle + \langle 1,0\rangle$ and $\langle 0,0\rangle + \langle 2,0\rangle$. Then J is not a principal ideal of $\mathcal{B}$.

Proof. Suppose that J is principal. By Theorem 4.5, $J = I(l,K)$ for some $l \geqslant 1$ and some $\overline{p} \times \overline{0}$ matrix $K = (k_{i0})$ over $l\mathbb{Z}$. We may thus write

$$\langle 0,0\rangle + \langle 1,0\rangle = A + \sum_{t=1}^{r} \sum_{i=0}^{p} n_t k_{i0} \langle i + u_t, v_t\rangle \tag{4}$$

with $A \in \mathcal{F}_l$, $r \geqslant 0$, $n_t \in \mathbb{Z}$ and $u_{tj}, v_{tj} \in \mathbb{N}$. We may assume that r is minimum. Since $\langle 0,0\rangle + \langle 1,0\rangle \notin \mathcal{F}_l$, we must have $r \geqslant 1$. Suppose that $v_s - u_s = v_t - u_t$ for some distinct $s, t \in \{1, \ldots, r\}$. Then we may assume that $u_s - u_t = v_s - v_t = w \geqslant 0$, and

$$\sum_{i=0}^{p} n_s k_{i0} \langle i + u_s, v_s\rangle + \sum_{i=0}^{p} n_t k_{i0} \langle i + u_t, v_t\rangle = \sum_{i=0}^{p} (n_s + n_t) k_{i0} \langle i + u_s, v_s\rangle + W$$

for some $W \in \mathcal{F}_l$, contradicting the minimality of r. Hence we may assume that $v_1 - u_1 < \ldots < v_r - u_r$ and all n_t are nonzero. Since $\langle 0,0\rangle + \langle 1,0\rangle \notin \mathcal{F}_l$, we must have $k_{i0} \neq 0$ for some i. Let m (or M) denote the minimum (respectively, maximum) $i \in \{0, \ldots, p\}$ such that $k_{i0} \neq 0$.

Recall the notation α and ω from (2). In view of (4) and all the above assumptions, to determine the image of $\langle 0,0\rangle + \langle 1,0\rangle$ by α and ω we must compute the minimum and maximum values of $v_t - i - u_t$ for all i and t with $k_{i0} \neq 0$. It follows easily that

$$-1 = (\langle 0,0\rangle + \langle 1,0\rangle)\alpha = -M + v_1 - u_1,$$
$$0 = (\langle 0,0\rangle + \langle 1,0\rangle)\omega = -m + v_r - u_r.$$

Since $-M \leqslant -m$ and $v_1 - u_1 \leqslant v_r - u_r$, one of the following two cases must occur:
(i) $M = m$, $v_r - u_r = v_1 - u_1 + 1$,
(ii) $M = m + 1$, $v_r - u_r = v_1 - u_1$.

Case (i). This implies that k_{m0} is the unique nonzero k_{i0} and $r = 2$. Hence

$$\langle 0,0\rangle + \langle 1,0\rangle = A + n_1 k_{m0}\langle m + u_1, v_1\rangle + n_2 k_{m0}\langle m + u_2, v_2\rangle.$$

It follows that $\langle 0,0\rangle$ overlaps with $\langle m+u_2, v_2\rangle$, and $\langle 1,0\rangle$ overlaps with $\langle m+u_1, v_1\rangle$. In particular, we obtain $n_1 k_{m0} = 1$ and so $k_{m0} = \pm 1$. It follows from the definition of $I(l, K)$ that $\langle m, 0\rangle \in I(l, K) = J$. By ([4], Theorem 4.4), $\mathcal{B}$ can be presented as a unitary ring by $\langle x, y;\ xy = 1\rangle$, where x and y correspond to the generators $\langle 0,1\rangle$ and $\langle 1,0\rangle$. Since $1^2 = 1$ in $\mathbb{Z}_2$, defining

$$\varphi\colon\ \mathcal{B} \to \mathbb{Z}_2, \quad \langle 0,1\rangle, \langle 1,0\rangle \mapsto 1$$

we obtain a homomorphism. Since J is generated by $\langle 0,0\rangle + \langle 1,0\rangle = 1 + \langle 1,0\rangle$ and $\langle 0,0\rangle + \langle 2,0\rangle = 1 + \langle 1,0\rangle^2$, it follows that $J \subseteq \ker\varphi$; however, $\langle m,0\rangle = \langle 1,0\rangle^m \notin \ker\varphi$, a contradiction.

Case (ii). Then $r = 1$ and we may omit subscripts by writing

$$\langle 0,0\rangle + \langle 1,0\rangle = A + nk_{m0}\langle m + u, v\rangle + nk_{m+1,0}\langle m + 1 + u, v\rangle.$$

Similarly to Case (i), we obtain $nk_{m0} = nk_{m+1,0} = 1$ and so $k_{m0} = k_{m+1,0} = \pm 1$. Since $J = I(l, K)$ it follows easily from the definition of $I(l, K)$ that J is generated by $[0,0]$ and $\langle m,0\rangle + \langle m+1,0\rangle$. Again using the presentation $\langle x, y;\ xy = 1\rangle$ of $\mathcal{B}$, and the identity $(-1)^2 = 1$ in $\mathbb{Z}$, defining

$$\psi\colon\ \mathcal{B} \to \mathbb{Z}, \quad \langle 0,1\rangle, \langle 1,0\rangle \mapsto -1$$

we obtain a homomorphism. Taking into account that J is generated by $[0,0] = 1 - \langle 1,0\rangle\langle 0,1\rangle$ and that

$$\langle m,0\rangle + \langle m+1,0\rangle = \langle 1,0\rangle^m + \langle 1,0\rangle^{m+1},$$

it follows that $J \subseteq \ker\psi$; however,

$$\langle 0,0\rangle + \langle 2,0\rangle = 1 + \langle 1,0\rangle^2 \notin \ker\psi,$$

another contradiction. Therefore J is not principal. □

5. Other properties of $\mathcal{B}$

We collect here all the remaining information we have on $\mathcal{B}$. They amount to a few salient features which indicate the peculiar nature of the ring $\mathcal{B}$.

We have seen in ([4], Lemma 4.2) that $\mathcal{F}_k$, for $k = 1, 2, \ldots$ exhaust all ideals of $\mathcal{F}$. It is easy to prove that

$$\mathcal{I}_k = \mathcal{B}(k\langle 0,0\rangle)\mathcal{B}, \quad \mathcal{F}_k = \mathcal{B}(k[0,0])\mathcal{B}.$$

Proposition 5.1. *Let $1 \leqslant k \mid l$ and $1 \leqslant m \mid n$. Then*

(i) $\mathcal{I}_k \subseteq \mathcal{I}_m \Leftrightarrow \mathcal{F}_k \subseteq \mathcal{F}_m \Leftrightarrow m \mid k$.

(ii) $\mathcal{B}/\mathcal{F}_k \cong \mathcal{B}/\mathcal{F}_m \Leftrightarrow \mathcal{F}/\mathcal{F}_k \cong \mathcal{F}/\mathcal{F}_m \Leftrightarrow k = m$.

P r o o f. (i) Immediate.

(ii) If $X \in \mathcal{F}$, then the additive order of $X + \mathcal{F}_k$ divides k and equals k in the case of $[0,0] + \mathcal{F}_k$. If $X \in \mathcal{B} \setminus \mathcal{F}$, then the additive order of $X + \mathcal{F}_k$ is infinite. Therefore $\mathcal{B}/\mathcal{F}_k \cong \mathcal{B}/\mathcal{F}_m$ if and only if $k = m$, and the same argument yields the remaining equivalence. $\square$

We now discuss the ring $\mathcal{B}$ with respect to some basic ring theoretic properties. Recall that a ring R is *prime* if for any ideals I and J of R, $IJ = (0)$ implies that either $I = (0)$ or $J = (0)$; and that R is *semiprimitive* if its Jacobson radical is equal to (0). We show first that $\mathcal{B}$ shares these properties.

Proposition 5.2. *The ring $\mathcal{B}$ is prime and semiprimitive.*

P r o o f. Let I and J be nonzero ideals of $\mathcal{B}$. By Theorem 4.5, there exist $k, l \geqslant 1$ such that $\mathcal{F}_k \subseteq I$ and $\mathcal{F}_l \subseteq J$. But then $IJ \supseteq \mathcal{F}_k\mathcal{F}_l \supseteq \mathcal{F}_{kl}$ and $IJ \neq (0)$. Consequently, $\mathcal{B}$ is a prime ring.

By contradiction, suppose that $\mathcal{B}$ is not semiprimitive. Then the Jacobson radical $\mathrm{Jac}(\mathcal{B})$ is a nonzero ideal of $\mathcal{B}$ and must thus contain $\mathcal{F}_k$ for some $k \geqslant 1$. We recall that given a unitary ring R, $a \in R$ is *quasi-invertible* if $1 - a$ is invertible, and an ideal J of R is quasi-invertible if all its elements are quasi-invertible. Since the Jacobson radical of a unitary ring can be characterized as the greatest quasi-invertible ideal of the ring ([6], Proposition 2.5.4), and since $-k[0,0] \in \mathcal{F}_k \subseteq \mathrm{Jac}(\mathcal{B})$, it follows that

$$1 - (-k)[0,0] = (1+k)[0,0] + \langle 1,1\rangle$$

is invertible. By Proposition 3.1, this implies that

$$(1+k)[0,0] + \langle 1,1\rangle = A^0 + \langle n+1, n+1\rangle$$

for an invertible $A \in M_n(\mathbb{Z})$. Since A has the determinant equal to $k+1$, it cannot be invertible, so a contradiction is reached and $\mathcal{B}$ is semiprimitive. $\square$

Let R be a ring. Then R is *left Artinian* (or *Noetherian*) if it satisfies the minimal (respectively, maximal) condition on left ideals. The "right" concepts pertain to right ideals. In view of ([6], Proposition 2.1.11), R is *primitive* if and only if there exists a proper left ideal L of R such that $L + J = R$ for every proper ideal J of R.

Next we prove that $\mathcal{B}$ does not satisfy any of the usual chain conditions.

Proposition 5.3. *The ring $\mathcal{B}$ is not left nor right Artinian or Noetherian or primitive.*

Proof. For every $k \in \mathbb{N}$, define

$$L_k = \{(x_{ij}) \in \mathcal{B};\ x_{ij} = 0 \text{ for all } i \geqslant 0,\ j \geqslant k\}.$$

It is straightforward to check that L_k is a left ideal of $\mathcal{B}$. Moreover,

$$L_0 \subset L_1 \subset L_2 \subset \dots$$

and so $\mathcal{B}$ is not left Noetherian. In fact, the above argument shows that there is no left Noetherian subring of $\mathcal{A}$ containing $\mathcal{F}$.

Dually, we may consider the right ideals

$$R_k = \{(x_{ij}) \in \mathcal{B};\ x_{ij} = 0 \text{ for all } i \geqslant k,\ j \geqslant 0\}$$

and prove that R is not right Noetherian.

By the Hopkins-Levitzki Theorem and its dual (see [6], Theorem 2.7.2), it follows that $\mathcal{B}$ is not left nor right Artinian. This can also be checked directly in view of the fact that

$$\mathcal{F}_2 \supset \mathcal{F}_{2^2} \supset \mathcal{F}_{2^3} \supset \dots$$

is an infinite descending chain of ideals of $\mathcal{B}$.

Let L be a proper left ideal of $\mathcal{B}$. We shall show that $L + \mathcal{F}_k \neq \mathcal{B}$ for some $k \geqslant 1$.

Let S denote the subsemigroup of $\mathcal{B}$ defined by

$$S = \bigcup_{n \geqslant 0} \{A^0 + \langle n+1, n+1\rangle;\ A \in M_n(\mathbb{Z})\}.$$

For $A \in M_n(\mathbb{Z})$, we denote by $\det A$ the determinant of A (cf. [3], Section XIII.4). Given

$$X = A^0 + \langle n+1, n+1\rangle \in S$$

with $A \in M_n(\mathbb{Z})$, we define the determinant of X to be $\det X = \det A$. If also $X = B^0 + \langle m+1, m+1\rangle$ with $B \in M_m(\mathbb{Z})$, say $m > n$, then

$$B = \begin{bmatrix} A & 0 \\ 0 & I_{m-n} \end{bmatrix},$$

hence $\det B = \det A$ and $\det X$ is well defined. Let

$$P = \{\det X;\ X \in L \cap S\}.$$

We shall prove that $P \subseteq k\mathbb{Z}$ for some $k > 1$. We may assume $P \neq \emptyset$ and $P \neq \{0\}$. Let $m > 0$ denote the greatest common divisor of all elements of P.

Suppose that $m = 1$. Then we may write

$$1 = r_1 p_1 + \ldots + r_t p_t$$

for some $r_1, \ldots, r_t \in \mathbb{Z}$ and $p_1, \ldots, p_t \in P$. By enlarging the finite matrices (if necessary), we may assume that for $i = 1, \ldots, t$ we have $p_i = \det X_i$, where $X_i = A_i^0 + \langle n+1, n+1\rangle$ and $A_i \in M_n(\mathbb{Z})$. By ([3], Proposition XIII.4.16), there exists $\widetilde{A_i} \in M_n(\mathbb{Z})$ such that $\widetilde{A_i} A_i = p_i I_n$. It follows that

$$I_n = \sum_{i=1}^{t} r_i p_i I_n = \sum_{i=1}^{t} r_i \widetilde{A_i} A_i$$

and we obtain

$$\begin{aligned} 1 &= I_n^0 + \langle n+1, n+1\rangle \\ &= \sum_{i=1}^{t} r_i \widetilde{A_i}^0 A_i^0 + \langle n+1, n+1\rangle \\ &= r_1 \widetilde{A_1}^0 A_1^0 + \langle n+1, n+1\rangle + \sum_{i=2}^{t} r_i \widetilde{A_i}^0 A_i^0 \\ &= (r_1 \widetilde{A_1}^0 + \langle n+1, n+1\rangle) X_1 + \sum_{i=2}^{t} r_i \widetilde{A_i}^0 X_i \in L, \end{aligned}$$

a contradiction, since L is a proper left ideal of $\mathcal{B}$. Hence $m > 1$. Since $P \subseteq m\mathbb{Z}$, we have $P \subseteq k\mathbb{Z}$ for some $k > 1$ in all cases.

Suppose that $L + \mathcal{F}_k = \mathcal{B}$. Then $X + A = 1$ for some $X \in L$ and $A \in \mathcal{F}_k$. Since $X = 1 - A$, we have $X \in L \cap S$ and so $\det X \in k\mathbb{Z}$. For every matrix Y over $\mathbb{Z}$, let $Y_{(k)}$ denote the matrix over $\mathbb{Z}_k$ obtained by projecting each entry of Y into $\mathbb{Z}_k$. It is a simple exercise to check that if $Y \in S$ then $\det Y_{(k)}$ is the projection in $\mathbb{Z}_k$ of $\det Y$. In particular, $\det X_{(k)} = 0$ in $\mathbb{Z}_k$. But

$$X_{(k)} = 1_{(k)} - A_{(k)} = 1_{(k)}$$

and we obtain $\det X_{(k)} = 1$ in $\mathbb{Z}_k$, a contradiction. Thus $L + \mathcal{F}_k \neq \mathcal{B}$ as required. $\square$

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