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# SOME RESULTS ABOUT DISSIPATIVITY OF KOLMOGOROV OPERATORS

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*Dedicated to Ivo Vrkoč on the occasion of his 70th birthday*

*Abstract.* Given a Hilbert space $H$ with a Borel probability measure $\nu$, we prove the $m$-dissipativity in $L^1(H,\nu)$ of a Kolmogorov operator $K$ that is a perturbation, not necessarily of gradient type, of an Ornstein-Uhlenbeck operator.

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*MSC 2000*: 47B25, 81S20, 37L40, 35K57, 70H15

## 1. Introduction

Let $H$ be a real separable Hilbert space and $\nu$ a Borel probability measure on $H$. We are given a linear operator $A\colon \mathfrak{D}(A) \subset H \to H$ that we suppose to be the infinitesimal generator of a strongly continuous semigroup $\mathrm{e}^{tA}$ on $H$, a linear operator $B \in L(H)$ and a nonlinear Borel mapping $F\colon H \to H$. We set $C = BB^*$.

Let us introduce the function space $\mathcal{E}_A(H)$ as the linear span of all real and imaginary parts of functions on $H$ of the form $x \to \mathrm{e}^{i\langle h,x\rangle}$, where $h \in \mathfrak{D}(A^*)$ and $A^*$ is the adjoint of $A$. It is well known that this space is dense in $L^p(H,\nu)$ for any $p \geqslant 1$.

We are concerned with the linear operator

$$\mathring{K}\varphi = L\varphi + \langle F(x), C^{1/2}D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H),$$

where $L$ is the Ornstein-Uhlenbeck operator

$$L\varphi = \frac{1}{2}\operatorname{Tr}[CD^2\varphi] + \langle x, A^*D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H).$$

In a sense this paper is a continuation of the paper [4]. The main difference is that here we do not assume that $\nu$ is absolutely continuous with respect to a Gaussian measure.

Let us state our assumptions. Concerning $A$ and $C$ we will assume

**Hypothesis 1.**

(i) *There exists* $\omega \geqslant 0$ *such that*

$$\langle Ax, x\rangle \leqslant -\omega |x|^2, \quad x \in \mathfrak{D}(A), \tag{1.1}$$

(ii) $\operatorname{Tr} Q < +\infty$, *where*

$$Qx = \int_0^{+\infty} \mathrm{e}^{tA} C \mathrm{e}^{tA^*} x \, \mathrm{d}t, \quad x \in H,$$

and concerning $F$ we will assume

**Hypothesis 2.**

(i) *There exists a constant* $c > 0$ *such that*

$$\int_H (|x|^2 + |F(x)|^2)\, \nu(\mathrm{d}x) \leqslant c, \tag{1.2}$$

(ii) *for any* $\varphi \in \mathcal{E}_A(H)$ *we suppose*

$$\int_H \mathring{K} \varphi \, \mathrm{d}\nu = 0, \tag{1.3}$$

(iii) $\mathring{K}$ *is dissipative in* $L^p(H, \nu)$, $\forall p \geqslant 1$,

(iv) *there exist a sequence* $(F_n) \subset \mathcal{C}_b^2(H; H)$ *such that* $F_n(x) \to F(x)$ $\nu$-a.e. *and a constant* $c_1 > 0$ *such that*

$$\int_H |F_n(x)|^2 \, \nu(\mathrm{d}x) \leqslant c_1.$$

It is well known that the operator $\mathring{K}$ is closable in $L^p(H, \nu)$ since it is dissipative in it, as stated in (iii). Let us denote its closure in $L^p(H, \nu)$ by $K_p$. Our goal is to show that $K_p$ is dissipative on $L^p(H, \nu)$, $p \geqslant 1$ and that $\nu$ is an infinitesimally invariant measure for $K_p$. The main result of the paper is Theorem 3.6, where we show that $K_1$ is $m$-dissipative on $L^1(H, \nu)$.

In this section we assume that Hypothesis 1 holds. Let $\mathcal{C}_b(H)$ be the space of uniformly continuous and bounded functions $\varphi\colon H \to \mathbb{R}$. Moreover, for any integer $k \geqslant 0$ let us define $\mathcal{C}_{b,k}(H)$ as the space of all $\varphi\colon H \to \mathbb{R}$ such that the mapping

$$H \to \mathbb{R}, \quad x \to \frac{\varphi(x)}{1+|x|^k}$$

belongs to $\mathcal{C}_b(H)$. We set

$$\|\varphi\|_{b,k} = \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^k}.$$

Obviously one has $\mathcal{C}_{b,k}(H) \subset \mathcal{C}_{b,k+1}(H)$.

Denoting by $N_{Q_t}$ the Gaussian measure with mean 0 and covariance operator

$$Q_t x = \int_0^t \mathrm{e}^{sA} C \mathrm{e}^{sA^*} x \, \mathrm{d}s, \quad x \in H,$$

let $\mathcal{R}_t$ be the Ornstein-Uhlenbeck "semigroup"

$$\mathcal{R}_t\varphi(x) = \int_H \varphi(\mathrm{e}^{tA}x + y)\, N_{Q_t}(\mathrm{d}y), \quad \varphi \in \mathcal{C}_{b,k}(H),\ k \geqslant 0. \tag{2.1}$$

It is not difficult to show that for all $t \geqslant 0$ and for all $k \geqslant 0$, $\mathcal{R}_t$ maps $\mathcal{C}_{b,k}(H)$ into itself, see [1]. Following [1][1], we define the infinitesimal generator $L$ of $\mathcal{R}_t$ through its resolvent

$$(\lambda - L)^{-1}\varphi(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathcal{R}_t\varphi(x)\, \mathrm{d}t, \quad x \in H,\ \lambda > 0. \tag{2.2}$$

Thus for any $\lambda > 0$, $(\lambda - L)^{-1}$ maps $\mathcal{C}_{b,k}(H)$ into itself. Since the image of the resolvent is independent of $\lambda$ we can set, see [1],

$$\mathfrak{D}\big(L, \mathcal{C}_{b,k}(H)\big) = (\lambda - L)^{-1}\big(\mathcal{C}_{b,k}(H)\big), \quad k \geqslant 0.$$

As noticed in [1], $\mathcal{R}_t$ is not a strongly continuous semigroup on $\mathcal{C}_{b,k}(H)$ for any $k \geqslant 0$. Let us denote by $\mathfrak{X}_k$ the maximal closed subspace of $\mathcal{C}_{b,k}(H)$ where $\mathcal{R}_t$ is strongly continuous, that is

$$\mathfrak{X}_k = \big\{\varphi \in \mathcal{C}_{b,k}(H)\colon \lim_{t\to 0} \mathcal{R}_t\varphi = \varphi \ \text{ in } \ \mathcal{C}_{b,k}(H)\big\}.$$

[1] See also [2] and [7].

To characterize $\mathfrak{X}_k$ it is useful to introduce an auxiliary family $(\mathcal{G}_t)$ of linear operators on $\mathcal{C}_{b,k}(H)$:

$$\mathcal{G}_t\varphi(x) = \int_H \varphi(x+y)\, N_{Q_t}(\mathrm{d}y), \quad \varphi \in \mathcal{C}_{b,k}(H).$$

They are related to $(\mathcal{R}_t)$ by

$$\mathcal{R}_t\varphi(x) = (\mathcal{G}_t\varphi)(\mathrm{e}^{tA}x), \quad \varphi \in \mathcal{C}_{b,k}(H).$$

**Proposition 2.1.** *Let $\varphi \in \mathcal{C}_{b,k}(H)$. Then the following statements are equivalent:*

(i) $\lim_{t\to 0} \mathcal{R}_t\varphi = \varphi$ *in* $\mathcal{C}_{b,k}(H)$.

(ii) $\lim_{t\to 0} \varphi(\mathrm{e}^{tA}\cdot) = \varphi$ *in* $\mathcal{C}_{b,k}(H)$.

P r o o f. We first show that for any $\varphi \in \mathcal{C}_{b,k}(H)$ we have

$$\lim_{t\to 0} \mathcal{G}_t\varphi = \varphi \ \text{ in } \ \mathcal{C}_{b,k}(H). \tag{2.3}$$

Let $\varphi \in \mathcal{C}_{b,k}(H)$ and set $\psi(x) = \varphi(x)/(1+|x|^k)$. We may assume that $\psi \in \mathcal{C}_b^1(H)$.Then we have

$$\mathcal{G}_t\varphi(x) - \varphi(x) = \int_H \big[(1+|x+y|^k)\psi(x+y) - (1+|x|^k)\psi(x)\big]\, N_{Q_t}(\mathrm{d}y).$$

Consequently,

$$\frac{|\mathcal{G}_t\varphi(x) - \varphi(x)|}{1+|x|^k} \leqslant \int_H \left|\frac{1+|x+y|^k}{1+|x|^k} - 1\right| \|\psi\|_0\, N_{Q_t}(\mathrm{d}y) + \|\psi\|_1 \int_H |y|\, N_{Q_t}(\mathrm{d}y).$$

Therefore (2.3) follows.

We now prove that (i) $\Rightarrow$ (ii). In fact we have

$$|\varphi(\mathrm{e}^{tA}x) - \varphi(x)| \leqslant |\varphi(\mathrm{e}^{tA}x) - \mathcal{G}_t\varphi(\mathrm{e}^{tA}x)| + |\mathcal{R}_t\varphi(x) - \varphi(x)|.$$

So (i) $\Rightarrow$ (ii). The converse can be proved similarly. □

**Remark 2.2.** Since for any $\varphi_h = \mathrm{e}^{i\langle h,x\rangle}$ we have

$$\mathcal{R}_t\varphi_h = \mathrm{e}^{-1/2\langle Q_t h,h\rangle}\varphi_{\mathrm{e}^{tA^*}h},$$

it follows that $\mathcal{R}_t$ maps $\mathcal{E}_A(H)$ into itself. Properties of the space $\mathcal{E}_A(H)$ follow also from the results in [3] and [10].

**Corollary 2.3.**

(i) $\mathcal{E}_A(H) \subset \mathfrak{D}\big(L, \mathcal{C}_{b,k}(H)\big)$ *for all* $k \geqslant 1$,

(ii) $\mathcal{E}_A(H) \subset \mathfrak{X}_1$, *and consequently,*

$$L\varphi = \frac{1}{2}\operatorname{Tr}[CD^2\varphi] + \langle x, A^*D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H). \tag{2.4}$$

(iii) *If* $\varphi \in \mathcal{E}_A(H)$, *then we have* $L\varphi \in \mathfrak{X}_2$.

Proof. Taking in account the definition of $\mathcal{E}_A(H)$, we need only to prove the corollary in the case of the functions $\sin[\langle x,h\rangle]$ and $\cos[\langle x,h\rangle]$. Moreover, since the proof for the cosine function is just the same as for the sine, we are reduced to make the proof only for $\varphi_h(x) = \sin[\langle x,h\rangle]$. Hence we have

$$L\varphi_h = -\frac{1}{2}\sin[\langle x,h\rangle]\,|h|^2 + \cos[\langle x,h\rangle]\langle x, A^*h\rangle, \tag{2.5}$$

which yields (i). Let us prove (ii). We have

$$\frac{\varphi_h(\mathrm{e}^{tA}x) - \varphi_h(x)}{1+|x|} = \frac{\sin[\langle \mathrm{e}^{tA}x, h\rangle] - \sin[\langle x,h\rangle]}{1+|x|}.$$

Consequently,

$$\frac{|\varphi_h(\mathrm{e}^{tA}x) - \varphi_h(x)|}{1+|x|} \leqslant \frac{|\langle x, \mathrm{e}^{tA^*}h\rangle - \langle x,h\rangle|}{1+|x|} \leqslant \frac{|x|}{1+|x|}|\mathrm{e}^{tA^*}h - h|.$$

This implies

$$\lim_{t\to 0}\sup_{x\in H}\frac{|\varphi_h(\mathrm{e}^{tA}x) - \varphi_h(x)|}{1+|x|} = 0,$$

and so $\varphi_h \in \mathfrak{X}_1$ by Proposition 2.1.

Finally, (iii) follows by a similar argument, when taking into account (2.5). □

### 2.1. Approximations by exponential functions.

This subsection is devoted to the study of a kind of approximations of functions of $\mathcal{C}_b(H)$, and moreover of $\mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$, by functions of $\mathcal{E}_A(H)$, which we need in the sequel.

These approximations are not possible by using simple sequences, but $k$-sequences, $k \in \mathbb{N}$, that is sequences $\{\varphi_n\} = \{\varphi_{n_1,\ldots,n_k}\}$ depending on $k$ indices. We say that $\{\varphi_n\}$ is convergent to $\varphi$ if

$$\lim_{n\to\infty}\varphi_n := \lim_{n_1\to\infty}\ldots\lim_{n_k\to\infty}\varphi_{n_1,\ldots,n_k}(x) = \varphi(x), \quad x \in H.$$

**Lemma 2.4.** *For any $\varphi \in \mathcal{C}_b(H)$ there exists a 3-sequence $(\varphi_n) = (\varphi_{n_1,n_2,n_3}) \subset \mathcal{E}_A(H)$ such that*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \forall x \in H, \tag{2.6}$$

*and*

$$\|\varphi_n\|_{b,0} \leqslant \|\varphi\|_{b,0}. \tag{2.7}$$

Proof. Let $\varphi \in \mathcal{C}_b(H)$ and let $(P_{n_1})_{n_1\in\mathbb{N}}$ be a sequence of finite dimensional projection operators on $H$ strongly convergent to the identity. Then for each $n_1 \in \mathbb{N}$ there exists[2] a sequence $(\varphi_{n_1,n_2})_{n_2\in\mathbb{N}} \subset \mathcal{E}(H)$ such that

$$\lim_{n_2\to\infty} \varphi_{n_1,n_2}(x) = \varphi(P_{n_1}x), \quad x \in H,$$

and

$$|\varphi_{n_1,n_2}(x)| \leqslant |\varphi(P_{n_1}x)| \leqslant \|\varphi\|_{b,0}.$$

Now set

$$\varphi_{n_1,n_2,n_3}(x) = \varphi_{n_1,n_2}\big(n_3(n_3 - A^*)^{-1}x\big), \quad x \in H.$$

Then $\varphi_n = \varphi_{n_1,n_2,n_3} \subset \mathcal{E}_A(H)$, $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, $\forall x \in H$, and

$$|\varphi_{n_1,n_2,n_3}(x)| = \big|\varphi_{n_1,n_2}\big(n_3(n_3 - A^*)^{-1}x\big)\big| \leqslant \|\varphi_{n_1,n_2}\|_{b,0} \leqslant \|\varphi\|_{b,0}.$$

Therefore the 3-sequence $(\varphi_{n_1,n_2,n_3})$ fulfils (2.6) and (2.7) as required. □

Now we want to show that any function $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ can be approximated pointwise in the graph norm by functions in $\mathcal{E}_A(H)$ with uniformly bounded norm.

**Proposition 2.5.** *For any $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ there exist a 4-sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ and $C(\varphi) > 0$ such that for all $x \in H$ we have*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} L\varphi_n(x) = L\varphi(x), \tag{2.8}$$

*and*

$$\sup_{x\in H}\left\{\frac{|\varphi_n(x)| + |L\varphi_n(x)|}{1+|x|^2}\right\} \leqslant C(\varphi). \tag{2.9}$$

[2] For example, first we can approximate $\varphi_{n_1}$ by functions with support contained in squares larger and larger, for each of which we can use multiple Fourier series; then we can apply a diagonal procedure.

P r o o f. Set $f=\varphi-L\varphi$ and let $(f_n)=(f_{n_1,n_2,n_3})\subset\mathcal{E}_A(H)$ be a 3-sequence fulfilling (2.6) and (2.7) (with $\varphi$ replaced by $f$). Setting $\varphi_n=(1-L)^{-1}f_n$, we have

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x),\qquad\forall x\in H,$$
$$\lim_{n\to\infty}L\varphi_n(x)=L\varphi(x),\quad\forall x\in H,$$

and

$$\|\varphi_n\|_{b,0}\leqslant\|f\|_{b,0}\leqslant(2\|\varphi\|_{b,0}+\|L\varphi\|_{b,0}),$$
$$\|L\varphi_n\|_{b,0}\leqslant(\|\varphi\|_{b,0}+\|L\varphi\|_{b,0}).$$

Next, set for any $M,N\in\mathbb{N}$

$$\varphi_{n,M,N}(x)=\frac{1}{M}\sum_{h=1}^{N}\sum_{k=1}^{M}\mathrm{e}^{-(h+k/M)}\mathcal{R}_{h+k/M}f_n(x),$$

so that

$$|\varphi_{n,M,N}(x)|\leqslant\|f\|_0$$

and

$$L\varphi_{n,M,N}(x)=\frac{1}{M}\sum_{h=1}^{N}\sum_{k=1}^{M}\mathrm{e}^{-(h+k/M)}\mathcal{R}_{h+k/M}Lf_n(x).$$

Now, by Corollary 2.3 it follows that $Lf_n\in\mathfrak{X}_2$ so that $\mathcal{R}_tf_n$ is continuous on $t$ in the topology of $\mathcal{C}_{b,2}(H)$. Therefore for any $n=(n_1,n_2,n_3)$ we have

$$\lim_{M,N\to\infty}\sup_{x\in H}\frac{1}{1+|x|^2}\left|\int_0^{+\infty}\mathrm{e}^{-t}\mathcal{R}_tLf_n(x)\,\mathrm{d}t-\frac{1}{M}\sum_{h=1}^{N}\sum_{k=1}^{M}\mathrm{e}^{-(h+\frac{k}{M})}\mathcal{R}_{h+\frac{k}{M}}Lf_n(x)\right|=0.$$

Therefore for any $\varepsilon\in(0,1]$ there exist $M_\varepsilon,N_\varepsilon$ such that

$$|L\varphi_n(x)-L\varphi_{n,M_\varepsilon,N_\varepsilon}(x)|\leqslant\varepsilon(1+|x|^2),\quad x\in H.$$

Consequently,

$$\lim_{\varepsilon\downarrow0}L\varphi_{n,M_\varepsilon,N_\varepsilon}(x)=L\varphi_n(x),$$

and

$$|L\varphi_{n,M_\varepsilon,N_\varepsilon}(x)|\leqslant|L\varphi_n(x)|+\varepsilon(1+|x|^2)\leqslant2\|f\|_0+|x|^2.$$

Now the conclusion follows easily. □

In a similar way we prove

**Proposition 2.6.** *For any* $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_{b,1}(H)\big)$ *there exist a 4-sequence* $(\varphi_n) = (\varphi_{n_1,n_2,n_3,n_4}) \subset \mathcal{E}_A(H)$ *and* $C(1,\varphi) > 0$ *such that for all* $x \in H$ *we have*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} D\varphi_n(x) = D\varphi(x), \quad \lim_{n\to\infty} L\varphi_n(x) = L\varphi(x) \tag{2.10}$$

*and*

$$\sup_{x\in H}\left\{\frac{|\varphi_n(x)| + |D\varphi_n(x)| + |L\varphi_n(x)|}{1+|x|^2}\right\} \leqslant C(1,\varphi). \tag{2.11}$$

**Proposition 2.7.** *Assume in addition that* $C^{-1}$ *is bounded. Then for any* $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ *there exist a 4-sequence* $(\varphi_n) \subset \mathcal{E}_A(H)$ *and* $C(\varphi) > 0$ *such that for all* $x \in H$ *we have*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} D\varphi_n(x) = D\varphi(x), \quad \lim_{n\to\infty} L\varphi_n(x) = L\varphi(x) \tag{2.12}$$

*and*

$$\sup_{x\in H}\left\{\frac{|\varphi_n(x)| + |D\varphi_n(x)| + |L\varphi_n(x)|}{1+|x|^2}\right\} \leqslant C(\varphi). \tag{2.13}$$

Proof. Let $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$. By Proposition 2.5 we know that there exist a 4-sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ and $C(\varphi) > 0$ such that (2.8) and (2.9) hold. Moreover, if $C^{-1}$ is bounded then $\mathcal{R}_t$ is strong Feller and, for any $k = 0, 1, \ldots$, there exists $c_k > 0$ such that

$$\frac{|D\mathcal{R}_t f(x)|}{1+|x|^k} \leqslant c_k t^{-1/2} \|f\|_{b,k}, \quad k = 0, 1, \ldots$$

By the Laplace transform we obtain

$$\frac{|D(\lambda - L)^{-1} f(x)|}{1+|x|^k} \leqslant \sqrt{\pi/\lambda}\, c_k \|f\|_{b,k}, \quad k = 0, 1, \ldots$$

Now set $\varphi_n - L\varphi_n = f_n$. Then we have

$$\frac{|D\varphi_n(x)|}{1+|x|^2} \leqslant \sqrt{\pi}\, c_2 \|f\|_{b,2}.$$

Since

$$\|f\|_{b,2} \leqslant \|\varphi_n\|_{b,2} + \|L\varphi_n\|_{b,2},$$

the conclusion follows from (2.8) and (2.9). □

## 3. $m$-dissipativity of $K_1$ on $L^1(H,\nu)$

**Proposition 3.1.** *For all $\varphi \in \mathcal{E}_A(H)$ we have*

$$\int_H \mathring{K}\varphi\,\varphi\,\mathrm{d}\nu = -\frac{1}{2}\int_H |C^{1/2}D\varphi|^2\,\mathrm{d}\nu. \tag{3.1}$$

P r o o f. In fact, if $\varphi \in \mathcal{E}_A(H)$ then we have $\varphi^2 \in \mathcal{E}_A(H)$ and

$$\mathring{K}(\varphi^2) = 2\varphi\mathring{K}\varphi + |C^{1/2}D\varphi|^2.$$

Then integrating both sides with respect to $\nu$ and using (1.3), the conclusion follows. □

Since, by definition, $\mathcal{E}_A(H)$ is a core for $K_2$, (3.1) implies that the linear operator

$$D_C\colon\ \mathcal{E}_A(H) \subset \mathfrak{D}(K_2) \to L^2(H,\nu;H), \quad \varphi \to C^{1/2}D\varphi,$$

is continuous and consequently can be extended to all $\mathfrak{D}(K_2)$. We denote again by $D_C$ the extension. By Proposition 3.1 we get

**Corollary 3.2.** *For all $\varphi \in \mathfrak{D}(K_2)$ we have*

$$\int_H K_2\varphi\,\varphi\,\mathrm{d}\nu = -\frac{1}{2}\int_H |D_C\varphi|^2\,\mathrm{d}\nu. \tag{3.2}$$

Let us now consider the problem

$$\mathrm{d}X_n = \big(AX_n + C^{1/2}F_n(X_n)\big)\,\mathrm{d}t + B\,\mathrm{d}W_t, \quad X_n(0) = x. \tag{3.3}$$

Since $F_n \in \mathcal{C}_b^2(H)$, problem (3.3) has a unique mild solution that we will denote by $X_n(t,x)$, see e.g. [5]. Moreover, $X_n(t,x)$ is differentiable with respect to $x$ and, setting $\eta_n^h(t,x) = DX_n(t,x)h$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_n^h(t,x) = A\eta_n^h(t,x) + C^{1/2}DF_n\big(X_n(t,x)\big)\eta_n^h(t,x), \quad \eta_n^h(t,x) = h. \tag{3.4}$$

Now we consider the equation

$$\lambda\varphi_n - L\varphi_n - \langle F_n(x), C^{1/2}D\varphi_n\rangle = f. \tag{3.5}$$

**Lemma 3.3.** *Let $f \in \mathcal{C}_b^2(H)$ and $\lambda > 0$. Then equation (3.5) has a unique solution $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b^1(H)\big) \cap \mathcal{C}_b^1(H)$ given by*

$$\varphi_n(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathbb{E}\big[f\big(X_n(t,x)\big)\big]\,\mathrm{d}t, \quad x \in H. \tag{3.6}$$

P r o o f. Let $f \in \mathcal{C}_b^1(H)$ and

$$\varphi_n(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathbb{E}\big[f\big(X_n(t,x)\big)\big]\,\mathrm{d}t.$$

Clearly $\varphi_n \in \mathcal{C}_b^1(H)$ since $|\eta_n^h(t,x)| \leqslant \mathrm{e}^{t\|C^{1/2}F_n\|_0}$, and we have

$$\langle D\varphi_n(x), h\rangle = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathbb{E}\big[\big\langle Df\big(X_n(t,x)\big), \eta_n^h(t,x)\big\rangle\big]\,\mathrm{d}t.$$

Let us prove that $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$. Set

$$Z(t,x) = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-s)A}B\,\mathrm{d}W(s),$$

so that

$$X_n(t,x) = Z(t,x) + \int_0^t \mathrm{e}^{(t-s)A}C^{1/2}F_n\big(X_n(s,x)\big)\,\mathrm{d}s, \quad t \geqslant 0.$$

For any $h > 0$ we have

$$\begin{aligned}
&\frac{1}{h}\big(\mathcal{R}_h\varphi_n(x) - \varphi_n(x)\big)\\
&\quad= \frac{1}{h}\mathbb{E}\big[\varphi_n\big(Z(h,x)\big) - \varphi_n(x)\big]\\
&\quad= \frac{1}{h}\mathbb{E}\bigg[\varphi_n\bigg(X_n(h,x) - \int_0^h \mathrm{e}^{(h-s)A}C^{1/2}F_n\big(X_n(s,x)\big)\,\mathrm{d}s\bigg) - \varphi_n(x)\bigg]\\
&\quad= \frac{1}{h}\mathbb{E}\big[\varphi_n\big(X_n(h,x)\big) - \varphi_n(x)\big]\\
&\qquad - \frac{1}{h}\mathbb{E}\bigg[\bigg\langle D\varphi_n\big(X_n(h,x)\big), \int_0^h \mathrm{e}^{(h-s)A}C^{1/2}F_n\big(X_n(s,x)\big)\,\mathrm{d}s\bigg\rangle\bigg] + o(h).
\end{aligned}$$

As $h \to 0$ we find that $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ and

$$L\varphi_n = \lambda\varphi_n - \langle C^{1/2}F_n, D\varphi_n\rangle.$$

If $f \in \mathcal{C}_b^2(H)$ we prove, by proceeding in the same way as above, that $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b^1(H)\big)$. $\square$

**Lemma 3.4.** *Let* $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b^1(H)\big)$*. Then* $\varphi \in \mathfrak{D}(K_1)$ *and*

$$K_1\varphi = L\varphi + \langle F, C^{1/2} D\varphi\rangle. \tag{3.7}$$

P r o o f. By Proposition 2.6 there exist a 4-sequence $(\varphi_k) \subset \mathcal{E}_A(H)$ and $M > 0$ such that

$$\varphi_k(x) \to \varphi(x),\ \ D\varphi_k(x) \to D\varphi(x),\ \ L\varphi_k(x) \to L\varphi(x), \quad x \in H,$$

and

$$|\varphi_k(x)| + |D\varphi_k(x)| \leqslant M, \quad |L\varphi_k(x)| \leqslant M(1 + |x|^2), \quad x \in H.$$

It follows that

$$K_1\varphi_k(x) \to L\varphi(x) + \langle F(x), C^{1/2} D\varphi(x)\rangle, \quad x \in H,$$

and

$$|K_1\varphi_k(x)| \leqslant M(1 + |x|^2) + M|F(x)|\|C^{1/2}\|.$$

Now the conclusion follows from (1.2) and the dominated convergence theorem. □

**Lemma 3.5.** *Let* $f \in \mathcal{C}_b^1(H)$ *and* $\lambda > 0$*. Then the solution* $\varphi_n$ *to* (3.5) *belongs to* $\mathfrak{D}(K_1)$ *and we have*

$$K_1\varphi_n = L\varphi_n + \langle F_n(x), C^{1/2} D\varphi_n\rangle. \tag{3.8}$$

P r o o f. By Lemma 3.3 we have $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b^1(H)\big)$ and by Lemma 3.4 we know that $\varphi_n \in \mathfrak{D}(K_1)$. Thus the conclusion follows. □

**Theorem 3.6.** $K_1$ *is* $m$*-dissipative on* $L^1(H, \nu)$*.*

P r o o f. Let $f \in \mathcal{C}_b^2(H)$ and let $\varphi_n$ be the solution to (3.5):

$$\lambda\varphi_n - L\varphi_n - \langle F_n(x), C^{1/2} D\varphi_n\rangle = f.$$

Then Lemma 3.5 yields $\varphi_n \in \mathfrak{D}(K_1)$ and

$$K_1\varphi_n = L\varphi_n + \langle F(x), C^{1/2} D\varphi_n\rangle.$$

Therefore

$$\lambda\varphi_n - K_1\varphi_n = f + \langle F_n(x) - F(x), C^{1/2}D\varphi_n\rangle. \tag{3.9}$$

Taking into account 3.2 we obtain

$$\lambda\int_H \varphi_n^2 \,\mathrm{d}\nu + \frac{1}{2}\int_H |C^{1/2}D\varphi_n|^2 \,\mathrm{d}\nu = \int_H f\varphi_n \,\mathrm{d}\nu + \int_H \varphi_n\langle F_n - F, C^{1/2}D\varphi_n\rangle \,\mathrm{d}\nu.$$

Moreover, in view of 3.6, $\|\varphi_n\|_0 \leqslant \lambda^{-1}\|f\|_0$,

$$\begin{aligned}
\lambda\int_H \varphi_n^2 \,\mathrm{d}\nu &+ \frac{1}{2}\int_H |C^{1/2}D\varphi_n|^2 \,\mathrm{d}\nu \\
&\leqslant \frac{1}{\lambda}\|f\|_0^2 + \frac{1}{\lambda}\|f\|_0 \int_H |F_n - F|\,|C^{1/2}D\varphi_n| \,\mathrm{d}\nu \\
&\leqslant \frac{1}{\lambda}\|f\|_0^2 + \frac{1}{4}\int_H |C^{1/2}D\varphi_n|^2 \,\mathrm{d}\nu + \frac{4}{\lambda^2}\|f\|_0^2 \int_H |F_n - F|^2 \,\mathrm{d}\nu.
\end{aligned}$$

Consequently, there exists a constant $M_1$ independent of $n$ and such that

$$\int_H |C^{1/2}D\varphi_n|^2 \,\mathrm{d}\nu \leqslant M_1.$$

It follows that

$$\lim_{n\to\infty} \langle F_n(x) - F(x), C^{1/2}D\varphi_n\rangle = 0$$

in $L^1(H,\nu)$ and so

$$\lim_{n\to\infty} \lambda\varphi_n - K_1\varphi_n = f.$$

Therefore the closure of the image of $\lambda - \overline{K}$ contains $\mathcal{C}_b^2(H)$ and so it is dense in $L^1(H,\nu)$. Now the conclusion follows from a classical result due to Lumer and Phillips. □

## 4. Gradient systems

We assume here, in addition to Hypotheses 1 and 2, that $A$ is self-adjoint and commuting with $C$. In this case the Ornstein-Uhlenbeck semigroup $\mathcal{R}_t$ is symmetric. We will denote by $\mu$ the Gaussian measure $N_Q$ of mean 0 and covariance operator $Q$. Moreover, we recall that for any $\varphi \in \mathfrak{D}(L)$ and any $\psi \in W_C^{1,2}(H,\mu)$ the following identity holds:

$$\int_H L\varphi\psi \,\mathrm{d}\mu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\psi\rangle \,\mathrm{d}\mu. \tag{4.1}$$

We are given a probability measure $\nu$ of the form

$$\nu(\mathrm{d}x) = \varrho(x)\,\mu(\mathrm{d}x),$$

where $\varrho$ fulfils

**Hypothesis 3.**
(i) $\varrho \geqslant 0, \quad \varrho \in L^1(H,\mu) \quad |x|^2\varrho \in L^1(H,\mu)$
(ii) $\sqrt{\varrho} \in W_C^{1,2}(H,\mu)$ *and* $\varrho \in W_C^{1,2}(H,\mu)$.

We notice that under Hypothesis 3 we have

$$C^{1/2}D\log\varrho \in L^2(H,\nu;H). \tag{4.2}$$

In fact,

$$\int_H |C^{1/2}D\log\varrho|^2\,\mathrm{d}\nu = \int_H \frac{|C^{1/2}D\varrho|^2}{\varrho}\,\mathrm{d}\mu = 4\int_H |C^{1/2}D\sqrt{\varrho}|^2\,\mathrm{d}\mu.$$

We set

$$U = -\frac{1}{2}\log\varrho, \quad F = -C^{1/2}DU = \frac{1}{2}C^{1/2}D\log\varrho.$$

We are going to show that Hypothesis 2 is fulfilled.

(i) follows from (4.2) and the assumption $|x|^2\varrho \in L^1(H,\mu)$.

(ii) is established by the following Proposition:

**Proposition 4.1.** *Under Hypothesis* 3 *we have*

$$\int_H \mathring{K}\varphi\,\mathrm{d}\nu = 0, \quad \varphi \in \mathcal{E}_A(H). \tag{4.3}$$

P r o o f. We have

$$\int_H \mathring{K}\varphi\,\mathrm{d}\nu = \int_H L\varphi\varrho\,\mathrm{d}\mu - \int_H \langle C^{1/2}DU, C^{1/2}D\varphi\rangle\,\mathrm{d}\nu.$$

However, in view of (4.1) we have

$$\int_H L\varphi\,\varrho\,\mathrm{d}\mu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\varrho\rangle\,\mathrm{d}\mu = \int_H \langle C^{1/2}DU, C^{1/2}D\varphi\rangle\,\mathrm{d}\nu,$$

and the conclusion follows. □

(iii) follows from the following Proposition:

**Proposition 4.2.** *Under Hypothesis* 3, $\mathring{K}$ *is symmetric. Moreover,*

$$\int_H (\mathring{K}\varphi)\psi \, \mathrm{d}\nu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\psi\rangle \, \mathrm{d}\nu, \quad \varphi, \psi \in \mathcal{E}_A(H). \tag{4.4}$$

Proof. For all $\varphi, \psi \in \mathcal{E}_A(H)$ we have

$$\int_H (\mathring{K}\varphi)\psi \, \mathrm{d}\nu = \int_H L\varphi(\psi\varrho) \, \mathrm{d}\mu - \int_H \langle DU, C^{1/2}D\varphi\rangle\psi \, \mathrm{d}\nu.$$

However, $\psi\varrho \in W^{1,2}_C(H,\nu)$, and so by (4.1) we have

$$\int_H (L\varphi)\psi\varrho \, \mathrm{d}\mu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\psi\rangle \, \mathrm{d}\mu - \frac{1}{2}\int_H \langle C^{1/2}D\varphi, D\log\varrho\rangle \, \mathrm{d}\mu,$$

and the conclusion follows easily □

**Remark 4.3.** By Proposition 4.2 it follows that $K_2$ is dissipative in $L^2(H,\mu)$. By [6] it follows that $K_p$ is dissipative in $L^p(H,\nu)$ for all $p \geqslant 1$.

Finally, to prove (iv) we need suitable approximations for $F$. To this end it is convenient to introduce the Sobolev space $W^{1,2}(H,\nu)$, in which $D_h$, the partial derivative in the direction $e_h$, is closable.

We need the following integration-by-parts formula.

**Proposition 4.4.** *Assume that Hypotheses* 1, 2 *and* 3 *hold. Let* $\varphi, \psi \in \mathcal{E}_A(H)$, $h \in \mathbb{N}$. *Then we have*

$$\int_H (D_h\varphi)\psi \, \mathrm{d}\nu = -\int_H \varphi(D_h\psi) \, \mathrm{d}\nu + \frac{1}{\lambda_h}\int_H x_h\varphi\psi \, \mathrm{d}\nu + 2\int_H \varphi\psi(D_hU) \, \mathrm{d}\nu. \tag{4.5}$$

Proof. In fact we have

$$\int_H (D_h\varphi)\psi \, \mathrm{d}\nu = \int_H (D_h\varphi)\psi\varrho \, \mathrm{d}\mu.$$

Since $\psi\varrho \in W^{1,2}(H,\mu)$ we have

$$\int_H (D_h\varphi)\psi \, \mathrm{d}\nu - \int_H (D_h\varphi)\psi\varrho \, \mathrm{d}\mu - \int_H \varphi D_h(\psi\varrho) \, \mathrm{d}\mu + \frac{1}{\lambda_h}\int_H x_h\varphi\psi \, \mathrm{d}\nu,$$

and the conclusion follows. □

Proposition 4.4 implies, by standard arguments, that the mapping

$$D\colon \mathcal{E}_A(H) \subset L^2(H,\nu) \to L^2(H,\nu;H)$$

is closable; we denote its closure again by $D$.

Let us define the space $W^{1,2}(H,\nu)$ as the subspace of $L^2(H,\nu)$ consisting of all functions $\varphi \in \mathfrak{D}(D)$ such that

$$\int_H |D\varphi|^2 \, \mathrm{d}\nu < +\infty.$$

Now, since $U \in W^{1,2}(H,\nu)$, there is a sequence $(U_N) \subset \mathcal{E}_A(H)$ such that

$$U_N \to U \ \text{ in } \ L^2(H,\nu), \quad DU_N \to DU \ \text{ in } \ L^2(H,\nu;H).$$

Hence we can apply the previous results.

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