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DISTINGUISHED COMPLETION OF A DIRECT PRODUCT OF LATTICE ORDERED GROUPS

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Abstract. The distinguished completion $E(G)$ of a lattice ordered group G was investigated by Ball [1], [2], [3]. An analogous notion for MV-algebras was dealt with by the author [7].

In the present paper we prove that if a lattice ordered group G is a direct product of lattice ordered groups G_i $(i \in I)$, then $E(G)$ is a direct product of the lattice ordered groups $E(G_i)$.

From this we obtain a generalization of a result of Ball [3].

Keywords: lattice ordered group, distinguished completion, direct product

MSC 2000: 06F15

1. Preliminaries

For lattice ordered groups we apply the notation as in Conrad [4]. We recall the following basic definitions (cf. [3]).

1.1. Definition. Let G and H be lattice ordered groups such that H is an extension of G. Suppose that

(i) G is a dense ℓ-subgroup of H;

(ii) if $h_1, h_2 \in H$ and $h_1 < h_2$, then there are $g_1, g_2 \in G$ such that $g_1 < g_2$ and the interval $[g_1, g_2]$ of H is projective to a subinterval of $[h_1, h_2]$ in H.

Under these conditions H is said to be a distinguished extension of G.

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1.2. Definition. A lattice ordered group G is called distinguished if it has no proper distinguished extension.

1.3. Definition. Let G and H be lattice ordered groups such that
(i) H is a distinguished extension of G;
(ii) the lattice ordered group H is distinguished.
Then H is said to be a distinguished completion of G.

In [3] it was proved that each lattice ordered group G possesses a distinguished completion which is determined uniquely up to isomorphisms leaving all elements of G fixed.

2. The lattice ordered group $E(G)$

We recall some notation and results from [3] which we shall apply below.

First, let G be a distributive lattice and let $\operatorname{Int} G$ be the set of all intervals in G. For $[a, b]$ and $[c, d]$ in $\operatorname{Int} G$ we write

$$[a, b] \sim [c, d]$$

if the intervals $[a, b]$ and $[c, d]$ are projective. Further, we put

$$[a, b] \leqslant [c, d]$$

if $[a, b]$ is projective to a subinterval of $[c, d]$. We denote

$$\langle a, b\rangle = \{[a_1, b_1] \in \operatorname{Int} G\colon\ [a_1, b_1] \sim [a, b]\},$$
$$S(G) = \{\langle a, b\rangle :\ [a, b] \in \operatorname{Int} G\}.$$

We also set $\langle a, b\rangle \leqslant \langle c, d\rangle$ if $[a, b] \leqslant [c, d]$. Then $\leqslant$ is a correctly defined relation of partial order on $S(G)$ and with respect to this relation, $S(G)$ turns out to be a meet-semilattice with the least element $\langle g, g\rangle$, where g is an arbitrary element of G. We denote $\langle g, g\rangle = \overline{0}$. We put

$$\langle a, b\rangle^{\perp} = \{\langle c, d\rangle \in S(G)\colon\ \langle a, b\rangle \wedge \langle c, d\rangle = \overline{0}\}.$$

For $X \subseteq S(G)$ we denote

$$X^{\perp} = \bigcap\{\langle c, d\rangle^{\perp} :\ \langle c, d\rangle \in X\}.$$

Let $B(G)$ be the system

$$\{\emptyset \neq X \subseteq S(G)\colon\ X = X^{\perp\perp}\};$$

this system is partially ordered by the set-theoretical inclusion. Then $B(G)$ is a Boolean algebra such that

$$\bigwedge_{i\in I} X_i = \bigcap_{i\in I} X_i,$$

$$\bigvee_{i\in I} X_i = \Big(\bigcup_{i\in I} X_i\Big)^{\perp\perp}.$$

Further, for $X \in B(G)$, $X^{\perp}$ is the complement of X in $B(G)$.

For each $g \in G$ we put

$$\varphi(g) = \{\langle a, b\rangle \in S(G)\colon\ g \vee a = g \vee b\}.$$

Then φ is an isomorphism of the lattice G into $B(G)$. If g and $\varphi(g)$ are identified, then G can be viewed as a sublattice of $B(G)$.

Now suppose that G is a lattice ordered group; we use the notation as above. Let $g \in G$, $\langle a, b\rangle \in S(G)$ and $X \subseteq S(G)$. We put

$$\langle a, b\rangle + g = \langle a + g, b + g\rangle\,,$$
$$X + g = \{\langle a, b\rangle + g\colon\ \langle a, b\rangle \in X\};$$

the meanings of $g + \langle a, b\rangle$ and $g + X$ are analogous. Further, we set

$$X' = \{u + \langle a, b\rangle + v\colon\ \langle a, b\rangle \in X \text{ and } u, v \in G^-\}^{\perp\perp}.$$

2.1. Definition. We denote by $E(G)$ the system of all $\emptyset \neq X \subseteq S(G)$ such that

(i) $X' = X$;

(ii) for each $g \in G$ with $g > 0$, $g + X \neq X \neq X + g$.

This system is partially ordered by the set-theoretical inclusion.

2.2. Definition. For $\langle a, b\rangle$ and $\langle c, d\rangle$ in $S(G)$ we put

$$\langle a, b\rangle + \langle c, d\rangle = \langle (a + d) \vee (b + c), b + d\rangle\,.$$

Further, for X_1 and X_2 in $E(G)$ we set

$$X_1 + X_2 = \{\langle s_1, t_1\rangle + \langle s_2, t_2\rangle :\ \langle s_i, t_i\rangle \in X_i \quad (i = 1, 2)\}.$$

2.3. Theorem (cf. [3]). *$E(G)$ is a lattice ordered group. Moreover, it is a distinguished completion of G.*

3. Direct product decompositions (finite case)

Let I be a nonempty set and for each $i \in I$ let G_i be a lattice ordered group. The direct product

$$\prod_{i \in I} G_i$$

is defined in the usual way. If $G = \prod\limits_{i \in I} G_i$ and $g \in G$, then the component of G in G_i will be denoted by g_i.

Let $i(0) \in I$ and $x \in G_{i(0)}$. Then the element x is identified with the element $g \in G$ such that

$$g_i = \begin{cases} x & \text{for } i = i(0), \\ 0 & \text{for } i \neq i(0). \end{cases}$$

This means that all direct product decompositions we are dealing with are internal (in the sense of [5]). Hence under this convention, each G_i is an ℓ-subgroup of G.

In the present section we deal with the case when the set I is finite, i.e.,

$$G = G_1 \times G_2 \times \ldots \times G_n.$$

We start with the assumption that

$$G = A \times B. \tag{1}$$

For $g \in G$ we denote by g_A or g_B the component of g in A or in B, respectively.

The following lemma is a consequence of the fact that the lattice G is distributive; we omit the proof.

3.1. Lemma. *Let $[a, b]$ and $[c, d]$ be intervals in G. Put*

$$u_1 = a \wedge c, \quad v_1 = b \wedge d, \quad u_2 = a \vee c, \quad v_2 = b \vee d.$$

Then the following conditions are equivalent:

(i) *$[a, b]$ and $[c, d]$ are projective.*

(ii) *The relations*

$$\begin{gathered} a \wedge v_1 = u_1 = c \wedge v_1, \quad a \vee v_1 = b, \quad c \vee v_1 = d, \\ b \wedge u_2 = a, \quad d \wedge u_2 = c, \quad b \vee u_2 = v_2 = d \vee u_2 \end{gathered}$$

are valid.

Since the lattice operations in a direct product are performed componentwise, from 3.1 we conclude

3.2. Lemma. *Let $[a,b]$ and $[c,d]$ be intervals in G. Then the following conditions are equivalent:*

(i) $[a,b] \sim [c,d]$.

(ii) $[a_A, b_A] \sim [c_A, d_A]$ *and* $[a_B, b_B] \sim [c_B, d_B]$.

Let $S(G)$ be as above; also, let the relation $\leqslant$ in $S(G)$ be as in Section 2. The symbols $S(A)$ and $S(B)$ are defined analogously.

For each $\langle a,b\rangle \in S(G)$ we put

$$\varphi_1(\langle a,b\rangle) = (\langle a_A, b_A\rangle, \langle a_B, b_B\rangle).$$

3.3. Lemma. *φ_1 is an isomorphism of the partially ordered set $S(G)$ onto $S(A) \times S(B)$.*

P r o o f. This is a consequence of the relation (1) and of 3.2. $\square$

For each nonempty subset X of $S(G)$ we put

$$\varphi_2(X) = (X^A, X^B),$$

where

$$X^A = \{\langle a_A, b_A\rangle : \langle a,b\rangle \in X\},$$
$$X^B = \{\langle a_B, b_B\rangle : \langle a,b\rangle \in X\}.$$

Then for each $g \in G$ we have

$$\varphi_2(g + X) = (g_A + X^A, g_B + X^B), \tag{2}$$

and similarly for $\varphi_2(X + g)$.

Further, from 3.3 we obtain by a simple calculation that the implication

$$\varphi_2(X_1) = \varphi_2(X_2) \Rightarrow X_1 = X_2 \tag{3}$$

is valid.

3.4. Lemma. *Let $\emptyset \neq X \subseteq S(G)$. Then*

$$\varphi_2(X^\perp) = ((X^A)^\perp, (X^B)^\perp).$$

P r o o f. This follows from 3.2 and 3.3. $\square$

Since the group operation in G is performed componentwise, from 3.4 we obtain

3.5. Lemma. *Let $\emptyset \neq X \subseteq S(G)$. Then the following conditions are equivalent:*
(i) $X' = X$.
(ii) $(X^A)' = X^A$ *and* $(X^B)' = X^B$.

In fact, the first equation in (ii) is taken with respect to the lattice ordered group A, and similarly, the second with respect to B.

3.6. Lemma. *Let X be as in 3.5. Then the following conditions are equivalent:*
(i) *For each $g \in G$ with $0 < g$ we have $g + X \neq X \neq X + g$.*
(ii) *If $0 < a \in A$ and $0 < b \in B$, then*

$$a + X^A \neq X^A \neq X^A + a, \quad b + X^B \neq X^B \neq X^B + b.$$

Proof. Suppose that (i) holds. Let $a \in A$, $a > 0$. In view of the above convention we have $a \in G$ and $a_A = a$, $a_B = 0$. Thus in view of (2)

$$\varphi_2(a + X) = (a + X^A, X^B).$$

According to (i), $a + X \neq X$. If $a + X^A = X^A$, then we would have

$$\varphi_2(a + X) = \varphi_2(X),$$

whence in view of (3), $a + X = X$, which is a contradiction. Thus $a + X \neq X$. Similarly we obtain the other relations from (ii).

Conversely, suppose that (ii) is valid. Let $0 < g \in G$. Then $g_A \geqslant 0$, $g_B \geqslant 0$ and either $g_A > 0$ or $g_B > 0$. E.g., let $g_A > 0$. Hence $g_A + X^A \neq X^A$. Thus (2) holds.

If $g + X = X$, then $\varphi_2(g + X) = (X^A, X^B)$, whence $g_A + x^A = X^A$, which is a contradiction. Therefore $g + X \neq X$. Analogously we obtain $X + g \neq X$. $\square$

From 3.5 and 3.6 we conclude

3.7. Lemma. *Let $\emptyset \neq X \subseteq S(G)$.*
(i) *If $X \in E(G)$, then $X^A \in E(A)$ and $X^B \in E(B)$.*
(ii) *If $X^A \in E(A)$ and $X^B \in E(B)$, then $X \in E(G)$.*

From 3.3 we infer that for each $P \subseteq S(A)$ and each $Q \subseteq S(B)$ there exists a uniquely determined $X \subseteq S(G)$ such that

$$\varphi_2(X) = (P, Q).$$

Thus if we restrict ourselves, by the application of φ_2, to elements of $E(G)$ only, then from 3.7 we obtain

3.8. Lemma. *φ_2 is a one-to-one mapping of the set $E(G)$ onto $E(A) \times E(B)$.*

In view of 2.2 we have an operation $+$ on $S(G)$, and similarly on $S(A)$ and on $S(B)$. From this definition and from 3.3 we obtain that φ_2 is an isomorphism with respect to the operation $+$ (where φ_2 is taken as in 3.8).

Finally, 3.3 and 3.8 yield also that φ_2 is an isomorphism of the partially ordered set $E(G)$ onto $E(A) \times E(B)$.

Summarizing, we have

3.9. Lemma. *Let* (1) *be valid. Then the lattice ordered group $E(G)$ is isomorphic to the direct product $E(A) \times E(B)$.*

By the obvious induction we obtain

3.10. Proposition. *Let G be a lattice ordered group which is isomorphic to the direct product $G_1 \times G_2 \times \ldots \times G_n$. Then $E(G)$ is isomorphic to the direct product $E(G_1) \times E(G_2) \times \ldots \times E(G_n)$.*

Hence, up to isomorphisms, we can write

$$E(G) = E(G_1) \times E(G_2) \times \ldots \times E(G_n).$$

4. Direct product decompositions (infinite case)

Now let us suppose that we have a direct product decomposition

$$G = \prod_{i \in I} G_i, \tag{1}$$

where the set I can be infinite. Then for each fixed $i(0) \in I$ there exists a direct product decomposition

$$G = G_{i(0)} \times G'_{i(0)},$$

where

$$G'_{i(0)} = \prod G_i \quad (i \in I \setminus \{i(0)\}.$$

Thus in view of 3.10 we can write

$$E(G) = E(G_{i(0)}) \times E(G'_{i(0)}). \tag{2}$$

In fact, the relation (2) is meant in the sense that $E(G)$ has a direct product decomposition of the form $E(G) = H_1 \times H_2$, where H_1 is isomorphic to $E(G_{i(0)})$ and H_2 is isomorphic to $E(G'_{i(0)})$. We prefer the simpler formulation from (2), and analogously at similar places below. Cf. also the convention introduced in Section 3.

Moreover, if $i(1)$ and $i(2)$ are distinct elements of I, then there exists a direct product decomposition

$$G = G_{i(1)} \times G_{i(2)} \times C,$$

where

$$C = \prod G_i \quad (i \in I \setminus \{i(1), i(2)\}.$$

Hence

$$E(G) = E(G_{i(1)}) \times E(G_{i(2)}) \times E(C).$$

This yields that the relation

$$E(G_{i(1)}) \cap E(G_{i(2)}) = \{0\} \tag{3}$$

is valid whenever $i(1)$ and $i(2)$ are distinct elements of I.

We apply the following result which is a consequence of the facts expressed in the diagram on p. 143 of [2].

4.1. Proposition (cf. [2]). *For each lattice ordered group G, $E(G)$ is laterally complete.*

4.2. Lemma. $\bigcap\limits_{i\in I} E(G'_i) = \{0\}$.

P r o o f. By way of contradiction suppose that

$$\bigcap_{i\in I} E(G'_i) = G^0 \neq \{0\}.$$

Then there exists $0 < g^0 \in G^0$. Each $E(G'_i)$ is a convex ℓ-subgroup of $E(G)$, whence G^0 is a convex ℓ-subgroup of $E(G)$ as well. Further, G is a dense ℓ-subgroup of $E(G)$, thus there is $0 < g \in G$ with $g \leqslant g^0$. We obtain $g \in G^0$, hence $g \in G'_i$ for each $i \in I$. This yields that $g_i = 0$ for each $i \in I$. But then, in view of (1), we get $g = 0$, which is a contradiction. $\square$

4.3. Lemma. *Let H be a laterally complete lattice ordered group. Let I be a nonempty set and for each $i \in I$ let*

$$H = H_i \times H'_i$$

where

a) $H_{i(1)} \cap H_{i(2)} = \{0\}$ *whenever* $i(1)$ *and* $i(2)$ *are distinct elements of* I*;*

b) $\bigcap\limits_{i\in I} H_i' = \{0\}$.

Then $H = \prod\limits_{i\in I} H_i$.

P r o o f. Put $\prod\limits_{i\in I} H_i = H^*$. Consider the mapping $\psi\colon H \to H^*$ such that $\psi(h) = (\ldots, h_i, \ldots)_{i\in I}$, where h_i is the component of h in H_i. Then ψ is a homomorphism of H into H^*. Let $h \in H$ be such that $\psi(h) = 0$. Then $h_i = 0$ for each $i \in I$, whence

$$h \in \bigcap_{i\in I} H_i',$$

and thus, according to b), $h = 0$. Therefore ψ is an isomorphism of H into H^*.

Let $0 \leqslant h^* \in H^*$ and let h_i^* denote the component of h^* in H_i. If $i(1)$ and $i(2)$ are distinct elements of I, then in view of a) we have

$$h^*_{i(1)} \wedge h^*_{i(2)} = 0.$$

Thus $(h_i)^*_{i\in I}$ is an orthogonal indexed system of elements of H. Since H is laterally complete, there exists $h \in H$ with

$$h = \bigvee_{i\in I} h_i^*.$$

For each $i \in I$ there is also $h' \in H$ with

$$h' = \bigvee h_j^* \quad (j \in I \setminus \{i\}).$$

Thus we have

$$h = h_i^* \vee h'.$$

If $0 \leqslant x \in H_i$ and $j \in I \setminus \{i\}$, then $x \wedge h_j^* = 0$ and hence (in view of the infinite distributivity of H) $x \wedge h' = 0$; therefore $h_i' = 0$. Clearly $(h_i^*)_i = h_i^*$, thus

$$h_i = (h_i^*)_i \vee h_i' = h_i^*.$$

Hence $h = h^*$ and therefore $(H^*)^+ \subseteq \psi(H)$. This obviously implies that $H^* \subseteq \psi(H)$. Then $\psi(H) = H^*$, which completes the proof. $\square$

4.4. Proposition. *Let* (1) *be valid. Then*

$$E(G)=\prod_{i\in I}E(G_i).$$

P r o o f. This is a consequence of 4.1, 4.2 and 4.3. (Again, the above equality is meant in the sense of an isomorphism.) □

For a lattice ordered group G let $G^\wedge$ be the Dedekind completion of G. If G is linearly ordered, then $G^\wedge$ is linearly ordered as well.

4.5. Proposition (cf. Ball [3], 4.4)**.** *Let G be a linearly ordered group. Then $E(G)=G^\wedge$.*

The following result extends Proposition 4.5.

4.6. Proposition. *Let* (1) *be valid. Suppose that each G_i is a linearly ordered group. Then*

$$E(G)=\prod_{i\in I}G_i{}^\wedge.$$

P r o o f. In view of 4.5, for each $i\in I$ we have $E(G_1)=G_i{}^\wedge$. Now it suffices to apply 4.4. □

The following result was proved in [5] under the assumption that G is abelian, but the proof remains valid also without this assumption.

4.7. Proposition (cf. [5], Theorem 2.7)**.** *Let* (1) *be valid. Then*

$$G^\wedge=\prod_{i\in I}G_i{}^\wedge.$$

4.8. Proposition. *Let* (1) *be valid. Suppose that all G_i are linearly ordered. Then $E(G)=G^\wedge$.*

P r o o f. This is a consequence of 4.6 and 4.7. □

4.9. Proposition. *Let H be a distinguished lattice ordered group. Suppose that*

$$H=\prod_{i\in I}H_i. \tag{4}$$

Then all H_i are distinguished.

P r o o f. In view of 4.4 we have

$$H = \prod_{i \in I} E(H_i), \tag{5}$$

since $E(H) = H$. If $i(1)$ and $i(2)$ are distinct elements of I, then from (5) we infer

$$E(H_{i(1)}) \cap E(H_{i(2)}) = \{0\}.$$

Because $E(H_{i(1)})$ is a direct factor of H, the relation (4) yields

$$E(H_{i(1)}) = \prod_{i \in I} (H_i \cap E(H_{i(1)}). \tag{6}$$

If $i \neq i(1)$, then $H_i \cap E(H_{i(1)}) = \{0\}$, whence in view of (6)

$$E(H_{i(1)}) = H_{i(1)} \cap E(H_{i(1)}) = H_{i(1)}.$$

Therefore $H_{i(1)}$ is distinguished. □

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