Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 3, 585–608

Persistent URL: http://dml.cz/dmlcz/127671

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ESTIMATES FOR THE ENERGY INTEGRAL OF QUASIREGULAR MAPPINGS ON RIEMANNIAN MANIFOLDS AND ISOPERIMETRY

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(Received October 15, 1998)

Abstract. The rate of growth of the energy integral of a quasiregular mapping $f\colon \mathscr{X} \to \mathscr{Y}$ is estimated in terms of a special isoperimetric condition on $\mathscr{Y}$. The estimate leads to new Phragmén-Lindelöf type theorems.

Keywords: Phragmén-Lindelöf type theorems, quasiregular mappings, isoperimetry

MSC 2000: 30C65

1. Introduction

Let $D \subset C$ be an unbounded domain and let $w = f(z)$ be a holomorphic function continuous on the closure $\overline{D}$. The Phragmén-Lindelöf principle [16] traditionally refers to the alternatives of the following type:

α) If $\operatorname{Re} f(z) \leqslant 1$ everywhere on ∂D, then either $\operatorname{Re} f(z)$ grows with a certain rate as $z \to \infty$, or $\operatorname{Re} f(z) \leqslant 1$ for all $z \in D$;

β) If $|f(z)| \leqslant 1$ on ∂D, then either $|f(z)|$ grows with a certain rate as $|z| \to \infty$ or $|f(z)| \leqslant 1$ for all $z \in D$.

Here the rate of growth of the quantities $\operatorname{Re} f(z)$ and $|f(z)|$ depends on the "width" of the domain D near infinity and, in fact, the "narrower" the domain the higher the rate of growth.

It is not difficult to prove that these conditions are equivalent to the following conditions:

α_1) If $\operatorname{Re} f(z) = 1$ on ∂D and $\operatorname{Re} f(z) \geqslant 1$ in D, then either $\operatorname{Re} f(z)$ grows with a certain rate as $z \to \infty$ or $f(z) \equiv \text{const}$;

β_1) If $|f(z)| = 1$ on ∂D and $|f(z)| \geqslant 1$ in D then either $|f(z)|$ grows with a certain rate as $z \to \infty$ or $f(z) \equiv \text{const}$.

Let D be an unbounded domain in $\mathbb{R}^n$ and let $f = (f_1, f_2, \ldots, f_n)\colon D \to \mathbb{R}^n$ be a quasiregular mapping. We assume that $f \in C^0(\overline{D})$. It seems natural to consider the Phragmén-Lindelöf alternative under the following assumptions:

a) $f_1(x)\big|_{\partial D} = 1$ and $f_1(x) \geqslant 1$ everywhere in D,

b) $\sum\limits_{i=1}^{p} f_i^2(x)\big|_{\partial D} = 1$ and $\sum\limits_{i=1}^{p} f_i^2(x) \geqslant 1$ on D, $1 < p < n$,

c) $|f(x)| = 1$ on ∂D and $|f(x)| \geqslant 1$ on D.

Several formulations of the Phragmén-Lindelöf theorem under various assumptions can be found in [14], [17], [2], [6], [12], [13]. However, these results are mainly of qualitative character. Here we give a new approach to Phragmén-Lindelöf type theorems for quasiregular mappings, based on isoperimetry, which leads to nearly sharp results. Our approach can be used to prove Phragmén-Lindelöf type results for quasiregular mappings of Riemannian manifolds.

Let $\mathscr{Y}$ be an n-dimensional noncompact Riemannian C^2-manifold with piecewise smooth boundary $\partial\mathscr{Y}$ (possibly empty). A function $u \in C^0(\overline{\mathscr{Y}}) \cap W_n^1(\mathscr{Y})$ is called a *growth function* with $\mathscr{Y}$ as a *domain of growth* if (i) $u \geqslant 1$, (ii) $u\big|\partial\mathscr{Y} = 1$ if $\partial\mathscr{Y} \neq \emptyset$, and $\sup\limits_{y \in \mathscr{Y}} u(y) = +\infty$.

We consider a quasiregular mapping $f\colon \mathscr{X} \to \mathscr{Y}$, where $\mathscr{X}$ is a noncompact Riemannian C^2-manifold, $\dim \mathscr{X} = n$ and $\partial\mathscr{X} \neq \emptyset$. We assume that $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$. In what follows by the Phragmén-Lindelöf principle, we mean an alternative of the form: either the function $u(f(x))$ has a certain rate of growth in $\mathscr{X}$ or $f(x) \equiv \text{const}$.

By choosing the domain of growth $\mathscr{Y}$ and the growth function $u(y)$ in a special way we can obtain several formulations of Phragmén-Lindelöf theorems for quasiregular mappings. In view of the examples in [14], the best results are obtained if an n-harmonic function is chosen as a growth function. In the case a) the domain of growth is $\mathscr{Y} = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n\colon\ y_1 \geqslant 0\}$ and as the function of growth it is natural to choose $u(y) = y_1 + 1$; in the case b) the domain $\mathscr{Y}$ is the set $\{y = (y_1, \ldots, y_n) \in \mathbb{R}^n\colon\ \sum\limits_{i=1}^{p} y_i^2 \geqslant 1\}$, $1 < p < n$, and $u(y) = \Big(\sum\limits_{i=1}^{p} y_i^2\Big)^{(n-p)/(2(n-1))}$; in the case c) the domain of growth is $\mathscr{Y} = \{y \in \mathbb{R}^n\colon\ |y| > 1\}$ and $u(y) = \log|y| + 1$.

Our approach is based on isoperimetric conditions for $\mathscr{Y}$ in a certain metric $\mathrm{d}s_u$, defined by the growth function. For manifolds of dimension $\dim \mathscr{X} = \dim \mathscr{Y} = n > 2$ there are many different isoperimetry types, which are not equivalent to one another, see [3].

Let $\mathscr{X}$ be a Riemannian manifold with boundary $\partial\mathscr{X}$ (possibly empty). Throughout the paper, we will assume that the manifold is orientable and of class C^2. By $T(\mathscr{X})$ we denote the tangent bundle and by $T_x(\mathscr{X})$ the tangent space at the point $x \in \mathscr{X}$. For each pair of vectors $x, y \in T_x(\mathscr{X})$ the symbol $\langle x, y\rangle$ denotes their inner product.

Below we shall use standard notation for function classes on manifolds. Thus, for example, the symbol $L^p_{\mathrm{loc}}(D)$ stands for the set of all Lebesgue measurable functions on an open set $D \subset \mathscr{X}$ which are locally L^p-integrable. The symbol $W^1_{p,\mathrm{loc}}(D)$ stands for the set of functions in $L^p_{\mathrm{loc}}(D)$ that have generalized partial derivatives in the sense of Sobolev of class $L^p_{\mathrm{loc}}(D)$.

If $\mathscr{X}$ and $\mathscr{Y}$ are Riemannian manifolds of class C^2 and $F\colon D \to \mathscr{Y}$, $D \subset \mathscr{X}$, is a mapping, then we shall say that $F \in L^p_{\mathrm{loc}}(D)$ if for an arbitrary function $\varphi \in C^0(\mathscr{Y})$ we have $\varphi\circ F \in L^p_{\mathrm{loc}}(D)$. The mapping F is in the class $W^1_{p,\mathrm{loc}}(D)$, if $\varphi\circ F \in W^1_{p,\mathrm{loc}}(D)$ for every $\varphi \in C^1(\mathscr{Y})$.

Let $\mathscr{X}$ and $\mathscr{Y}$ be Riemannian manifolds of dimension n. A mapping $f\colon \mathscr{X} \to \mathscr{Y}$ of class $W^1_{n,\mathrm{loc}}(\mathscr{X})$ is called a quasiregular mapping if f satisfies

$$|f'(x)|^n \leqslant K J_f(x) \tag{1}$$

almost everywhere on $\mathscr{X}$. Here $f'(x)\colon T_x(\mathscr{X}) \to T_{f(x)}(\mathscr{Y})$ is the formal derivative of $f(x)$ and $J_f(x)$ is the Jacobian of f at the point $x \in \mathscr{X}$.

The least constant $K \geqslant 1$ in the inequality (1) is called the outer dilatation of f and denoted by $K_O(f)$. If F is quasiregular, then the least constant $K \geqslant 1$, for which we have

$$J_f(x) \leqslant K l(f'(x))^n$$

almost everywhere on $\mathscr{X}$, is called the inner dilatation of the mapping $f\colon \mathscr{X} \to \mathscr{Y}$ and denoted by $K_I(f)$. Here

$$l(f'(x)) = \min_{|h|=1} |f'(x)h|.$$

The quantity

$$K(f) = \max\{K_O(f), K_I(f)\}$$

is called the maximal dilatation of f and if $K(f) \leqslant K$, then the mapping f is called K-quasiregular.

If $f(x)\colon \mathscr{X} \to \mathscr{Y}$ is a quasiregular homeomorphism, then the mapping f is called quasiconformal. In this case the inverse mapping f^{-1} is also quasiconformal in the domain $f(\mathscr{X}) \subset \mathscr{Y}$ and $K(f^{-1}) = K(f)$ ([8], [18]).

An elementary example of a quasiconformal mapping is a bilipschitz mapping. A homeomorphism $f\colon \mathscr{X} \to \mathscr{Y}$ of a class $W^1_{n,\,\mathrm{loc}}(\mathscr{X})$ is called a (locally) bilipschitz mapping if f satisfies almost everywhere on $\mathscr{X}$

$$\frac{1}{L} \leqslant |f'(x)| \leqslant L, \quad L \equiv \mathrm{const}.$$

Let $\mathscr{X}$ be a Riemannian manifold and let

$$A\colon T(\mathscr{X}) \to T(\mathscr{X})$$

be a mapping defined a.e. on the tangent bundle $T(\mathscr{X})$. Suppose that for a.e. $x \in \mathscr{X}$ the mapping A is continuous on the fiber T_x, i.e. for a.e. $x \in \mathscr{X}$ the function $A(x,\xi)\colon \xi \in T_x \to T_x$ is defined and continuous; the mapping $x \to A_x(X)$ is measurable for all measurable vector fields X (see [8]).

Suppose that for a.e. $x \in \mathscr{X}$ and for all $\xi \in T_x$ the following inequalities are valid:

$$\nu_1 |\xi|^n \leqslant \langle \xi, A(x,\xi) \rangle \tag{2}$$

and

$$|A(x,\xi)| \leqslant \nu_2 |\xi|^{n-1}, \tag{3}$$

where $0 < \nu_1 \leqslant \nu_2 < +\infty$ are constants (see [8]).

We consider the equation

$$\operatorname{div} A(x, \nabla f) = 0. \tag{4}$$

Solutions to (4) are understood in the weak sense, that is, solutions are $W^1_{n,\mathrm{loc}}$-functions satisfying the integral identity

$$\int_{\mathscr{X}} \langle \nabla\theta, A(x,\nabla f) \rangle \, \mathrm{d}v_{\mathscr{X}} = 0 \tag{5}$$

for all $\theta \in W^1_n(\mathscr{X})$ with compact support in $\mathscr{X}$ where $\mathrm{d}v_{\mathscr{X}}$ is the volume form on $\mathscr{X}$.

A function f in $W^1_{n,\mathrm{loc}}(\mathscr{X})$ is a *subsolution* of (4) in $\mathscr{X}$ if

$$\operatorname{div} A(x,\nabla f) \geqslant 0 \tag{6}$$

weakly in $\mathscr{X}$, i.e.

$$\int_{\mathscr{X}} \langle \nabla\theta, A(x,\nabla f) \rangle \, \mathrm{d}v_{\mathscr{X}} \leqslant 0 \tag{7}$$

whenever $\theta \in W^1_n(\mathscr{X})$ is nonnegative with compact support in $\mathscr{X}$.

A basic example of an equation satisfying (2)–(4) is the n-Laplace equation

$$\operatorname{div}(|\nabla f|^{n-2}\nabla f) = 0. \tag{8}$$

3. Exhaustion functions

Let $\mathscr{X}$ be a noncompact Riemannian manifold with a boundary $\partial\mathscr{X}$ (possibly empty) and let $h\colon \mathscr{X} \to (0,+\infty)$ be a locally Lipschitz function. For $t \in (0,+\infty)$ we denote by

$$B_h(t) = \{x \in \mathscr{X} : h(x) < t\}, \quad \Sigma_h = \Sigma_h(t) = \{x \in \mathscr{X} : h(x) = t\}$$

the h-balls and h-spheres, respectively.

We say that a function h is an *exhaustion function for the manifold* $\mathscr{X}$ if the following tree conditions are satisfied:

(i) for all $t \in (0,+\infty)$ the h-ball $\overline{B_h(t)}$ is compact;

(ii) there exists a compact set $K \subset \mathscr{X}$ such that $|\nabla h(x)| > 0$ for a.e. $x \in \mathscr{X} \setminus K$;

(iii) for every sequence $t_1 < t_2 < \ldots < \infty$ with $\lim\limits_{k\to\infty} t_k = +\infty$, the sequence of h-balls $\{B_h(t_k)\}$ generates an exhaustion of $\mathscr{X}$, i.e.

$$B_h(t_1) \subset B_h(t_2) \subset \ldots \subset B_h(t_k) \subset \ldots \quad \text{and} \quad \bigcup_k B_h(t_k) = \mathscr{X}.$$

The coarea formula or the Kronrod-Federer formula [11], [5, § 3.2] are useful tools for various estimates involving an exhaustion function.

3.1. Theorem. *Let Φ be a nonnegative Borel-measurable function in a domain $D \subset \mathscr{X}$ and u a locally Lipschitz function on D. Then*

$$\int_D \Phi(x)|\nabla u(x)|\,\mathrm{d}v_{\mathscr{X}} = \int_0^\infty \mathrm{d}t \int_{E_t} \Phi(x)\,\mathrm{d}H$$

where H is the surface measure on $E_t = \{x \in \mathscr{X} : |u(x)| = t\}$.

3.2. Example. Let $\mathscr{X} = \mathbb{R}^n$ be an n-dimensional Euclidean space. We fix an integer k, $1 \leqslant k \leqslant n$, and set

$$d_k(x) = \left(\sum_{i=1}^k (x_i)^2\right)^{1/2}.$$

Now $|\nabla d_k(x)| = 1$ everywhere in $\mathbb{R}^n \setminus K$ where $K = \{x \in \mathbb{R}^n : \, d_k(x) = 0\}$. We call the set

$$B_k(t) = \{x \in \mathbb{R}^n : \, d_k(x) < t\}$$

a k-ball and the set

$$\Sigma_k(t) = \{x \in \mathbb{R}^n : \, d_k(x) = t\}$$

a k-sphere in $\mathbb{R}^n$.

We say that an unbounded domain $D \subset \mathbb{R}^n$ is *k-admissible* if for each $t > \inf_{x \in D} d_k(x)$ the set $D \cap B_k(t)$ is precompact.

It is clear that every unbounded domain $D \subset \mathbb{R}^n$ is n-admissible. In the general case the domain D is k-admissible if and only if the function $d_k(x)$ is an exhaustion function of D. It is not difficult to see that if a domain $D \subset \mathbb{R}^n$ is k-admissible, then it is l-admissible for all $k < l < n$.

Let A satisfy (2) and (3) and let $h\colon \mathscr{X} \to (0, \infty)$ be an exhaustion function satisfying the following conditions:

a$_1$) there exists a compact set $K \subset \mathscr{X}$ such that h is a solution of (4) in $\mathcal{X} \setminus K$;

a$_2$) for a.e. $t_1, t_2 \in (0, \infty)$, $t_1 < t_2$,

$$\int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle \mathrm{d}H = \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle \mathrm{d}H.$$

Here $\mathrm{d}H$ is the element of the $(n-1)$-dimensional Hausdorff measure on Σ_h. Exhaustion functions with these properties will be called *the special exhaustion functions of $\mathscr{X}$ with respect to A.* In most cases the mapping A will be the n-Laplace operator

$$A(x, h) = |h|^{n-2} h$$

and then we may omit A from the above definition.

Since the unit vector $\nu = \nabla h / |\nabla h|$ is orthogonal to the h-sphere Σ_h, the condition a$_2$) means that the flux of the vector field $A(x, \nabla h)$ through h-spheres $\Sigma_h(t)$ is constant.

Suppose that the function $A(x, \xi)$ is continuously differentiable. If

b$_1$) $h \in C^2(\mathcal{X} \setminus K)$ and satisfies equation (4), and

b$_2$) at every point $x \in \mathscr{X}$ where $\partial \mathscr{X}$ has a tangent plane $T_x(\partial \mathscr{X})$ the condition

$$\langle A(x, \nabla h(x)), \nu \rangle = 0$$

is satisfied where ν is a unit vector of the inner normal to the boundary $\partial \mathscr{X}$, then h is a special exhaustion function of the manifold $\mathscr{X}$.

The proof of this statement is simple. Consider the domain

$$\mathscr{X}(t_1,t_2)=\{x\in\mathscr{X}:\ t_1<h(x)<t_2\},\quad 0<t_1<t_2<\infty,$$

with the boundary $\partial\mathcal{X}(t_1,t_2)$. Using the Gauss formula, we have

$$\begin{aligned}\int_{\Sigma_h(t_2)}\Big\langle\frac{\nabla h}{|\nabla h|},A(x,\nabla h)\Big\rangle\,\mathrm{d}H-\int_{\Sigma_h(t_1)}\Big\langle\frac{\nabla h}{|\nabla h|},A(x,\nabla h)\Big\rangle\,\mathrm{d}H\\ =\int_{\partial\mathcal{X}(t_1,t_2)\cup\bigcup\limits_{i=1,2}\Sigma_h(t_i)}\langle\nu,A(x,\nabla h)\rangle\,\mathrm{d}H\\ =\int_{\partial\mathscr{X}(t_1,t_2)}\langle\nu,A(x,\nabla h)\rangle\,\mathrm{d}H\\ =\int_{\mathscr{X}(t_1,t_2)}\operatorname{div}A(x,\nabla h)\,\mathrm{d}v_{\mathscr{X}}=0.\end{aligned}$$

This computation provides the validity of property a$_2$).

3.3. Example. Fix $1\leqslant k<n$. Let Δ be a bounded domain in the plane $x_1=\ldots=x_k=0$ with a piecewise smooth boundary and let

$$D=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n:\ (x_{k+1},\ldots,x_n)\in\Delta\}=\mathbb{R}^{n-k}\times\Delta \tag{9}$$

be the cylinder domain with the base Δ.

The domain D is k-admissible. The k-spheres $\Sigma_k(t)$ are orthogonal to the boundary ∂D and therefore $\langle\nabla d_k,\nu\rangle=0$ everywhere on the boundary, where d_k is as in Example 3.2.

Let $h=\varphi(d_k)$ where φ is a C^2-function. We have $\nabla h=\varphi'\nabla d_k$ and

$$\begin{aligned}\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(|\nabla h|^{n-2}\frac{\partial h}{\partial x_i}\Big)&=\sum_{i=1}^k\frac{\partial}{\partial x_i}\Big((\varphi')^{n-1}\frac{\partial d_k}{\partial x_i}\Big)\\ &=(n-1)(\varphi')^{n-2}\varphi''+\frac{k-1}{d_k}(\varphi')^{n-1}.\end{aligned}$$

From the equation

$$(n-1)\varphi''+\frac{k-1}{d_k}\varphi'=0$$

we conclude that the function

$$h(x)=(d_k(x))^{(n-k)/(n-1)} \tag{10}$$

satisfies the equation (8) in $D\setminus K$ and thus it is a special exhaustion function of the domain D.

3.4. Example. Let (r, θ), where $r \geqslant 0$, $\theta \in \mathbb{S}^{n-1}(1)$, be the spherical coordinates in $\mathbb{R}^n$. Let $U \subset \mathbb{S}^{n-1}(1)$, $\partial U \neq \emptyset$, be an arbitrary domain on the unit sphere $\mathbb{S}^{n-1}(1)$. We fix $0 \leqslant r_1 < \infty$ and consider the domain

$$D = \{(r, \theta) \in \mathbb{R}^n : \ r_1 < r < \infty, \ \theta \in U\}. \tag{11}$$

As above it is easy to verify that the given domain is n-admissible and the function

$$h(|x|) = \log \frac{|x|}{r_1} \tag{12}$$

is a special exhaustion function of the domain D.

3.5. Example. Fix $1 \leqslant n \leqslant p$. Let $x_1, x_2, \ldots, x_n$ be an orthonormal system of coordinates in $\mathbb{R}^n$, $1 \leqslant n < p$. Let $D \subset \mathbb{R}^n$ be an unbounded domain with piecewise smooth boundary and let $\mathscr{B}$ be a $(p-n)$-dimensional compact Riemannian manifold with or without boundary. We consider the manifold $\mathscr{M} = D \times \mathscr{B}$.

We denote by $x \in D$, $b \in \mathscr{B}$, and $(x, b) \in \mathscr{M}$ the points of the corresponding manifolds. Let $\pi \colon D \times \mathscr{B} \to D$ and $\eta \colon D \times \mathscr{B} \to \mathscr{B}$ be the natural projections of the manifold $\mathscr{M}$.

Assume now that the function h is a function on the domain D satisfying the conditions $b_1)$, $b_2)$ and the equation (8). We consider the function $h^* = h \circ \pi \colon \mathscr{M} \to (0, \infty)$.

We have

$$\nabla h^* = \nabla(h \circ \pi) = (\nabla_x h) \circ \pi$$

and

$$\begin{aligned} \operatorname{div}(|\nabla h^*|^{p-2} \nabla h^*) &= \operatorname{div}(|\nabla(h \circ \pi)|^{p-2} \nabla(h \circ \pi)) \\ &= \operatorname{div}(|\nabla_x h|^{p-2} \circ \pi (\nabla_x h) \circ \pi) \\ &= \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|\nabla_x h|^{p-2} \frac{\partial h}{\partial x_i} \right) \right) \circ \pi. \end{aligned}$$

Because h is a special exhaustion function of D we have

$$\operatorname{div}(|\nabla h^*|^{p-2} \nabla h^*) = 0.$$

Let $(x, b) \in \partial\mathscr{M}$ be an arbitrary point where the boundary $\partial\mathscr{M}$ has a tangent hyperplane and let ν be a unit normal vector to $\partial\mathscr{M}$.

If $x \in \partial D$, then $\nu = \nu_1 + \nu_2$ where the vector $\nu_1 \in \mathbb{R}^k$ is orthogonal to ∂D and ν_2 is a vector from $T_b(\mathscr{B})$. Thus

$$\langle \nabla h^*, \nu \rangle = \langle (\nabla_x h) \circ \pi, \nu_1 \rangle = 0,$$

because h is a special exhaustion function on D and satisfies the property b_2) on ∂D. If $b \in \partial \mathscr{B}$, then the vector ν is orthogonal to $\partial \mathscr{B} \times \mathbb{R}^n$ and

$$\langle \nabla h^*, \nu \rangle = \langle (\nabla_x h) \circ \pi, \nu \rangle = 0,$$

because the vector $(\nabla_x h) \circ \pi$ is parallel to $\mathbb{R}^n$.

The other requirements for a special exhaustion function for the manifold $\mathscr{M}$ are easy to verify.

Therefore, the function

$$h^* = h^*(x,b) = h \circ \pi \colon \mathscr{M} \to (0,\infty) \tag{13}$$

is a special exhaustion function on the manifold $\mathscr{M} = D \times \mathscr{B}$.

4. Estimates for the energy integral

Let $\mathscr{Y}$ be a noncompact Riemannian manifold of dimension n. We denote by $\mathrm{d}s_{\mathscr{Y}}$ the element of length on $\mathscr{Y}$. Let u be a locally Lipschitz function in $\mathscr{Y}$ such that $u \geqslant 1$ and $u \not\equiv 1$.

We assume that $u\big|_{\partial\mathscr{Y}} = 1$ if $\partial\mathscr{Y} \neq \emptyset$ and $\sup\limits_{y\in\mathscr{Y}} u(y) = \infty$, i.e. $u(y)$ is a growth function on $\mathscr{Y}$.

We consider the metric $\mathrm{d}s = \mathrm{d}s_u = |\nabla u(y)|\,\mathrm{d}s_{\mathscr{Y}}$. Here $\nabla u(y)$ is the gradient of u. If $\nabla u(y)$ is not defined at a point $y \in \mathscr{Y}$, then we set $|\nabla u(y)| = 1$. For an arbitrary domain $G \subset \mathscr{Y}$ we will denote by $\partial' G = \partial G \setminus \partial\mathscr{Y}$ the boundary of G with respect to $\mathscr{Y}$. Now

$$V_{u,\mathscr{Y}}(G) = \int_G |\nabla u(y)|^n \,\mathrm{d}v$$

denotes the volume in the metric $\mathrm{d}s$, and

$$A_{u,\mathscr{Y}}(\partial' G) = \int_{\partial' G} |\nabla u(y)|^{n-1} H(\mathrm{d}s_{\mathscr{Y}})$$

is the area of $\partial' G$ in the metric $\mathrm{d}s$. Here $H(\mathrm{d}s_{\mathscr{Y}})$ refers to the $(n-1)$-dimensional Hausdorff measure on $\partial' G$.

We consider isoperimetric profiles of the Riemannian manifold $\mathscr{Y}$ with the metric $\mathrm{d}s_u$. An isoperimetric profile of the pair $(\mathscr{Y}, \mathrm{d}s_u)$ is the function

$$\theta_{u,\mathscr{Y}}\colon [0,v) \to \mathbb{R}_+, \quad v = V_{u,\mathscr{Y}}(\mathscr{Y}),$$

defined by

$$\theta_{u,\mathscr{Y}}(\tau) = \inf\{A_{u,\mathscr{Y}}(\partial' G)\colon G \subset \mathscr{Y} \quad \text{a compact domain} \\ \text{with } H(\partial' G) < \infty, \quad V_{u,\mathscr{Y}}(G) = \tau\},$$

i.e. the isoperimetric profile $\theta_{u,\mathscr{Y}}$ is the best function among the functions θ satisfying

$$\theta(V_{u,\mathscr{Y}}) \leqslant A_{u,\mathscr{Y}}(\partial' G). \tag{14}$$

In the special case of surfaces in $\mathbb{R}^n$ this definition goes back to Ahlfors [1, p. 188]; for applications of the isoperimetric method to quasiconformal mappings on manifolds see [7], [15].

In general, the isoperimetric profile $\theta_{u,\mathscr{Y}}(\tau)$ is difficult to compute. It is also difficult to estimate the isoperimetric profile in terms of the curvature and other geometric data. We describe some of these cases below.

4.1. Example. Let $\mathscr{Y} = \mathbb{R}^n = \{y = (y_1, \ldots, y_n)\}$ be the Euclidean space. We choose the growth function $u(y) = |y| + 1$; now $|\nabla u(y)| = 1$ here. The classical isoperimetric inequality says that if $G \subset \mathbb{R}^n$ is a compact domain with smooth boundary ∂G, then

$$c_n (V_{u,\mathscr{Y}}(G))^{(n-1)/n} \leqslant A_{u,\mathscr{Y}}(\partial G)$$

where

$$c_n = \omega_{n-1}^{1/n} n^{(n-1)/n}$$

and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere $\mathbb{S}^{n-1}(0,1)$.

Hence, we have

$$\theta_{\mathbb{R}^n}(\tau) = c_n \tau^{(n-1)/n}. \tag{15}$$

4.2. Example. Let $\mathscr{Y}$ be a complete, simply connected, n-dimensional Riemannian manifold with nonpositive sectional curvature. We consider the growth function $u(y) = \operatorname{dist}(y,a) + 1$ where $a \in \mathscr{Y}$ is a fixed point. We have $|\nabla u(y)| = 1$ for $y \neq a$ and therefore,

$$V_{u,\mathscr{Y}}(G) = \operatorname{vol}(G) \quad \text{and} \quad A_{u,\mathscr{Y}}(\partial G) = \operatorname{area}(\partial G).$$

By [9], [4] for every n there is a constant $\overline{c}_n < c_n$ such that

$$\overline{c}_n(\mathrm{vol}(G))^{(n-1)/n} \leqslant \mathrm{area}(\partial G)$$

and it follows that

$$\theta_{u,\mathscr{Y}}(\tau) > \overline{c}_n \tau^{(n-1)/n}. \tag{16}$$

4.3. Example. Let $\mathscr{Y}$ be a complete, simply connected Riemannian manifold, $\dim \mathscr{Y} = n$. Let $u(y) = \mathrm{dist}(y, a) + 1$, $a \in \mathscr{Y}$, be a growth function on $\mathscr{Y}$. If the sectional curvature $K_{\mathscr{Y}}$ of $\mathscr{Y}$ satisfies $K_{\mathscr{Y}} \leqslant k < 0$, $k = \mathrm{const}$, then

$$(n-1)\sqrt{(-k)}\,\mathrm{vol}(G) \leqslant \mathrm{area}(\partial G)$$

([19, p. 504]; [3, 34.2.6]) and thus

$$\theta_{u,\mathscr{Y}}(\tau) \geqslant (n-1)\sqrt{(-k)}\tau. \tag{17}$$

The case $\partial\mathscr{Y} \neq \emptyset$ is more complicated. The following proposition is sometimes helpful in this problem.

Let $u = u(y)$ be a growth function in $\mathscr{Y}$ and suppose that u is a locally Lipschitz subsolution of (4) in $\mathscr{Y}$. We assume that A satisfies (2) and (3) with the structure constants ν_1, ν_2.

4.4. Proposition. *Let $b\colon \mathscr{M} \to \mathscr{Y}$ be a bilipschitz mapping of the manifold $\mathscr{M}$ onto the manifold $\mathscr{Y}$. If the domain of growth $\mathscr{Y}$ satisfies the isoperimetric inequality* (14) *with the function θ, then the function $u^* = u \circ b$ is also a growth function in $\mathscr{M}$ with the isoperimetric profile*

$$\theta_{u^*,\mathscr{M}}(t) = \frac{1}{k_b}\theta_{u,\mathscr{Y}}(t). \tag{18}$$

Moreover, u^* is a subsolution of an equation of the type (4), *with the structure constants*

$$\nu_1' = \nu_1/k_b, \quad \nu_2' = \nu_2. \tag{19}$$

Here k_b is the maximal dilatation of the mapping b.

Proof. We observe first that by [8, Theorem 14.42] the function u^* is a subsolution of some equation of the type (4) with structure constants (19).

Let $G \subset \mathscr{M}$ be an arbitrary precompact domain and let $G' = b(G)$ be its image. By the definition we have

$$V_{u^*,\mathscr{M}}(G) = \int_G |\nabla u^*(m)|^n \, \mathrm{d}v.$$

At almost every point $m \in \mathscr{M}$ of the manifold $\mathscr{M}$ we have ([8, Theorem 14.28])

$$\nabla_y u(y) = b'(m)^* \nabla_m u^*(m).$$

Thus

$$V_{u^*,\mathscr{M}}(G) \leqslant k_b \int_{G'} |\nabla u(y)|^n \, \mathrm{d}v = k_b V_{u,\mathscr{Y}}(G').$$

Similarly,

$$\begin{aligned} A_{u,\mathscr{Y}}(\partial' G') &= \int_{\partial' G'} |\nabla u(y)|^{n-1} H(\mathrm{d}s_{\mathscr{Y}}) \\ &\leqslant \int_{\partial' G} |\nabla_y u(b(m))|^{n-1} |b'(m)|^{n-1} H(\mathrm{d}s_{\mathscr{M}}) \\ &\leqslant \int_{\partial' G} |\nabla u^*(m)|^{n-1} H(\mathrm{d}s_{\mathscr{M}}) \\ &= A_{u^*,\mathscr{M}}(\partial' G). \end{aligned}$$

Therefore

$$\theta(V_{u*,\mathscr{M}}(G)) \leqslant \theta(k_b V_{u,\mathscr{Y}}(G')) \leqslant k_b A_{u,\mathscr{Y}}(\partial' G') \leqslant k_b A_{u^*,\mathscr{M}}(\partial' G)$$

so that the relation (18) indeed holds. □

4.5. Example. We assume that the domain of growth $\mathscr{Y} \subset \mathbb{R}^n$ is the half-space $y_1 \geqslant 0$ and $u(y) = y_1 + 1$. In this case inequality (14) is a simple corollary of the classical isoperimetric inequality in $\mathbb{R}^n$, connecting the volume of a domain and the area of its boundary.

Here, as is easy to see, we have

$$\theta(t) = L t^{(n-1)/n}, \tag{20}$$

where $L = (\omega_{n-1}/2)^{1/n} n^{(n-1)/n}$ and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere $\mathbb{S}^{n-1}(0,1) \subset \mathbb{R}^n$.

Using the results of Section 4.4, we conclude: Every manifold, which is bilipschitz equivalent to a half-space in $\mathbb{R}^n$, has a growth function with the property of Proposition 4.4, satisfying the isoperimetric inequality (14) with the function $\theta(t) = \frac{L}{k_b} t^{(n-1)/n}$.

Let $\mathscr{X}$ be an n-dimensional Riemannian manifold with a boundary $\partial\mathscr{X}$ (possibly empty). We fix a locally Lipschitz exhaustion function $h\colon \mathscr{X} \to (0,\infty)$. Let $\overline{h} = \inf\limits_{x\in\mathscr{X}} h(x)$.

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a quasiregular mapping and let $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$. We assume that the manifold $\mathscr{Y}$ satisfies the isoperimetry condition (14) with the function θ.

We first observe that for almost all $t \in (\overline{h},\infty)$ the restriction of the mapping f to the h-sphere $\Sigma_h(t)$ is of the class $W^1_{n,\mathrm{loc}}$. We fix arbitrarily such a value $t \in (\overline{h},\infty)$ and denote by $B'(t)$ the image of the h-ball $B_h(t)$ under the mapping $y = f(x)$.

Because the mapping $f\colon \mathscr{X} \to \mathscr{Y}$ is quasiregular, it is open and discrete. For an arbitrary $y \in \overline{B}'(t)$ we denote by $N(y,t)$ the number of points $x \in \overline{B}_h(t)$ for which $f(x) = y$.

Let $\Sigma'(t) = f(\Sigma_h(t))$.

Using the θ-isoperimetry property of the manifold $\mathscr{Y}$ we have

$$\theta\left(\int_{B'(t)} |\nabla u(y)|^n \,\mathrm{d}v\right) \leqslant \int_{\partial' B'(t)} |\nabla u(y)|^{n-1} H(\mathrm{d}s_{\mathscr{Y}}).$$

The restriction of the mapping f to $\Sigma_h(t)$ has Lusin's property (N), and because $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$ we have $\partial' B'(t) \subset \Sigma'(t)$. Performing a change of variables we have

$$\begin{aligned}&\theta\left(\int_{B(t)} |\nabla_y u(f(x))|^n \mathscr{J}_f(x) N(f(x),t)^{-1} \,\mathrm{d}v\right)\\ &\leqslant \int_{\Sigma_h(t)} |\nabla_y u(f(x))|^{n-1} N(f(x),t)^{-1} H(\mathrm{d}s_{\mathscr{X}}).\end{aligned}$$

This inequality, condition (1) and Hölder's inequality yield

$$\begin{aligned}&\theta\left(K^{-1}\int_{B_h(t)} |\nabla_y u(f(x))|^n |f'(x)|^n N(f(x),t)^{-1} \,\mathrm{d}v\right)\\ &\leqslant \left(\int_{\Sigma_h(t)} N(f(x),t)^{-1} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{Y}})\right)^{1/n}\\ &\quad\times\left(\int_{\Sigma_h(t)} |\nabla_y u(f(x))|^n |f'(x)|^n N(f(x),t)^{-1} \frac{H(\mathrm{d}s_{\mathscr{X}})}{|\nabla h|}\right)^{(n-1)/n}.\end{aligned}$$

We set

$$J(t) = \int_{B_h(t)} |\nabla_y u(f(x))|^n |f'(x)|^n N(f(x),t)^{-1} \,\mathrm{d}v.$$

Using the Kronrod-Federer formula

$$J(t)=\int_{\overline{h}}^{t}\mathrm{d}\tau\int_{\Sigma_h(\tau)}|\nabla_y u(f(x))|^n|f'(x)|^n N(f(x),t)^{-1}\frac{H(\mathrm{d}s_{\mathscr{Y}})}{|\nabla h|}$$

we observe that for almost every $t\in(\overline{h},\infty)$

$$J'(t)=\int_{\Sigma_h(t)}|\nabla_y u(f(x))|^n|f'(x)|^n N(f(x),t)^{-1}\frac{H(\mathrm{d}s_{\mathscr{X}})}{|\nabla h|}.$$

Therefore

$$\theta^{n/(n-1)}\left(\frac{J(t)}{K}\right)\leqslant J'(t)\left(\int_{\Sigma_h(t)}|\nabla h|^{n-1}\frac{H(\mathrm{d}s_{\mathscr{X}})}{N(f(x),t)}\right)^{1/(n-1)}. \tag{21}$$

For an arbitrary $t>\overline{h}$ we set $N_f(t)=\inf\limits_{x\in B_h(t)}N(f(x),t)$. The inequality (21) attains the form

$$N_f^{1/(n-1)}(t)\theta^{n/(n-1)}\left(\frac{J(t)}{K}\right)\leqslant J'(t)\left(\int_{\Sigma_h(t)}|\nabla h|^{n-1}H(\mathrm{d}s_{\mathscr{X}})\right)^{1/(n-1)}. \tag{22}$$

The following statement characterizes the class of isoperimetric functions θ for which the mapping $f\colon\ \mathscr{X}\to\mathscr{Y}$ is trivial.

4.6. Theorem. *Let h be a special exhaustion function on $\mathscr{X}$ and assume that the manifold $\mathscr{X}$ satisfies the condition*

$$\int^{\infty}\mathrm{d}t\left(\int_{\Sigma_h(t)}|\nabla h|^{n-1}H(\mathrm{d}s_{\mathscr{X}})\right)^{1/(1-n)}=\infty. \tag{23}$$

If the manifold $\mathscr{Y}$ is θ-isoperimetric with the function $\theta(t)$ satisfying

$$\int^{\infty}\theta(t)^{n/(1-n)}\,\mathrm{d}t<\infty, \tag{24}$$

then each quasiregular mapping $f\colon\ \mathscr{X}\to\mathscr{Y}$, $f(\partial\mathscr{X})\subset\partial\mathscr{Y}$, is a constant.

Proof. We will use (22). Observing that $N_f(t)\geqslant 1$ and integrating the aforementioned differential inequality, for each $\tau>\overline{h}+1$ we get

$$\int_{\overline{h}+1}^{\tau}\mathrm{d}t\left(\int_{\Sigma_h(t)}|\nabla h|^{n-1}H(\mathrm{d}s_{\mathscr{X}})\right)^{1/(1-n)}\leqslant\int_{CJ(\overline{h}+1)}^{CJ(\tau)}\theta(t)^{n/(1-n)}\,\mathrm{d}t, \tag{25}$$

where $C=1/K$.

If $J(\tau)\not\equiv 0$, then the conditions (23) and (24) lead to a contradiction. Therefore, $J(\tau)\equiv 0$ and thus $f(x)\equiv\mathrm{const}$. □

This theorem is a version of Liouville's theorem for quasiregular mappings $f\colon \mathscr{X} \to \mathscr{Y}$ of Riemannian manifolds. A natural choice for the growth function u is the following.

Let $\mathscr{Y}$ be a Riemannian manifold with a non-empty boundary $\partial\mathscr{Y}$. We set $u(y) = \varrho(y, \partial\mathscr{Y}) + 1$, where $\varrho(y, \partial\mathscr{Y})$ is the distance from a point y to the boundary $\partial\mathscr{Y}$. Then $u(y) \geqslant 1$ and u is a locally Lipschitz function on $\mathscr{Y}$, and also $|\nabla u(y)| = 1$ almost everywhere on $\mathscr{Y}$.

If the boundary $\partial\mathscr{Y} = \emptyset$, then one may fix an arbitrary point $y_0 \in \mathscr{Y}$ and set $u(y) = \varrho(y, y_0) + 1$.

It is clear that the function $u(y)$ thus constructed has the properties of a growth function for the manifold $\mathscr{Y}$. In addition, for every subdomain $G \subset \mathscr{Y}$ with boundary $\partial' G = \partial G \setminus \partial\mathscr{Y}$ with respect to $\mathscr{Y}$ we have: $V_{u,\mathscr{Y}}(G)$ is the volume of G in the standard metric of $\mathbb{R}^n$ and $A_{u,\mathscr{Y}}(\partial' G)$ is the $(n-1)$-dimensional area.

The isoperimetric inequality (14) now takes the form

$$\theta\left(\int_G \mathrm{d}v\right) \leqslant \int_{\partial' G} H(\mathrm{d}s_{\mathscr{Y}}). \tag{26}$$

Thus we get

4.7. Corollary. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a quasiregular mapping such that $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$ if $\partial\mathscr{X} \neq \emptyset$. Let $h\colon \mathscr{X} \to (0,\infty)$ be an exhaustion function $\mathscr{X}$ satisfying condition* (23). *If the manifold $\mathscr{Y}$ satisfies* (26), *where the function θ has property* (24), *then $f \equiv \mathrm{const}$.*

From (17), (23) and (24) we have

4.8. Corollary. *Let $\mathscr{Y}$ be a complete, simply connected, n-dimensional Riemannian manifold with sectional curvature $K_{\mathscr{Y}} \leqslant k < 0$, $k = \mathrm{const}$. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a quasiregular mapping. If the manifold $\mathscr{X}$ satisfies* (23), *then $f \equiv \mathrm{const}$.*

We consider the case in which $h\colon \mathscr{X} \to (0,\infty)$ is a special exhaustion function on $\mathscr{X}$. Then the integral

$$I = \int_{\Sigma_h(t)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m, \nabla h) \right\rangle H(\mathrm{d}s_{\mathscr{X}})$$

is independent of t.

Using structural conditions (2), (3), we note that

$$\left\langle \frac{\nabla h}{|\nabla h|}, A(m, \nabla h) \right\rangle \geqslant \nu_1 |\nabla h|^{n-1}$$

and

$$\left\langle \frac{\nabla h}{|\nabla h|}, A(m, \nabla h) \right\rangle \leqslant |A(m, \nabla h)| \leqslant \nu_2 |\nabla h|^{n-1}.$$

Hence for every $t \in (0, \infty)$ we have that

$$\frac{1}{\nu_2} I \leqslant \int_{\Sigma_h(t)} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{X}}) \leqslant \frac{1}{\nu_1} I. \tag{27}$$

Condition (23) on the manifold $\mathscr{X}$ is fulfilled automatically. Hence we get

4.9. Corollary. *Suppose that the manifold $\mathscr{X}$ has a special exhaustion function $h\colon \mathscr{X} \to (0, \infty)$, and the manifold $\mathscr{Y}$ is θ-isoperimetric with a function $\theta(t)$ satisfying the condition* (24). *Then every quasiregular mapping $f\colon \mathscr{X} \to \mathscr{Y}$, $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$, is a constant.*

Relations (21), (25) are sources for Liouville theorems of various types. These theorems give an estimate for the minimal admissible speed of growth for the energy integral of a non-trivial quasiregular mapping $f\colon \mathscr{X} \to \mathscr{Y}$. Consider the following example.

Let $\mathscr{Y}$ be a manifold bilipschitz equivalent to a half-space in $\mathbb{R}^n$. As was shown in example (4.5), here the isoperimetric function has the form $\theta(t) = (L/k_b)t^{(n-1)/n}$, L is a constant from (20) and k_b is the maximal dilatation of the bilipschitz mapping. Then the integral on the right hand side of inequality (25) is computed and this inequality takes the form

$$J(\overline{h}+1) \leqslant J(\tau) \exp\left\{ -\left(\frac{L}{k_b}\right)^{n/(n-1)} \int_{\overline{h}+1}^{\tau} \mathrm{d}t \left(\int_{\Sigma_h(t)} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{X}}) \right)^{-1/(n-1)} \right\},$$

where

$$J(t) = \int_{B_h(t)} |f'(x)|^n \frac{\mathrm{d}v}{N(f(x), t)}$$

is a special case of the integral from Theorem 4.6.

If the exhaustion function h of the manifold $\mathscr{X}$ is a special exhaustion function then by virtue of (27) for every $\tau'' > \tau' \geqslant \overline{h} + 1$ we have

$$(28)\qquad (\tau'' - \tau')\left(\frac{I}{\nu_1}\right)^{1/(1-n)} \leqslant \int_{\tau'}^{\tau'} \mathrm{d}t \left(\int_{\Sigma_h(t)} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{X}})\right)^{1/(1-n)} \leqslant (\tau'' - \tau')\left(\frac{I}{\nu_2}\right)^{1/(1-n)}.$$

Here I is the flux of the vector field $A(x, \nabla h)$ through h-spheres $\Sigma_h(t)$.

Under these assumptions we have from (16)

4.10. Corollary. *Let $\mathscr{Y}$ be a complete, simply connected, n-dimensional Riemannian manifold with sectional curvature $K_{\mathscr{Y}} \leqslant 0$. If the manifold $\mathscr{X}$ has a special exhaustion function, then every quasiregular mapping $f\colon\ \mathscr{X} \to \mathscr{Y}$ with*

$$\lim_{\tau\to 0} J(\tau) \exp\left\{ -\overline{c}_n \left(\frac{I}{\nu_1}\right)^{-1/(n-1)} \tau \right\} = 0$$

is a constant.

From (18), (19), (20) we get

4.11. Corollary. *If a manifold $\mathscr{X}$ has a special exhaustion function h, and a manifold $\mathscr{Y}$ is bilipschitz equivalent to a half-space, then every quasiregular mapping $f\colon\ \mathscr{X} \to \mathscr{Y}$, $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$, with the property*

$$\liminf_{\tau\to\infty} J(\tau) \exp\left\{ -\left(\frac{L}{k_b}\right)^n \left(\frac{I}{\nu_1}\right)^{-1/(n-1)} \tau \right\} = 0$$

is a constant. Here k_b is the maximal dilatation of the bilipschitz mapping b.

5. Phragmén-Lindelöf theorem

Let $\mathscr{X}$, $\mathscr{Y}$ be noncompact Riemannian manifolds, $\dim\mathscr{X} = \dim\mathscr{Y} = n \geqslant 2$. Let $f\colon\ \mathscr{X} \to \mathscr{Y}$ be a quasiregular mapping with $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$ if the boundary $\partial\mathscr{X} \neq \emptyset$. Let $h\colon\ \mathscr{X} \to (0,\infty)$ be an exhaustion function on $\mathscr{X}$ and let $u(y) \geqslant 1$ be a growth function defined on the manifold $\mathscr{Y}$. Suppose that $u(y)$ satisfies the condition (7).

The function $u^*(x) = u(f(x))$ is a subsolution of some inequality of the form (2), (3), (6) with structure constants ${\nu'}_1 = \nu_1/K$, ${\nu'}_2 = \nu_2 K$.

Fix $\tau'' > \tau' > \overline{h} + 1$. We choose an arbitrary locally Lipschitz function

$$\varphi\colon\ (0,\infty) \to (0,1),\ \ \varphi(\tau) = 1\ \ for\ \ \tau \leqslant \tau',\ \ \varphi(\tau) = 0, \quad \text{for } \tau \geqslant \tau''.$$

The function $u^*(x) - 1$ is a solution of the differential inequality (6). Since $u^*(x) - 1 \geqslant 0$ and $(u^*(x) - 1)|_{\partial\mathscr{X}} = 0$, choosing $\theta(x) = (u^*(x) - 1)\varphi^n(h(x))$, as a test function in (7) we have

$$\begin{aligned}
&\int_{\mathscr{X}} \varphi^n(h)\langle \nabla u^*, A(x, \nabla u^*)\rangle \, \mathrm{d}v_{\mathscr{X}} \\
&\leqslant\ -n\int_{\mathscr{X}} (u^* - 1)\varphi^{n-1}(h)\varphi'(h)\langle \nabla h, A(x, \nabla u^*)\rangle \, \mathrm{d}v_{\mathscr{X}} \\
&\leqslant n\int_{\mathscr{X}} |u^* - 1|\varphi^{n-1}(h)|\varphi'(h)|\,|\nabla h|\,|A(x, \nabla u^*)| \, \mathrm{d}v_{\mathscr{X}} \\
&\leqslant n\Big(\int_{\mathscr{X}} \varphi^n(h)|A(x, \nabla u^*)|^{\frac{n}{n-1}}\Big)^{(n-1)/n} \mathrm{d}v_{\mathscr{X}} \Big(\int_{\mathscr{X}} |u^* - 1|^n|\varphi'(h)|^n|\nabla h|^n\Big)^{1/n} \mathrm{d}v_{\mathscr{X}}.
\end{aligned}$$

Using conditions (2), (3) with the aforementioned structure constants ν_1', ν_2', we obtain

$$c_1^n \int_{\mathscr{X}} \varphi^n(h)|\nabla u^*|^n \, \mathrm{d}v_{\mathscr{X}} \leqslant \int_{\mathscr{X}} |u^* - 1|^n|\varphi'(h)|^n|\nabla h|^n \, \mathrm{d}v_{\mathscr{X}},$$

where

$$c_1 = \frac{\nu_1'}{n\nu_2'} = \frac{\nu_1}{\nu_2}(nK^2)^{-1}.$$

The particular choice of the function φ yields

$$c_1^n \int_{B_h(\tau')} |\nabla u^*|^n \, \mathrm{d}v_{\mathscr{X}} \leqslant \int_{B_h(\tau'')\setminus B_h(\tau')} |u^* - 1|^n|\varphi'(h)|^n|\nabla h|^n \, \mathrm{d}v_{\mathscr{X}}.$$

Using the maximum principle we obtain

$$c_1^n \int_{B_h(\tau')} |\nabla f|^n \, \mathrm{d}v_{\mathscr{X}} \leqslant M^n(\tau'') \int_{B_h(\tau'')\setminus B_h(\tau')} |\varphi'(h(x))|^n|\nabla h|^n \, \mathrm{d}v_{\mathscr{X}}, \tag{29}$$

where

$$M(\tau) = \max_{\Sigma_h(\tau)} |u^*(x) - 1|.$$

We must find the minimum of the integral

$$I(\varphi) = \int_{B_h(\tau'')\setminus B_h(\tau')} |\varphi'(h(x))|^n|\nabla h|^n \, \mathrm{d}v_{\mathscr{X}}$$

in the class of admissible functions φ.

Integrating over the level sets of the function h, we get

$$I(\varphi) = \int_{\tau'}^{\tau''} |\varphi'(t)|^n \,\mathrm{d}t \int_{\Sigma_h(t)} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{X}}).$$

Let

$$\alpha(t) = \int_{\Sigma_h(t)} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{X}}).$$

Because $\varphi(\tau') = 1$, $\varphi(\tau'') = 0$, we get

$$1 \leqslant \int_{\tau'}^{\tau''} |\varphi'(t)| \,\mathrm{d}t \leqslant \left(\int_{\tau'}^{\tau''} \alpha(t)|\varphi'(t)|^n \,\mathrm{d}t\right)^{1/n} \left(\int_{\tau'}^{\tau''} \alpha(t)^{1/(1-n)} \,\mathrm{d}t\right)^{(n-1)/n}.$$

Thus

$$I(\varphi) \geqslant \left(\int_{\tau'}^{\tau''} \alpha(t)^{1/(1-n)} \,\mathrm{d}t\right)^{1-n}.$$

This inequality reduces to equality for the following special choice of the function φ:

$$\varphi(t) = \begin{cases} 1, & \text{for } t \leqslant \tau' \\ \beta(t), & \text{for } \tau' < t < \tau'' \\ 0, & \text{for } t \geqslant \tau'' \end{cases}$$

where

$$\beta(t) = \frac{\int_t^{\tau''} \alpha(t)^{1/(1-n)} \,\mathrm{d}t}{\int_{\tau'}^{\tau''} \alpha(t)^{1/(1-n)} \,\mathrm{d}t}.$$

Hence

$$\min_{\varphi} I(\varphi) = \left(\int_{\tau'}^{\tau''} \alpha(t)^{1/(1-n)} \,\mathrm{d}t\right)^{1-n}$$

and we get

$$c_1^n \int_{B_h(\tau')} |\nabla u^*|^n \leqslant M^n(\tau'')(\lambda(\tau'') - \lambda(\tau'))^{1-n} \tag{30}$$

where

$$\lambda(t) = \int_{\overline{h}+1}^{t} \mathrm{d}\tau \left(\int_{\Sigma_h(\tau)} |\nabla h|^{n-1} H(\mathrm{d}s_{\mathscr{X}})\right)^{1/(1-n)}.$$

We now assume that the minimal multiplicity of the mapping $f\colon \mathscr{X} \to \mathscr{Y}$ satisfies $N_f(t) \geqslant n_f \geqslant 1$ for all $t > \overline{h}$ where n_f is a constant. Integrating (22) we get for $\tau'' > \tau' > \overline{h} + 1$

$$\lambda(\tau'') - \lambda(\tau') \leqslant \frac{K}{n_f^{1/(n-1)}} \left(\Phi\left(\frac{1}{K} J(\tau'')\right) - \Phi\left(\frac{1}{K} J(\tau')\right)\right),$$

where

$$\Phi(t) = \int_1^t \theta(\tau)^{n/(1-n)}\,\mathrm{d}\tau.$$

Next we note that it follows from (30) that ($y = f(x)$)

$$\begin{aligned} J(t) &= \int_{B_h(t)} |\nabla_y u(f(x))|^n |f'(x)|^n N(f(x),t)^{-1}\,\mathrm{d}v_{\mathscr{X}} \\ &\leqslant \frac{1}{n_f}\int_{B_h(t)} |\nabla_y u(f(x))|^n |f'(x)|^n\,\mathrm{d}v_{\mathscr{X}} \\ &\leqslant \frac{1}{n_f}\int_{B_h(t)} |\nabla u^*|^n\,\mathrm{d}v_{\mathscr{X}} \\ &\leqslant \frac{1}{c_1^n n_f} M^n(\tau'')(\lambda(\tau'')-\lambda(\tau'))^{1-n}. \end{aligned}$$

Under the assumption that both integrals (23) and (24) diverge, we arrive at the main statement of this section.

5.1. Theorem. *Let $f\colon \mathscr{X}\to\mathscr{Y}$ be a quasiregular mapping, $f(\partial\mathscr{X})\subset\partial\mathscr{Y}$, and let the multiplicity $N_f(t)\geqslant n_f$ for all $t>\overline{h}$. Then either $M(t)$ grows so quickly that*

$$\frac{n_f^{1/(n-1)}}{K(f)} \leqslant \liminf_{t',t''\to\infty}\frac{1}{\lambda(t')}\Phi\left(\frac{c_2 M^n(t'')}{(\lambda(t'')-\lambda(t'))^{n-1}}\right) \tag{31}$$

or $f(x)\equiv\mathrm{const}$. Here

$$c_2 = c_1^{-n} n_f^{-1} K(f)^{-1} = n^n K^{2n-1}(f) n_f^{-1}\left(\frac{\nu_2}{\nu_1}\right)^n.$$

In the case when the exhaustion function $h\colon \mathscr{X}\to(0,\infty)$ is a special exhaustion function, the quantity $\lambda(t)$ has an estimate (27) in terms of the flux I of the vector field $A(m,\nabla h)$ through h-spheres. Here using (28), we can write

$$\left(\frac{I}{\nu_1}\right)^{1/(1-n)}(t-\overline{h}-1)\leqslant\lambda(t),\quad \forall t''>t'\geqslant\overline{h}+1,$$

and for all $t''>t'$ we have

$$\left(\frac{I}{\nu_1}\right)^{1/(1-n)}(t''-t')\leqslant\lambda(t'')-\lambda(t').$$

So the relation (31) may be essentially simplified.

Namely, setting $t''=t'+1$, we obtain

5.2. Corollary. *If in the conditions of Theorem* 5.1 *the exhaustion function* $h\colon \mathscr{X} \to (0,\infty)$ *is special, then either* $f(x) \equiv \mathrm{const}$, *or*

$$\left(\frac{In_f}{\nu_1}\right)^{1/(n-1)} K(f)^{-1} \leqslant \liminf_{t\to\infty} \frac{1}{t}\Phi(c_3 M^n(t)), \tag{32}$$

where $c_3 = c_2\nu_1^{-1} I$.

Suppose that the manifold $\mathscr{Y}$ satisfies the assumptions of Theorem 5.1. Fix integers $1 \leqslant k \leqslant n \leqslant p$ and consider a domain $D \subset \mathbb{R}^n$ of the form (9) for $k < n$, or of the form (11) for $k = n$. Let $\mathscr{B}$ be a $(p-n)$-dimensional compact Riemannian manifold with or without boundary. The function h^*, defined by relation (13), where h is given respectively by the equalities (10) or (12), is a special exhaustion function of the manifold $\mathscr{X} = D \times \mathscr{B}$.

Under these assumptions, using (32), we have

5.3. Corollary. *Let* $f(x,b)\colon \mathscr{X} \to \mathscr{Y}$ *be a quasiregular mapping,* $f(\partial\mathscr{X}) \subset \partial\mathscr{Y}$, *and let the multiplicity* $N_f(t) \geqslant n_f$ *for all* $t > \overline{h}$. *Then either* $f(x) \equiv \mathrm{const}$, *or* $M(t)$ *grows so quickly that for* $k < n$ *and* $M_k(t) = \max\limits_{d_k(x)=t} u^*(x,b)$ *we have*

$$\left(\frac{In_f}{\nu_1}\right)^{1/(n-1)} K(f)^{-1}\frac{n-1}{n-k} \leqslant \liminf_{t\to\infty} t^{(k-n)/(n-1)}\Phi(c_3 M_k^n(t)); \tag{33}$$

for $k = n$ *and* $M_n(t) = \max\limits_{|x|=t} u^*(x,b)$ *we have*

$$\left(\frac{In_f}{\nu_1}\right)^{1/(n-1)} K(f)^{-1} \leqslant \liminf_{t\to\infty} \frac{1}{\log t}\Phi(c_3 M_n^n(t)). \tag{34}$$

To prove the relation (33) it is sufficient to note that if a point $(x,b) \in \Sigma_h(t)$, then

$$d_k(x) = \frac{n-k}{n-1} t^{(n-1)/(n-k)}.$$

So setting $\frac{n-k}{n-1} t^{(n-1)/(n-k)} = \tau$, we have

$$\liminf_{t\to\infty} \frac{1}{t}\Phi(c_3 M^n(t+1)) = \liminf_{\tau\to\infty} \frac{n-k}{n-1}\tau^{(k-n)/(n-1)}\Phi(c_3 M_k^n(\tau)).$$

Sufficiency follows from (32).

In the case (34), denoting $\log\frac{\tau}{r_1} = t$, we find

$$\liminf_{t\to\infty} \frac{1}{t}\Phi(c_3 M^n(t+1)) = \liminf_{\tau\to\infty} \frac{1}{\log\tau}\Phi(c_3 M_n^n(\tau)).$$

We note a special case of this theorem when $\mathscr{X}$ is an unbounded domain in $\mathbb{R}^n$ and the growth domain $\mathscr{Y} \subset \mathbb{R}^n$ is the half-space $y_1 \geqslant 1$, $u(y) = y_1$. Using the notation of Example 4.5 we now have

$$\theta(t) = \left(\frac{\omega_{n-1}}{2}\right)^{1/n} n^{(n-1)/n} t^{(n-1)/n}, \quad \Phi(t) = \frac{1}{n}\left(\frac{2}{\omega_{n-1}}\right)^{1/(n-1)} \log t.$$

Let

$$h(x) = \left(\sum_{i=1}^{p} x_i^2\right)^{1/2}, \quad 1 \leqslant p \leqslant n, \quad M(t) = \max f_1(x), \quad x \in \Sigma_h(t).$$

Note that $|\nabla h(x)| \equiv 1$. Then

$$\lambda(t) = \int_{\overline{h}+1}^{t} |\Sigma_h(\tau)|^{1/(1-n)} \,\mathrm{d}\tau, \quad |\Sigma_h(\tau)| = \operatorname*{mes}_{n-1} \Sigma_h(\tau).$$

In the case when $\mathscr{X}$ is a cone in $\mathbb{R}^n$ with its vertex at $x = 0$, choosing $p = n$, $\tau'' = 2\tau'$, we arrive at

5.4. Corollary. *If $f = (f_1, \dots, f_n)\colon \mathscr{X} \to \mathbb{R}^n$ is a quasiregular mapping with a multiplicity $N_f(t) \geqslant n_f$ for all $t > \overline{h}$ and $f_1(x)\big|_{\partial\mathscr{X}} \leqslant 1$ then either $f_1(x) \leqslant 1$ everywhere in $\mathscr{X}$ or*

$$\liminf_{t\to\infty} \frac{\log M(t)}{\log t} \geqslant \left(\frac{\omega_{n-1}}{2}\right)^{1/(n-1)} \frac{n_f^{1/(n-1)}}{K(f)|\Sigma_h(1)|^{1/(n-1)}} \tag{35}$$

If $\mathscr{X}$ is a half-cylinder $\Delta \times \mathbb{R}^1_+$ in $\mathbb{R}^n$ where Δ is a bounded domain in the hyperplane $x_1 = 0$, setting $p = 1$, $\tau'' = \tau' + 1$ we get

5.5. Corollary. *If $f = (f_1, \dots, f_n)\colon \mathscr{X} \to \mathbb{R}^n$ is a quasiregular mapping with a multiplicity $N_f(t) \geqslant n_f$ for all $t > \overline{h}$ and $f_1(x)\big|_{\partial\mathscr{X}} \leqslant 1$, then either $f_1(x) \leqslant 1$ in $\mathscr{X}$ or*

$$\liminf_{t\to\infty} \frac{\log M(t)}{t} \geqslant \left(\frac{\omega_{n-1}}{2}\right)^{1/(n-1)} \frac{n_f^{1/(n-1)}}{K(f)|\Delta|^{1/(n-1)}}. \tag{36}$$

5.6. Remark. In the case of holomorphic functions, $K_O = 1$ and the relations (35), (36) are sharp. Observe that the minimal multiplicity contributes to the growth of the quantity $M(t)$. The least growth of $M(t)$ is attained for univalent functions $f\colon \mathscr{X} \to \mathbb{R}^2$ mapping the domain $\mathscr{X}$ conformally onto the half-plane $y_1 > 1$.

It is a very interesting question to study equality in (35) and (36) for quasiregular maps $f\colon \mathscr{X} \to \mathbb{R}^n$, $n > 2$. Does there exist a general principle to the effect that the least growth in the Phragmén-Lindelöf alternative for quasiregular maps is attained by univalent mappings? In general, does the following assertion hold?

Problem. Let $\mathscr{X} \subset \mathbb{R}^n$ be a simply connected domain, let $u(y)$ be a growth function in $\mathscr{Y}$ and let $f_i\colon \mathscr{X} \to \mathscr{Y}$, $f_i(\partial\mathscr{X}) \subset \partial\mathscr{Y}$, $i = 1, 2$, be a quasiregular map with equal inner and outer dilatation. Then if f_1 is univalent and f_2 is not univalent, we have

$$\liminf_{x\to\infty} \frac{u(f_2(x))}{u(f_1(x))} > 1.$$

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