

Ján Jakubík

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STATE-HOMOMORPHISMS ON  $MV$ -ALGEBRAS

JÁN JAKUBÍK, Košice

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*Abstract.* Riečan [12] and Chovanec [1] investigated states in  $MV$ -algebras. Earlier, Riečan [11] had dealt with analogous ideas in  $D$ -posets. In the monograph of Riečan and Neubrunn [13] (Chapter 9) the notion of state is applied in the theory of probability on  $MV$ -algebras.

We remark that a different definition of a state in an  $MV$ -algebra has been applied by Mundici [9], [10] (namely, the condition (iii) from Definition 1.1 above was not included in his definition of a state; in other words, only finite additivity was assumed).

Below we work with the definition from [13]; but, in order to avoid terminological problems we use the term “state-homomorphism” (instead of “state”). The author is indebted to the referee for his suggestion concerning terminology.

Let  $\mathcal{A}$  be an  $MV$ -algebra which is defined on a set  $A$  with  $\text{card } A > 1$ . In the present paper we show that there exists a one-to-one correspondence between the system of all state-homomorphisms on  $\mathcal{A}$  and the system of all  $\sigma$ -closed maximal ideals of  $\mathcal{A}$ .

For  $MV$ -algebras we apply the notation and the definitions as in Gluschkof [3].

The relations between  $MV$ -algebras and abelian lattice ordered groups (cf. Mundici [8]) are substantially used in the present paper.

*Keywords:*  $MV$ -algebra, state homomorphism,  $\sigma$ -closed maximal ideal

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## 1. PRELIMINARIES

We recall that an  $MV$ -algebra is an algebraic system

$$\mathcal{A} = (A; \oplus, *, \neg, 0, 1),$$

where  $A$  is a nonempty set,  $\oplus$  and  $*$  are binary operations,  $\neg$  is a unary operation, and  $0, 1$  are nullary operations on  $A$  such that the conditions (m<sub>1</sub>)–(m<sub>9</sub>) from [3] are satisfied.

Let us remark that in [1], [11] and [13] another system of axioms for an *MV*-algebra was applied. Both these systems are equivalent in a natural sense (for a formal description of this equivalence we can apply Marczewski's theory of weak automorphisms of algebraic systems; cf., e.g., Goetz [4]).

In what follows we assume that  $\text{card } A > 1$ .

Let  $x, y \in A$ . We put

$$x \vee y = (x * \neg y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

Then (cf. Mundici [8]) we obtain that  $(A; \vee, \wedge)$  is a distributive lattice with the least element 0 and the greatest element 1. This lattice will be denoted by  $\ell(\mathcal{A})$ .

Let  $X$  be a partially ordered set,  $x \in X$  and let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$  such that  $x_n \leq x_{n+1}$  for each  $n \in \mathbb{N}$ , and  $\sup\{x_n\}_{n \in \mathbb{N}} = x$ . Then we write  $x_n \nearrow x$ .

We denote by  $\mathbb{R}$  the additive group of all reals with the natural linear order. For  $x, y \in \mathbb{R}$  with  $x \leq y$  let  $[x, y]$  be the corresponding interval in  $\mathbb{R}$ .

**1.1. Definition.** Let  $\mathcal{A}$  be as above. A state-homomorphism on  $\mathcal{A}$  is a mapping  $m \rightarrow [0, 1]$  which satisfies the following conditions:

- (i)  $m(1) = 1$ .
- (ii) If  $a, b \in A$  and  $a \leq \neg b$ , then  $m(a \oplus b) = m(a) + m(b)$ .
- (iii) If  $a \in A$ ,  $a_n \in A$  for  $n \in \mathbb{N}$  and  $a_n \nearrow a$ , then  $m(a_n) \nearrow m(a)$ .

According to 9.1.6 and 9.1.7 in [13], the above definition of a state-homomorphism is equivalent to the definition of a state considered in [13]. (We remark that for  $x \in A$  the symbol  $\neg x$  has the same meaning as the symbol  $x^*$  in [13].)

The notion of a congruence relation on  $\mathcal{A}$  has the usual meaning (i.e., it is a binary relation on the set  $A$  which is compatible with each of the operations  $\oplus, *, \neg$ ).

The system of all congruence relations on  $\mathcal{A}$  will be denoted by  $\text{Con } \mathcal{A}$ ; this system is partially ordered in the usual way.

Let  $\varrho \in \text{Con } \mathcal{A}$  and  $x \in A$ . Put  $x(\varrho) = \{y \in A: y \varrho x\}$ . The set  $0(\varrho)$  is called an ideal of  $\mathcal{A}$ .

An ideal  $0(\varrho)$  of  $\mathcal{A}$  is called maximal if it satisfies the following conditions:

- (i) Whenever  $\varrho_1 \in \text{Con } \mathcal{A}$  and  $0(\varrho) \subseteq 0(\varrho_1) \neq A$ , then  $0(\varrho) = 0(\varrho_1)$ .
- (ii)  $A \neq 0(\varrho)$ .

A subset  $X$  of  $A$  is said to be  $\sigma$ -closed if, whenever  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $X$  and  $a$  is an element of  $A$  such that either  $\sup\{x_n\}_{n \in \mathbb{N}} = a$  or  $\inf\{x_n\}_{n \in \mathbb{N}} = a$ , then  $a \in X$ .

## 2. FACTOR $MV$ -ALGEBRAS

Let  $\mathcal{A}$  be as above and let  $\varrho \in \text{Con } \mathcal{A}$ . Then we can construct in the usual way the factor  $MV$ -algebra  $\mathcal{A}/\varrho$  (cf., e.g., [7]). The algebraic system  $\mathcal{A}/\varrho$  is an  $MV$ -algebra; let us denote its underlying set by  $A_1$ . The mapping  $x \rightarrow x(\varrho)$  of  $A$  onto  $A_1$  is a homomorphism of  $\mathcal{A}$  onto  $\mathcal{A}/\varrho$ .

Let  $\mathcal{B}$  be an  $MV$ -algebra and let  $\varphi$  be a homomorphism of  $\mathcal{A}$  onto  $\mathcal{B}$ . For  $x, y \in A$  we put  $x \varrho_\varphi y$  if  $\varphi(x) = \varphi(y)$ . Then  $\varrho_\varphi$  is a congruence relation on  $\mathcal{A}$  and the mapping  $f$  defined by

$$f(x(\varrho_\varphi)) = \varphi(x)$$

is an isomorphism of the  $MV$ -algebra  $\mathcal{A}/\varrho_\varphi$  onto  $\mathcal{B}$ .

For lattice ordered groups we apply the notation and definitions as in [2].

Let  $G$  be an abelian lattice ordered group with a strong unit  $u$ . Then  $\mathcal{A}_0(G, u)$  has the same meaning as in [5].

Without loss of generality we can suppose that  $\mathcal{A} = \mathcal{A}_0(G, u)$  (cf. Mundici [8]).

For  $\varrho \in \text{Con } A$  we denote by  $0(\varrho)_0$  the convex  $\ell$ -subgroup of  $G$  which is generated by the set  $0(\varrho)$ . Further, let  $\varrho_0$  be the congruence relation on  $G$  which is generated by the  $\ell$ -ideal  $0(\varrho)_0$ .

**2.1. Lemma.** *Let  $\varrho \in \text{Con } \mathcal{A}$ . Then the following conditions are equivalent:*

- (i)  $0(\varrho)$  is a maximal ideal in  $\mathcal{A}$ .
- (ii)  $0(\varrho)_0$  is a maximal  $\ell$ -ideal in  $G$ .

*Proof.* This is a consequence of 1.10 in [7]. □

**2.2. Lemma.**  *$\ell(\mathcal{A})$  is a chain if and only if  $G$  is linearly ordered.*

*Proof.* If  $G$  is linearly ordered, then it is clear that  $\ell(\mathcal{A})$  is linearly ordered as well. If  $G$  is not linearly ordered, then there exist  $g_1$  and  $g_2$  in  $G$  such that  $g_1 > 0$ ,  $g_2 > 0$  and  $g_1 \wedge g_2 = 0$ . Put  $a_i = g_i \wedge u$  ( $i = 1, 2$ ). Then  $a_i \in A$ ,  $a_i > 0$  ( $i = 1, 2$ ) and  $a_1 \wedge a_2 = 0$ , hence  $\ell(\mathcal{A})$  is not linearly ordered. □

In what follows we often speak of  $\mathcal{A}$  being linearly ordered meaning that  $\ell(\mathcal{A})$  is linearly ordered.

**2.3. Lemma.** *Let  $\varrho \in \text{Con } \mathcal{A}$ . Assume that  $0(\varrho)$  is a maximal ideal in  $\mathcal{A}$ . Then the  $MV$ -algebra  $\mathcal{A}/\varrho$  is linearly ordered.*

*Proof.* According to 2.1,  $0(\varrho)_0$  is a maximal  $\ell$ -ideal in  $G$ . Thus  $G/\varrho_0$  is linearly ordered. Now 2.2 and [7], Proposition 2.4 yield that  $\mathcal{A}/\varrho$  is linearly ordered. □

For the notion of an archimedean *MV*-algebra cf., e.g., [6].

**2.4. Lemma.** *Let  $\varrho$  be as in 2.3. Then the *MV*-algebra  $\mathcal{A}/\varrho$  is archimedean.*

*Proof.* By way of contradiction, suppose that  $\mathcal{A}/\varrho$  is not archimedean. Then in view of 2.4 in [7] the lattice ordered group  $G/\varrho_0$  is not archimedean. Moreover, according to 2.2 and 2.3,  $G/\varrho_0$  is linearly ordered. Then there exists an  $\ell$ -ideal  $X$  in  $G/\varrho_0$  such that  $0(\varrho_0) \neq X \neq G/\varrho_0$ . Thus the set

$$X_1 = \{x \in G : x(\varrho_0) \in X\}$$

is an  $\ell$ -ideal in  $G$  with  $0(\varrho_0) \subset X_1 \neq G$ . Hence  $0(\varrho_0)$  is not a maximal  $\ell$ -ideal in  $G$ , which contradicts 2.1.  $\square$

**2.5. Lemma.** *Let  $\varrho$  be as in 2.3. Then the lattice ordered group  $G/\varrho_0$  is isomorphic to an  $\ell$ -subgroup of the linearly ordered group  $\mathbb{R}$ .*

*Proof.* It is well-known that each archimedean linearly ordered group is isomorphic to an  $\ell$ -subgroup of  $\mathbb{R}$ . In the proof of 2.3 we have observed that  $G/\varrho_0$  is linearly ordered. Moreover, the argument performed in the proof of 2.4 shows that  $G/\varrho_0$  is archimedean.  $\square$

If  $\varrho$  is as in 2.3, then in view of 2.5 and [7], Proposition 2.4 there exists an  $\ell$ -subgroup  $\mathbb{R}_1$  of  $\mathbb{R}$  and an element  $0 < v \in \mathbb{R}_1$  such that  $\mathcal{A}/\varrho$  is isomorphic to  $\mathcal{A}_0(\mathbb{R}_1, v)$ .

It is clear that  $\mathcal{A}_0(\mathbb{R}_1, v)$  is a subalgebra of  $\mathcal{A}_0(\mathbb{R}, v)$ . Further, for each element  $v_1 \in \mathbb{R}$  with  $v_1 > 0$ , the *MV*-algebra  $\mathcal{A}_0(\mathbb{R}, v)$  is isomorphic to  $\mathcal{A}_0(\mathbb{R}, v_1)$ . In particular, we can put  $v_1 = 1$ . Thus we obtain

**2.6. Lemma.** *Let  $\varrho$  be as in 2.3. Then there exists an isomorphism  $\psi$  of  $\mathcal{A}/\varrho$  into the *MV*-algebra  $\mathcal{A}_0(\mathbb{R}, 1)$ .*

**2.7. Lemma.** *Let  $\varrho$  be as in 2.3 and let  $\psi$  be as in 2.6. Then the following conditions are fulfilled:*

(i<sub>1</sub>)  $\psi(u) = 1$ .

(ii<sub>1</sub>) *If  $a, b \in A$  and  $a \leq -b$ , then  $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$ .*

*Proof.* The relation (i<sub>1</sub>) is an immediate consequence of the fact that  $\psi$  is an isomorphism. Let  $a, b \in A$  and  $a \leq -b$ . The isomorphism  $\psi$  yields that  $\psi(a) \leq -\psi(b)$ . Since  $-\psi(b) = 1 - \psi(b)$ , we obtain that  $\psi(a) + \psi(b) \leq 1$ , whence

$$\psi(a) \oplus \psi(b) = \psi(a) + \psi(b).$$

Further, in view of 2.6 we have  $\psi(a \oplus b) = \psi(a) + \psi(b)$ , thus (ii<sub>1</sub>) holds.  $\square$

**2.8. Lemma.** *Let  $\varrho$  be as in 2.3. Assume that the  $\ell$ -ideal  $0(\varrho)$  is  $\sigma$ -closed. Then the following condition is valid:*

(iii<sub>1</sub>) *If  $a_n \in A$  for each  $n \in \mathbb{N}$ ,  $a \in A$  and  $a_n \nearrow a$ , then  $a_n(\varrho) \nearrow a(\varrho)$ .*

*P r o o f.* It is easy to verify that for each  $x \in A$ , the set  $x(\varrho)$  is  $\sigma$ -closed. Let  $a_n \nearrow a$ . Then  $a_n(\varrho) \leq a_{n+1}(\varrho) \leq a(\varrho)$  for each  $n \in \mathbb{N}$ . We have to show that

$$(1) \quad \bigvee_{n \in \mathbb{N}} a_n(\varrho) = a(\varrho)$$

is valid in  $\mathcal{A}/\varrho$ . By way of contradiction, suppose that (1) fails to hold. Thus there is  $b \in A$  such that  $a_n(\varrho) \leq b(\varrho)$  for each  $n \in \mathbb{N}$  and  $b(\varrho) < a(\varrho)$ . We have  $a \wedge b \in b(\varrho)$ , thus without loss of generality we can suppose that  $b \leq a$ . Then

$$(2) \quad (a_n \vee b) \wedge a \nearrow (a \vee b) \wedge a = a$$

is valid in  $\mathcal{A}$  and

$$(a_n \vee b) \wedge a \in b(\varrho)$$

for each  $n \in \mathbb{N}$ . Since  $b(\varrho)$  is  $\sigma$ -closed we obtain from (2) that the element  $a$  belongs to  $b(\varrho)$ , which is a contradiction.  $\square$

The mapping  $\psi$  considered above was constructed by means of  $\varrho$ . Let us now write  $\psi_\varphi$  instead of  $\psi$ .

From 2.6, 2.7 and 2.8 we obtain

**2.9. Proposition.** *Let  $\varrho \in \text{Con } \mathcal{A}$ . Suppose that the ideal  $0(\varrho)$  of  $\mathcal{A}$  is maximal and  $\sigma$ -closed. Then the mapping  $\psi_\varrho$  is a state-homomorphism in  $\mathcal{A}$ .*

### 3. MAXIMAL IDEAL CORRESPONDING TO A STATE-HOMOMORPHISM

Suppose that  $m$  is a state-homomorphism on the  $MV$ -algebra  $\mathcal{A}$ . Let  $G$  be as above.

We define a partial binary operation  $-$  on  $A$  as follows. If  $a_1, a_2 \in A$  and  $a_1 \leq a_2$ , then  $a_2 - a_1$  in  $A$  has the same meaning as  $a_2 - a_1$  in  $G$ ; otherwise,  $a_2 - a_1$  is not defined in  $A$ .

From 9.16 and 9.1.7 in [13] we obtain

**3.1. Lemma.** *If  $a, b \in A$  and  $a \leq b$ , then  $m(b - a) = m(b) - m(a)$ .*

Similarly as in the preceding section we consider the interval  $[0, 1]$  of  $\mathbb{R}$  as the underlying set of the  $MV$ -algebra  $\mathcal{B} = \mathcal{A}_0(\mathbb{R}, 1)$ .

Put  $B_1 = m(A)$ . In view of 3.1 and according to Proposition 3.1 of [3] we have

**3.2. Lemma.**

- (i)  $B_1$  is an underlying set of a subalgebra  $\mathcal{B}_1$  of  $\mathcal{B}$ ;
- (ii)  $m$  is a homomorphism of  $\mathcal{A}$  onto  $\mathcal{B}_1$ .

We remark that the corresponding proof in [1] is performed by using different set of operations on an  $MV$ -algebra than we are applying in the present paper, but the notions of a congruence relation and of a homomorphism in both settings are the same.

Consider the congruence relation  $\varrho_m$  on  $A$  which is defined by means of the homomorphism  $m$  (cf. Section 2 above). Since  $\mathcal{A}/\varrho_m$  is isomorphic to  $\mathcal{B}_1$ , we obtain

**3.3. Lemma.**  $\mathcal{A}/\varrho_m$  is linearly ordered and archimedean.

Thus according to 2.2 and [7], Proposition 2.4 we have

**3.4. Lemma.**  $G/(\varrho_m)_0$  is linearly ordered and archimedean.

From 3.4 we infer that  $G/(\varrho_m)_0$  has no non-trivial  $\ell$ -ideal. This yields that the  $\ell$ -ideal  $0((\varrho_m)_0)$  of  $G$  is maximal. Then 2.1 yields

**3.5. Lemma.**  $0(\varrho_m)$  is a maximal ideal of  $\mathcal{A}$ .

**3.6. Lemma.**  $0(\varrho_m)$  is a  $\sigma$ -closed subset of  $A$ .

*P r o o f.* a) Let  $(x_n)$  be a sequence in  $0(\varrho_m)$ ,  $x \in A$  and suppose that the relation

$$\bigvee_{n \in \mathbb{N}} x_n = x$$

is valid in  $\mathcal{A}$ . Denote  $y_n = x_1 \vee x_2 \vee \dots \vee x_n$  for each  $n \in \mathbb{N}$ . Then  $y_n \leq y_{n+1}$  for each  $n \in \mathbb{N}$  and

$$\bigvee_{n \in \mathbb{N}} y_n = x,$$

whence  $y_n \nearrow x$  in  $\mathcal{A}$ . Since  $m$  is a state-homomorphism on  $\mathcal{A}$  we obtain  $m(y_n) \nearrow m(x)$ . Clearly  $y_n \in 0(\varrho_m)$ , thus  $m(y_n) = 0$  for each  $n \in \mathbb{N}$  and hence  $m(x) = 0$ . Therefore  $x \in 0(\varrho_m)$ .

b) Let  $(z_n)$  be a sequence in  $0(\varrho_m)$ ,  $z \in A$ . Assume that

$$\bigwedge_{n \in \mathbb{N}} z_n = z$$

holds in  $\mathcal{A}$ . Then  $0 \leq z \leq z_n$  for each  $n \in \mathbb{N}$ . Since  $0(\varrho_m)$  is a convex sublattice of  $\ell(\mathcal{A})$  and  $0 \in 0(\varrho_m)$  we obtain  $z \in 0(\varrho_m)$ . □

**3.7. Lemma.** *Let  $G_1$  and  $G_2$  be  $\ell$ -subgroups of  $\mathbb{R}$  such that  $0, 1 \in G_i$  for  $i = 1, 2$ . Assume that  $\varphi$  is an isomorphism of  $G_1$  onto  $G_2$  with  $\varphi(1) = 1$ . Then  $G_1 = G_2$  and  $\varphi$  is the identity on  $G_1$ .*

*Proof.* By way of contradiction, suppose that  $\varphi$  fails to be the identical mapping on  $G_1$ . Hence there is  $0 < x \in G_1$  such that  $\varphi(x) = y \neq x$ . Then there exist positive integers  $n$  and  $m$  such that either (i)  $mx < n < my$ , or (ii)  $my < n < mx$ . Suppose that (i) holds. Then  $\varphi(mx) < \varphi(n)$ . Clearly  $\varphi(mx) = my$ ,  $\varphi(n) = n$ , whence  $my < n$ , which is a contradiction. The case (ii) is analogous.  $\square$

**3.8. Lemma.** *Let  $G_1$  and  $G_2$  be  $\ell$ -subgroups of  $\mathbb{R}$  such that  $0, 1 \in G_i$  for  $i = 1, 2$ . Put  $\mathcal{A}_0 = \mathcal{A}_0(G_1, 1)$ ,  $\mathcal{A}_2 = \mathcal{A}_0(G_2, 1)$ . Suppose that  $\varphi_0$  is an isomorphism of  $\mathcal{A}_1$  onto  $\mathcal{A}_2$ . Then  $\varphi_0$  is the identical mapping on  $\mathcal{A}_1$ .*

*Proof.* From the fact that  $\varphi_0$  is an isomorphism of  $\mathcal{A}_1$  onto  $\mathcal{A}_2$  we easily obtain that there exists an isomorphism  $\varphi$  of  $G_1$  onto  $G_2$  such that  $\varphi(x) = \varphi_0(x)$  for each  $x \in A_1$ . In particular, we have  $\varphi(1) = 1$ . Then it suffices to apply 3.7.  $\square$

**3.9. Lemma** ([7], Lemma 1.11). *Let  $\varrho_1$  and  $\varrho_2$  be congruence relations on  $\mathcal{A}$  such that  $0(\varrho_1) = 0(\varrho_2)$ . Then  $\varrho_1 = \varrho_2$ .*

Let us denote by

$\mathcal{I}$ —the set of all  $\sigma$ -closed maximal ideals of  $\mathcal{A}$ ;

$\mathcal{S}$ —the set of all state-homomorphisms on  $\mathcal{A}$ .

Consider a mapping  $f_1: \mathcal{I} \rightarrow \mathcal{S}$  defined by

$$f_1(X) = \psi_\varrho$$

for each  $X \in \mathcal{I}$ , where  $\varrho$  is a congruence relation on  $\mathcal{A}$  with  $0(\varrho) = X$  (cf. 2.9 and 3.9).

Further, let  $f_2$  be the mapping of  $\mathcal{S}$  into  $\mathcal{I}$  such that

$$f_2(m) = 0(\varrho_m)$$

for each  $m \in \mathcal{S}$  (cf. 3.5 and 3.6).

From the construction of  $\psi_\varrho$  we immediately obtain

$$f_2(f_1(X)) = X$$

for each  $X \in \mathcal{I}$ .



Also, 3.8 and the definition of  $f_2$  yield

$$f_1(f_2(m)) = m$$

for each  $m \in M$ .

Hence we have

**3.10. Theorem.** *Under the notation as above,  $f_1$  is a bijection of  $\mathcal{I}$  onto  $\mathcal{S}$  and  $f_2 = f_1^{-1}$ .*

The above results show that state-homomorphisms on the  $MV$ -algebra  $\mathcal{A}$  can be viewed—up to isomorphism—as mappings of the form

$$a \rightarrow a \oplus 0(\varrho) \quad (a \in A),$$

where  $0(\varrho)$  is a  $\sigma$ -closed maximal ideal of  $\mathcal{A}$ .

#### References

- [1] *F. Chovanec*: States and observables on  $MV$  algebras. *Tatra Mt. Math. Publ.* 3 (1993), 55–64.
- [2] *P. Conrad*: Lattice Ordered Groups. Tulane University, 1970.
- [3] *D. Gluschkof*: Cyclic ordered groups and  $MV$ -algebras. *Czechoslovak Math. J.* 43 (1993), 249–263.
- [4] *A. Goetz*: On weak automorphisms and weak homomorphisms of abstract algebras. *Coll. Math.* 14 (1966), 163–167.
- [5] *J. Jakubík*: Direct product decompositions of  $MV$ -algebras. *Czechoslovak Math. J.* 44 (1994), 725–739.
- [6] *J. Jakubík*: On archimedean  $MV$ -algebras. *Czechoslovak Math. J.* 48 (1998), 575–582.
- [7] *J. Jakubík*: Subdirect product decompositions of  $MV$ -algebras. *Czechoslovak Math. J.* 49(124) (1999), 163–173.
- [8] *D. Mundici*: Interpretation of  $AFC^*$ -algebras in Łukasiewicz sentential calculus. *J. Funct. Anal.* 65 (1986), 15–53.
- [9] *D. Mundici*: Averaging the truth-value in Łukasiewicz logic. *Studia Logica* 55 (1995), 113–127.
- [10] *D. Mundici*: Uncertainty measures in  $MV$ -algebras, and states of  $AFC^*$ -algebras. *Notas Soc. Mat. Chile* 15 (1996), 42–54.
- [11] *B. Riečan*: Fuzzy connectives and quantum models. In: *Cybernetics and System Research 92* (R. Trappl, ed.). World Scientific Publ., Singapore, 1992, pp. 335–338.
- [12] *B. Riečan*: On limit theorems in fuzzy quantum spaces. (Submitted).
- [13] *B. Riečan and T. Neubrunn*: *Integral, Measure and Ordering*. Kluwer Publ., Dordrecht, 1997.

*Author's address*: Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: musavke@mail.saske.sk.